PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 42, Fasc. 1 (2022), pp. 23–39
Published online 1.8.2022
doi:10.37190/0208-4147.00010

URBANIK TYPE SUBCLASSES OF FREE-INFINITELY DIVISIBLE TRANSFORMS

BY

ZBIGNIEW J. JUREK (WROCŁAW)

Abstract. For the class of free-infinitely divisible transforms we introduce three families of increasing Urbanik type subclasses. They begin with the class of free-normal transforms and end up with the whole class of free-infinitely divisible transforms. Those subclasses are derived from the ones of classical infinitely divisible measures for which random integral representations are known. Special functions like Hurwitz–Lerch, polygamma and hypergeometric functions appear in kernels of the corresponding integral representations.

2020 Mathematics Subject Classification: Primary 60E07, 60E10, 60H05; Secondary 33B15, 33C05.

Key words and phrases: characteristic function, infinite divisibility, selfdecomposability, s-selfdecomposability, Lévy–Khinchin formula, free-infinite divisibility, Nevanlinna–Pick functions, polygamma function, hypergeometric function.

Limit distribution theory is one of the main topics in probability theory. Historically, it began with the central limit theorem which says that properly normalized partial sums of *independent and identically distributed* (i.i.d.) random variables with finite second moment converge in distribution to the *standard normal (Gaussian)* variable. When we drop the assumption about moments but still assume i.i.d. variables, in the limit we get the class of *stable distributions*. Still further, if we assume that observations (variables) are only stochastically independent but after some normalization by positive constants the corresponding triangular array is *uniformly infinitesimal* then in the limit we obtain the class of *selfdecomposable distributions* (Lévy class L). Finally, limits of sums of arbitrary infinitesimal and row-wise independent triangular arrays coincide with the class of *infinitely divisible distributions*; see Feller [4, Chapter XVII] or Gnedenko and Kolmogorov [5, Sects. 17–19, 29–30 and 33] or Loeve [23, Sect. 23]. Thus we have

$$(\star) \qquad \text{(Gaussian)} \subset \cdots \subset \text{(selfdecomposable)} \subset \cdots \subset \text{(infinitely divisible)}$$

[Here it might be worthy to notice that the class of selfdecomposable measures can be obtained from *strongly mixing sequences* not necessarily stochastically independent; cf. Bradley and Jurek [3]. That possible direction of study is not pursued in this article.]

Urbanik [24, 25] refined the left hand side inclusion (below class L), and Jurek [13], [14] refined the right hand side inclusion by introducing new subclasses of limit distributions.

Later on, all the subclasses introduced were generalized to distributions on infinite-dimensional spaces and then described as distributions of some random integrals on arbitrary Banach spaces. In Jurek [10], the normalization of partial sums of random variables was done by bounded linear operators on a Banach space. Those and other multidimensional set-ups might be of some use in a generalization to multidimensional free-probability theory. [For the random integral representation conjecture see the link following the reference item [12].]

In this paper we give characterizations of the above mentioned results (i.e., refinements of the inclusions in $(\star)$ by adding new subclasses in place of "$\cdots$") for additive free-independence (free-additive convolution $\boxplus$). More precisely, we describe the corresponding free-independent (Voiculescu) transforms. Those transforms are considered only on the imaginary axis, which is enough for the identification of the corresponding measure; see Jurek [16] and Jankowski and Jurek [7].

In Theorem 1 and the auxiliary Lemma 1 we deal with general random integral mappings that lead to subsets of free-independent transforms. Propositions 1–4 provide applications of Theorem 1 to some specified mappings. A complete filtration of the class of all free-infinitely divisible transforms is given in Corollary 2. Finally, Theorem 2 shows an intrinsic relation between two classes of free-infinitely divisible transforms: one derived from *linear* scalings and the other from *nonlinear* scalings.

0. INTRODUCTION AND NOTATIONS

We will introduce Urbanik type subclasses of free-infinitely divisible Voiculescu transforms in a such way that

$$
\begin{aligned}
(0.1)\qquad & (\text{Gaussian}, \boxplus) \subset (\text{stable}, \boxplus) \subset \cdots \\
& \subset (\mathcal{U}^{\langle k+1\rangle}, \boxplus) \subset (\mathcal{U}^{\langle k\rangle}, \boxplus) \subset \cdots \subset (\mathcal{U}^{\langle 1\rangle}, \boxplus) \equiv (\mathcal{U}, \boxplus) \\
& \equiv (\mathbb{U}_1, \boxplus) \subset (\mathbb{U}_2, \boxplus) \subset \cdots \subset \overline{\bigcup_{k=1}^{\infty} (\mathbb{U}_k, \boxplus)} \equiv (\text{ID}, \boxplus),
\end{aligned}
$$

where the closure is in the pointwise convergence of Voiculescu transforms (the topology of weak convergence of measures) and $\boxplus$ is the free-additive convolution. The classes $(\mathcal{U}^{\langle k\rangle}, \boxplus)$ and $(\mathbb{U}_k, \boxplus)$ are the free-probability counterparts of the classical probability classes $(\mathcal{U}^{\langle k\rangle}, *)$ and $(\mathbb{U}_k, *)$ in $(\text{ID}, *)$. For each of these classes there are characterizations in terms of random integrals; for a general conjecture see the link following reference item [12].

For the above classes we have the following integral representations:

$$\text{(0.2)}\quad \text{(i) } \mu \in (\mathcal{U}^{\langle k\rangle}, *) \quad \text{iff} \quad \mu = \mathcal{L}\Big(\int_{(0,1]} t\, dY(\tau_k(t))\Big),$$

$$\tau_k(t) := \frac{1}{(k-1)!}\int_0^t (-\log v)^{k-1} dv,\ 0 < t \leqslant 1,\ k = 1, 2, \ldots;$$

$$\text{(ii) } \nu \in (\mathbb{U}_k, *) \quad \text{iff} \quad \mu = \mathcal{L}\Big(\int_{(0,1]} t\, dY(r_k(t))\Big),\ r_k(t) := t^k,\ 0 \leqslant t \leqslant 1,$$

and $(Y(t), t \geqslant 0)$ is a cadlag Lévy process and $\mathcal{L}(Z)$ denotes the probability distribution of a random variable Z. [We have $\int (-\log x)^k\, dx = \Gamma(k, -\log x) + \mathrm{const}$ (the incomplete Euler gamma function), but we do not use this identity here.]

The integrals in (i) and (ii) are particular examples of random integrals

$$\text{(0.3)}\qquad \rho = I^{h,r}_{(a,b]}(\mu) := \mathcal{L}\Big(\int_{(a,b]} h(t)\, dY(r(t))\Big), \qquad \mathcal{L}(Y(1)) = \mu \in \mathcal{D}^{h,r}_{(a,b]},$$

where h is a real function, r (a time change) is a monotone, non-negative function and $\mathcal{D}^{h,r}_{(a,b]}$ denotes the domain of the random integral $I^{h,r}_{(a,b]}$; for details see, for instance, [13, 14, 15, 17, 19]. To Y (resp. μ) we refer as the *background driving Lévy process* (BDLP) (resp. *the background driving probability distribution* (BDPD)) of the measure ρ.

The identification (isomorphism) between classical infinitely divisible characteristic functions $\phi_\mu(t)$ and their free-infinitely divisible counterparts, the transform $V_{\tilde{\mu}}(it)$ (or measure) is given as follows:

$$\text{(0.4)}\qquad (\mathrm{ID}, *) \ni \mu \mapsto V_{\tilde{\mu}}(it) = it^2 \int_0^\infty \log \phi_\mu(-u) e^{-tu}\, du, \qquad t > 0;$$

see Jurek [17, Corollary 6] and the random integral mapping $\mathcal{K}^{(e)}$ in [17] which was the origin for the identity (0.4). The need for such an identification arises when one wants to use Bercovici–Pata isomorphism but we do not have parameters a and m, in the Lévy–Khinchin or Bercovici–Voiculescu formula, for classical and free independence. That those two approaches coincide was shown in [17], [18], [20, Theorem 2.1]. Moreover, [20, p. 350] gives a diagram showing how one may connect two abstract semigroups.

From the above mapping (0.4) we infer the properties

$$V_{\widetilde{\mu * \nu}}(it) = V_{\tilde{\mu}}(it) + V_{\tilde{\nu}}(it) = V_{\tilde{\mu} \boxplus \tilde{\nu}}(it); \qquad V_{\widetilde{T_c\mu}}(it) = cV_{\tilde{\mu}}(it/c) \quad \text{for } c > 0,$$

and the last property is in sharp contrast with $\phi_{T_c\mu}(t) = \phi_\mu(ct)$ for the characteristic functions.

The fundamental Lévy–Khinchin characterization says that

$$\text{(0.5)}\qquad \mu\in \mathrm{ID}\quad \text{iff}\quad \phi_\mu(t)=\exp\Big[ita+\int_{\mathbb{R}}\Big(e^{itx}-1-\frac{itx}{1+x^2}\Big)\frac{1+x^2}{x^2}m(dx)\Big]$$
$$=\exp\Big[ita-\frac{1}{2}\sigma^2t^2+\int_{\mathbb{R}\setminus\{(0)\}}\Big(e^{itx}-1-\frac{itx}{1+x^2}\Big)M(dx)\Big],\qquad t\in\mathbb{R},$$

where the real number a, the finite Borel measure m with $m(\{0\})=\sigma^2$ and the Lévy spectral measure $M(dx)=\frac{1+x^2}{x^2}m(dx)$ are uniquely determined. In that case, we will simply write

$$\mu=[a,\sigma^2,M].$$

Here is the Voiculescu analogue for free-infinite divisibility of the Lévy–Khinchin formula:

$$\text{(0.6)}\qquad \nu\in(\mathrm{ID},\boxplus)\quad\text{iff}\quad V_\nu(it)=a+\int_{\mathbb{R}}\frac{1+itx}{it-x}\,m(dx),\qquad t\neq 0,$$

where the real number a and the finite Borel measure m are uniquely determined; for details see Voiculescu [26], Bercovici and Voiculescu [2], Barndorff–Nielsen and Thorbjorsen [1] and Jurek [16, 17].

Uniqueness of the parameters a and m in (0.5) and (0.6) gives a natural identification between classical and free-infinitely divisible transforms as mentioned above. On the other hand, if one knows that $\phi_\mu\in(\mathrm{ID},*)$ or $V_\nu(it)\in(\mathrm{ID},\boxplus)$, finding their corresponding parameters a and m might be, in general, quite difficult. In that case, for $(\mathrm{ID},*)$ and $(\mathrm{ID},\boxplus)$ we use the identification given by (0.4).

1. A BASIC THEOREM

We begin with a basic result which, later on, will allow us to introduce new subclasses of free-infinitely divisible transforms by specifying their corresponding random integral representations $I^{h,r}_{(a,b]}(\mu)$; see (0.3) above and (1.4) below.

For a real measurable function h, a positive monotone function r and an interval $(a,b]$, we define constants

$$\mathbf{c}:=\int_a^b h(s)\,dr(s),\qquad \mathbf{d}:=\int_a^b h^2(s)\,dr(s).$$

Furthermore, we define two functions, $\mathbf{g}_+$ and $\mathbf{g}_-$, depending on the monotonicity of r:

$$\mathbf{g}_+(z):=\int_a^b\frac{h(s)}{zh(s)+1}\,dr(s)\quad\text{and}\quad \mathbf{g}_-(z):=\int_a^b\frac{h(s)}{zh(s)-1}\,dr(s),\ z\in\mathbb{C},$$

for r respectively non-decreasing and non-increasing. For brevity, in what follows we write these as $\mathbf{g}_\pm(z)$.

THEOREM 1. *For an infinitely divisible measure $\mu = [a, \sigma^2, M] \in \mathcal{D}^{h,r}_{(a,b]}$ and $\rho := I^{h,r}_{(a,b]}(\mu)$ there exists a free-infinitely divisible counterpart $\tilde{\rho} \in (\mathrm{ID}, \boxplus)$ with Voiculescu transform*

$$V_{\tilde{\rho}}(it) = a\,\boldsymbol{c} \pm \frac{\sigma^2}{it}\,\boldsymbol{d} \pm \int\limits_{\mathbb{R}\setminus(0)} x\left[\boldsymbol{g}_{\pm}\left(\frac{ix}{t}\right) - \frac{\pm\boldsymbol{c}}{1+x^2}\right] M(dx), \qquad t > 0, \tag{1.1}$$

where the upper sign in each occurrence of $\pm$ is for r non-decreasing and the lower sign is for r non-increasing. This convention will be in force throughout the paper, and applies to each occurrence of $\mp$ as well.

Equivalently, by putting $m(dx) := \frac{x^2}{1+x^2}M(dx)$ on $\mathbb{R}\setminus\{0\}$ and $m(\{0\}) := \sigma^2$ we get a finite measure m such that

$$V_{\tilde{\rho}}(it) = a\,\boldsymbol{c} \pm \int\limits_{\mathbb{R}} \left[\boldsymbol{g}_{\pm}\left(\frac{ix}{t}\right) - \frac{\pm\boldsymbol{c}}{1+x^2}\right]\frac{1+x^2}{x}\, m(dx), \qquad t > 0, \tag{1.2}$$

where the integrand is equal to $\boldsymbol{d}(it)^{-1}$ at zero.

Moreover, if $h(s) > 0$, then we have the following relation between $\tilde{\rho}$ and $\tilde{\mu}$, the free-probability counterparts of ρ and its background driving measure μ:

$$V_{\tilde{\rho}}(it) = \int\limits_a^b h(s)V_{\tilde{\mu}}(it/h(s))\, dr(s) = \int\limits_a^b V_{\widetilde{T_{h(s)}\mu}}(it)\, dr(s), \qquad t > 0, \tag{1.3}$$

where $(T_c(\mu))(B) := \mu(c^{-1}B)$ for Borel sets B and $c > 0$.

Proof. The isomorphism (0.4) gives a one-to-one correspondence between the classical $\phi_\mu(t)$ and the free-infinitely divisible $V_{\tilde{\mu}}(it)$ transforms. The law (of the random integral) $\rho = I^{h,r}_{(a,b]}(\mu)$ has characteristic function

$$\log\phi_\rho(v) = \int\limits_a^b \log\phi_\mu(\pm v h(s))\, d(\pm r(s)), \tag{1.4}$$

by (0.3); for details see [19, p. 279] or [20].

Since by (0.5), $\log\phi_\mu(t) = ita - \frac{\sigma^2}{2}t^2 + \int_{\mathbb{R}\setminus\{(0)\}}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)M(dx)$, using the Fubini Theorem and (0.4) we get

$$\begin{aligned}
V_{\tilde{\rho}}(it) &= it^2\int\limits_0^\infty \log\phi_\rho(-u)e^{-tu}\, du \\
&= it^2\int\limits_0^\infty\int\limits_a^b \log\phi_\mu(-(\pm u)h(s))\, d(\pm r(s))e^{-tu}\, du \\
&= \int\limits_a^b it^2\int\limits_0^\infty [\log\phi_\mu(\mp u h(s))\, e^{-tu}\, du]\, d(\pm r(s))
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b ia(\mp h(s))it^2 \int_0^\infty ue^{-tu}\,du\,d(\pm r(s)) \\
&\quad - \int_a^b (\mp h(s))^2 \frac{1}{2}\sigma^2 it^2 \int_0^\infty u^2 e^{-tu}\,du\,d(\pm r(s)) \\
&\quad + \int_a^b \int_{\mathbb{R}\setminus\{(0)\}} it^2 \int_0^\infty \left(e^{\mp ih(s)xu} - 1 - \frac{x}{1+x^2}(\mp ih(s)u) \right) e^{-tu}\,du\,M(dx)\,d(\pm r(s)) \\
&= \int_a^b ia(\mp h(s))it^2 \frac{1}{t^2}\,d(\pm r(s)) - \int_a^b (h(s))^2 \frac{1}{2}\sigma^2 it^2 \frac{2}{t^3}\,d(\pm r(s)) \\
&\quad + \int_a^b \int_{\mathbb{R}\setminus\{(0)\}} it^2 \left(\frac{1}{t \pm ih(s)x} - \frac{1}{t} - \frac{i(\mp h(s))x}{1+x^2}\frac{1}{t^2} \right) M(dx)\,d(\pm r(s)) \\
&= a\mathbf{c} \pm \frac{\sigma^2}{it}\mathbf{d} + \int_{\mathbb{R}\setminus\{(0)\}} \int_a^b \left[\frac{\pm t\,h(s)x}{t \pm ixh(s)} + \frac{x}{1+x^2}(\mp h(s)) \right] d(\pm r(s))\,M(dx) \\
&= a\mathbf{c} \pm \frac{\sigma^2}{it}\mathbf{d} \pm \int_{\mathbb{R}\setminus\{(0)\}} x \int_a^b \left[\frac{h(s)}{1 \pm ixh(s)/t} + \frac{1}{1+x^2}(\mp h(s)) \right] dr(s)\,M(dx) \\
&= a\mathbf{c} \pm \frac{\sigma^2}{it}\mathbf{d} \pm \int_{\mathbb{R}} x \left[\mathbf{g}_\pm(ix/t) - \frac{\pm \mathbf{c}}{1+x^2} \right] M(dx),
\end{aligned}
$$

which proves (1.1).

To get (1.2), note that $g_\pm(0) = \pm\mathbf{c}$ and

$$
\lim_{x\to 0} \frac{\mathbf{g}_\pm(\frac{ix}{t}) - \frac{\pm\mathbf{c}}{1+x^2}}{x} = \lim_{x\to 0} \int_a^b \frac{(h(s))^2}{(ixh(s)/t \pm 1)^2} \frac{-i}{t}\,dr(s) = \frac{1}{it}\mathbf{d}.
$$

Similarly,

$$
\begin{aligned}
V_{\tilde{\rho}}(it) &= \int_a^b it^2 \int_0^\infty [\log \phi_\mu(-h(s)u)e^{-tu}\,du]\,dr(s) \\
&= \int_a^b it^2 \int_0^\infty \left[\log \phi_\mu(-w)\,e^{-tw/h(s)} \frac{dw}{h(s)} \right] dr(s) = \int_a^b h(s) V_{\tilde{\mu}}(it/h(s)\,dr(s) \\
&= \int_a^b V_{\widetilde{T_{h(s)}\mu}}(it)\,dr(s),
\end{aligned}
$$

as by (0.4), $V_{\widetilde{T_c\mu}}(it) = cV_{\tilde{\mu}}(it/c)$ for $c > 0$; this completes the proof. ∎

The kernel $\mathbf{g}_+(z)$ from Theorem 1 admits the following representation:

LEMMA 1. *For r non-decreasing, the function $\mathbf{g}_+(z) := \int_a^b \frac{h(s)}{zh(s)+1}\,dr(s)$ maps the upper half-plane $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$ and is analytic with Pick–Nevanlinna representation*

$$\mathbf{g}_+(z) = \int_a^b \frac{h(s)}{1+h^2(s)}\,dr(s) + \int_{\mathbb{R}} \frac{1+zx}{z-x}\Big(\int_a^b \frac{h^2(s)}{1+h^2(s)}\delta_{-1/h(s)}(dx)\,dr(s)\Big).$$

Proof. First, note that $\Im(\mathbf{g}_+(z)) = -\Im(z)\int_a^b \frac{h^2(s)}{|1+zh(s)|^2}\,dr(s)$, where the integral is positive because r is non-decreasing. This means that $\mathbf{g} : \mathbb{C}^+ \to \mathbb{C}^-$ is an analytic function with

$$\frac{d^n}{dz^n}\mathbf{g}_+(z) = (-1)^n n! \int_a^b \Big(\frac{h(s)}{1+zh(s)}\Big)^{n+1} dr(s), \qquad n = 0, 1, \ldots.$$

Second, for $b \in \mathbb{R}$ we have the explicit Pick–Nevanlinna representation

$$\frac{1}{z+b} = u_b + \int_{\mathbb{R}} \frac{1+zx}{z-x}\,m_b(dx), \qquad u_b := \frac{b}{1+b^2}, \qquad m_b(A) := \frac{1}{1+b^2}\delta_{-b}(A),$$

and taking $b := 1/h(s)$ and integrating the above with respect to $dr(s)$ we get

$$\mathbf{g}_+(z) = u + \int_{\mathbb{R}} \frac{1+zx}{z-x}\,m(dx),$$

where $u := \int_a^b \frac{h(s)}{1+h^2(s)}\,dr(s)$ is the shift parameter and

$$m(A) := \int_a^b \frac{h^2(s)}{1+h^2(s)}\delta_A\Big(-\frac{1}{h(s)}\Big)\,dr(s) = \int_a^b \frac{h^2(s)}{1+h^2(s)}\delta_{-1/h(s)}(A)\,dr(s)$$

is a mixture of point-mass Dirac measures, $k(s)\delta_{f(s)}(A)$. This finishes the proof. ■

2. THE CLASSES $(\mathcal{U}^{\langle k\rangle}, \boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS

For the one-parameter semigroup $(U_r, r > 0)$ of non-linear *shrinking operations* (for short, *s-operations*) $U_r : \mathbb{R} \to \mathbb{R}$ defined

$$U_r(0) := 0, \qquad U_r(x) := \max\{|x| - r, 0\}\frac{x}{|x|}, \qquad x \neq 0,$$

Jurek [8, 9] introduced the class $\mathcal{U}$ of limiting distributions of sequences

$$U_{r_n}(X_1) + U_{r_n}(X_1) + \cdots + U_{r_n}(X_n) + x_n,$$

where the terms $U_{r_n}(X_j),\ 1 \leqslant j \leqslant n$, are uniformly infinitesimal and the random variables $X_n, n = 1, 2, \ldots$, are stochastically independent. The measures $\mu \in \mathcal{U}$ were termed *s-selfdecomposable.*

REMARK 1. Note that, in mathematical finance, for $X > 0$, the s-operation $U_r(X) = (X - r)_+$ is called *the European call option* on a stock X with exercise price r.

Jurek [15] introduced and characterized the following subclasses of the class $(\mathrm{ID}, *)$ of classical infinitely divisible measures:

$$\cdots \subset \mathcal{U}^{\langle k+1\rangle} \subset \mathcal{U}^{\langle k\rangle} \subset \cdots \subset \mathcal{U}^{\langle 1\rangle} \equiv \mathcal{U} \subset \mathrm{ID},$$

and measures $\mu \in \mathcal{U}^{\langle k\rangle}$ were called *k-times s-selfdecomposable*. Furthermore, as mentioned in the Introduction, taking the time change

$$\tau_k(t) := \frac{1}{(k-1)!} \int_0^t (-\log v)^{k-1}\, dv, \tag{2.1}$$

we get

$$(\mathcal{U}^{\langle k\rangle}, *) = I_{(0,1]}^{t,\tau_k(t)}(\mathrm{ID}), \text{ and } I_{(0,1]}^{t,\tau_k(t)}(\nu) = I_{(0,1]}^{t,t}(I_{(0,1]}^{t,t}(\ldots (I_{(0,1]}^{t,t}(\nu))) \ (k \text{ times});$$

see [15, Proposition 4 and Corollary 2], and [19] for more general theory of compositions of random integrals.

Here are the free-infinitely divisible counterparts of k-times s-selfdecomposable probability measures:

PROPOSITION 1. *For $k = 1, 2, \ldots$, a measure $\tilde{\nu}$ is a free-probability counterpart of $\nu = [a, \sigma^2, M] \in (\mathcal{U}^{\langle k\rangle}, *)$, that is, $\tilde{\nu} \in (\mathcal{U}^{\langle k\rangle}, \boxplus)$, if and only if*

$$V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \frac{\sigma^2}{3^k}\frac{1}{it} + \int_{\mathbb{R}\setminus(0)} x\left[\Phi\left(\frac{x}{it}, k, 2\right) - \frac{1}{1+x^2}\frac{1}{2^k}\right] M(dx). \tag{2.2}$$

Equivalently,

$$V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \int_{\mathbb{R}} \left[\Phi\left(\frac{x}{it}, k, 2\right) - \frac{1}{1+x^2}\frac{1}{2^k}\right] \frac{1+x^2}{x}\, m(dx), \tag{2.3}$$

where $a \in \mathbb{R}$, $m(dx) := \frac{x^2}{1+x^2} M(dx)$ on $\mathbb{R} \setminus \{0\}$ and $m(\{0\}) := \sigma^2$, is a finite Borel measure m and $\Phi(z, s, v) := \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s}$, $|z| < 1$, $v \neq 0, -1, -2, \ldots$, is the Hurwitz–Lerch function. Finally, the integrand in (2.3) *is equal to $(3^k it)^{-1}$ at zero.*

Proof. Taking into account (2.1) and Theorem 1, we get $\mathbf{c} = 2^{-k}$ and $\mathbf{d} = 3^{-k}$. Furthermore, to find $\mathbf{g}_+(z)$, by Gradshteyn and Ryzhik [6, formula (9.556)], the Hurwitz–Lerch function has the following integral representation: if $\Re v > 0$, or $|z| \leqslant 1$, $z \neq 1$, $\Re s > 0$, or $z = 1$, $\Re s > 1$ then

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}}\, dt.$$

Hence

$$\mathbf{g}_{+}(z) = \frac{1}{(k-1)!}\int_0^1 \frac{s(-\log s)^{k-1}}{1+zs}\,ds$$
$$= \frac{1}{(k-1)!}\int_0^\infty \frac{w^{k-1}e^{-2w}}{1+zw}\,dw = \Phi(-z,k,2),$$

which completes the proof. ■

REMARK 2. (i) For the class $(\mathcal{U}^{\langle 1\rangle},\boxplus)$ of free s-selfdecomposable measures we may use the identity $\Phi(-ix/t,1,2) = it(-x - it\log(1+ix/t))$. The characterization of the class $(\mathcal{U}^{\langle 1\rangle}\boxplus) \equiv (\mathcal{U},\boxplus)$ was given in [18, Proposition 1(b)]. Note a misprint there: $(it)^2$ should read t^2 in part (b).

(ii) Putting $k=0$ in Propositon 1, we get $(\mathcal{U}^{\langle 0\rangle},\boxplus) \equiv (\mathrm{ID},\boxplus)$, by (0.6).

Here are relations between consecutive classes $(\mathcal{U}^{\langle k\rangle},\boxplus)$:

COROLLARY 1. *Define* $\mathbb{D}f(t) := 2f(t) - t\frac{d}{dt}f(t)$. *Then for* $k \geqslant 1$,

$$\mathbb{D} : (\mathcal{U}^{\langle k\rangle)},\boxplus) \to (\mathcal{U}^{\langle k-1\rangle},\boxplus), \quad \textit{where} \quad (\mathcal{U}^{\langle 0\rangle},\boxplus) := (\mathrm{ID},\boxplus).$$

Hence $\mathbb{D}^k : (\mathcal{U}^{\langle k\rangle},\boxplus) \to (\mathrm{ID},\boxplus)$.

Proof. Let $V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \frac{1}{3^k}\frac{\sigma^2}{it} \in (\mathcal{U}^{\langle k\rangle},\boxplus)$. Then

$$\mathbb{D}(V_{\tilde{\nu}}(it)) = \frac{a}{2^{k-1}} + \frac{2\sigma^2}{3^k}\,\frac{1}{it} - t\frac{\sigma^2}{3^k}(-1)i(it)^{-2}$$
$$= \frac{a}{2^{k-1}} + \frac{1}{3^{k-1}}\,\frac{\sigma^2}{it} \in (\mathcal{U}^{\langle k-1\rangle},\boxplus).$$

Since by Wolframalpha.com

$$\frac{d}{dt}\left[\Phi\left(\frac{x}{it},k,2\right)\right] = -t^{-1}\left(\Phi\left(\frac{x}{it},k-1,2\right) - 2\Phi\left(\frac{x}{it},k,2\right)\right),$$

for the Poisson part in (2.2) we have

$$\mathbb{D}\left[\Phi\left(\frac{x}{it},k,2\right) - \frac{1}{2^k}\,\frac{1}{1+x^2}\right]$$
$$= 2\Phi\left(\frac{x}{it},k,2\right) - \frac{1}{2^{k-1}}\,\frac{1}{1+x^2} - t\frac{d}{dt}\left(\Phi\left(\frac{x}{it},k,2\right)\right)$$
$$= 2\Phi\left(\frac{x}{it},k,2\right) - \frac{1}{2^{k-1}}\,\frac{1}{1+x^2} + \Phi\left(\frac{x}{it},k-1,2\right) - 2\Phi\left(\frac{x}{it},k,2\right)$$
$$= \Phi\left(\frac{x}{it},k-1,2\right) - \frac{1}{2^{k-1}}\,\frac{1}{1+x^2},$$

which is the kernel in (2.2) corresponding to the free-infinitely divisible measure in $(\mathcal{U}^{\langle k-1\rangle},\boxplus)$. This completes the proof. ■

3. THE CLASSES $(\mathbb{U}_k, \boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS

For a fixed k, a probability measure μ is in $(\mathbb{U}_k, *)$ if there exists a sequence $\nu_n \in (\mathrm{ID}, *), n = 1, 2, \ldots,$ such that

$$\rho_n := T_{1/n}(\nu_1 * \nu_2 * \cdots * \nu_n)^{*n^{-k}} \Rightarrow \mu \quad \text{as } n \to \infty \tag{3.1}$$

(see [13, Theorem 1.1 and Corollary 1.1]; take there $Q = I$, Borel measures ν_k on the real line and $\beta = k$).

The class of all possible limits in (3.1) is denoted by $(\mathbb{U}_k, *)$ and measures $\mu \in (\mathbb{U}_k, *)$ are referred to as *k-times s-selfdecomposable measures*. Note that for $k = 1$ we get the class $(\mathbb{U}, *)$ of s-selfdecomposable measures.

Furthermore, the subclasses $(\mathbb{U}_k, *)$ form an increasing filtration of the class of infinitely divisible measures, and all subclasses admit random integral representations:

$$\text{if } 0 \leqslant k \leqslant l \text{ then } (\mathbb{U}_0, *) \subset (\mathbb{U}_k, *) \subset (\mathbb{U}_l, *) \subset (\mathrm{ID}, *); \tag{3.2}$$

in particular,

$$(\mathbb{U}_0, *) \equiv (L_0, *) \text{ (selfdecomposable measures; see Section 4)};$$
$$(\mathbb{U}_1, *) \equiv (\mathcal{U}^{\langle 1 \rangle}, *) \text{ (s-selfdecomposable measures; see Section 2)};$$
$$(\mathbb{U}_k, *) = I^{t,t^k}_{(0,1]}(\mathrm{ID}); \quad \overline{\bigcup_{k=1}^{\infty} (\mathbb{U}_k, *)} = (\mathrm{ID}, *).$$

Here are transforms of free-infinitely divisible counterparts of measures from the classes $(\mathbb{U}_k, *)$:

PROPOSITION 2. *For $k \geqslant 1$, a measure $\tilde{\nu}$ is a free-probability counterpart of $\nu = [a, \sigma^2, M] \in (\mathbb{U}_k, *)$, that is, $\tilde{\nu} \in (\mathbb{U}_k, \boxplus)$, if and only if for $t > 0$,*

$$\begin{aligned} V_{\tilde{\nu}}(it) &= \frac{k}{k+1} a + \frac{k}{k+2} \frac{\sigma^2}{it} \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \left[kit\Phi\left(\frac{x}{it}, 1, k\right) - it - \frac{k}{k+1} \frac{x}{1+x^2} \right] M(dx) \\ &= \frac{k}{k+1} a + \int_{\mathbb{R}} \left[kit \left(\Phi\left(\frac{x}{it}, 1, k\right) - k^{-1} \right) - \frac{k}{k+1} \frac{x}{1+x^2} \right] \frac{1+x^2}{x^2} m(dx) \end{aligned} \tag{3.3}$$

where M is an arbitrary Lévy measure; $m(dx) := \frac{x^2}{1+x^2} M(dx)$ on $\mathbb{R} \setminus \{0\}$, and $m(\{0\}) := \sigma^2$, is a finite measure; the integrand in (3.3) is $\frac{k}{k+2} \frac{1}{it}$ at zero; and $\Phi(z, s, v)$ is the Hurwitz–Lerch function.

Proof. Since $\mathbb{U}_k = I^{t,t^k}_{(0,1]}(\mathrm{ID})$, we take $a = 0, b = 1$, $h(t) = t$ and $r(t) = t^k$ in Theorem 1. Thus $\mathbf{c} = k/(k+1)$, $\mathbf{d} = k/(k+2)$ and

$$\mathbf{g}_+(z) = k\int_0^1 \frac{s^k}{1+zs}\,ds = \frac{k}{k+1}\,{}_2F_1(1, k+1; k+2; -z)$$

by Gradshteyn and Ryzhik [6, 3.194(5)] ($|\arg(1+z)| < \pi$) where ${}_2F_1$ denotes *the hypergeometric function* defined as

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad (x)_n := x(x+1)\dots(x+n-1),\ c \neq -\mathbb{N};$$

where $(x)_n$ is the *Pochhammer symbol* with $(x)_0 := 1$.

Consequently,

$$\begin{aligned}
\mathbf{g}_+(z) &= \frac{k}{k+1}\,{}_2F_1(1, k+1; k+2; -z)\\
&= k(k+1)^{-1}\sum_{n=0}^{\infty}\frac{(1)_n(k+1)_n}{(k+2)_n}\frac{(-z)^n}{n!}\\
&= k\sum_{n=0}^{\infty}\frac{(-z)^n}{k+n+1} = k(-z)^{-1}\sum_{j=1}^{\infty}\frac{(-z)^j}{k+j}\\
&= k(-z)^{-1}\left[\sum_{n=0}^{\infty}\frac{(-z)^n}{k+n} - \frac{1}{k}\right] = k(-z)^{-1}[\Phi(-z, 1, k) - k^{-1}].
\end{aligned}$$

Finally, $x\mathbf{g}(\frac{ix}{t}) = itk(\Phi(x/(it), 1, k) - k^{-1})$ and this completes the proof. ∎

COROLLARY 2. *If* $\tilde{\nu}_k \in (\mathbb{U}_k, \boxplus)$ *then*

$$\lim_{k\to\infty} V_{\tilde{\nu}_k}(it) = a + \int_{\mathbb{R}} \frac{1+itx}{it-x}\,m(dx) = V(it) \in (\mathrm{ID}, \boxplus) \quad \textit{for } t > 0.$$

In other words, $\overline{\bigcup_{k=1}^{\infty}(\mathbb{U}_k, \boxplus)} = (\mathrm{ID}, \boxplus)$.

Proof. Note that as $k \to \infty$ then

$$\begin{aligned}
k\Phi\left(\frac{x}{it}, 1, k\right) &= k\sum_{n=0}^{\infty}\left(\frac{x}{it}\right)^n\frac{1}{k+n} = \sum_{n=0}^{\infty}\left(\frac{x}{it}\right)^n\frac{1}{1+n/k}\\
&\to \sum_{n=0}^{\infty}\left(\frac{x}{it}\right)^n = \frac{it}{it-x},
\end{aligned}$$

and

$$\left[kit\Phi\left(\frac{x}{it}, 1, k\right) - it - \frac{k}{k+1}\frac{x}{1+x^2}\right]\frac{1+x^2}{x^2}$$
$$\to \left[it\frac{it}{it-x} - it - \frac{x}{1+x^2}\right]\frac{1+x^2}{x^2} = \frac{1+itx}{it-x},$$

which proves the corollary. ∎

REMARK 3. For any $\beta \geqslant -2$, the classes $(\mathbb{U}_\beta, \boxplus)$ are well defined by (3.1); cf. [13, 14]. Here we have restricted indices to the natural numbers to have the sequence of inclusions as announced in (0.1). Proposition 2 holds true when one replaces $k \geqslant 1$ by $\beta > 0$. Furthermore, for $\beta = 0$ we get the selfdecomposable distributions as discussed below.

4. URBANIK TYPE CLASSES $(L_k, \boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS

Urbanik [24, 25] introduced a filtration of convolution semigroups of *selfdecomposable measures* (Lévy class L_0) in such a way that

$$\text{(Gaussian)} \subset \text{(stable)} \subset L_\infty \subset \cdots \subset L_{k+1} \subset L_k \subset \cdots \subset L_0 \subset \cdots \subset \text{ID}. \tag{4.1}$$

Then using the extreme points method he found their descriptions in terms of characteristic functions. Measures $\mu \in (L_k, *)$ are called *k-times selfdecomposable*.

Later on, all the above classes were described in terms of random integrals. Namely, taking

$$r_k(t) := t^{k+1}/(k+1)!, \quad h(t) := e^{-t}, \quad t \in (0, \infty),$$

we have the representations

$$L_k = I_{(0,\infty)}^{e^{-t}, r_k(t)}(\text{ID}_{\log^{k+1}}),$$
$$\text{ID}_{\log^{k+1}} := \Big\{\nu \in \text{ID} : \int_{\mathbb{R}} \log^{k+1}(1+|x|)\nu(dx) < \infty\Big\}.$$

Furthermore, from the integral representations one easily gets their characteristic functions in the same form as in [24, 25]; see [11, Corollary 2.11 and Theorem 3.1].

Here are the free-infinitely divisible analogues of the Urbanik classes $(L_k, *)$:

PROPOSITION 3. *For $k = 0, 1, \ldots$, a measure $\tilde{\nu}$ is a free-probability counterpart of $\nu = [a, \sigma^2, M] \in (L_k, *)$, that is, $\tilde{\nu} \in (L_k, \boxplus)$, if and only if*

$$V_{\tilde{\nu}}(it) = a + \frac{1}{2^{k+1}} \frac{\sigma^2}{it} + \int_{\mathbb{R}\setminus(0)} \left(it \operatorname{Li}_{k+1}\left(\frac{x}{it}\right) - \frac{x}{1+x^2}\right) M(dx), \quad t > 0, \tag{4.2}$$

where the Lévy measure M has $\int_{(|x|>1)} \log^{k+1}(1+|x|)\, M(dx) < \infty$.

Equivalently,

$$V_{\tilde{\nu}}(it) = a + \int_{\mathbb{R}} \left[it \operatorname{Li}_{k+1}\left(\frac{x}{it}\right) - \frac{x}{1+x^2}\right] \frac{m(dx)}{\log^{k+1}(1+|x|^{2/(k+1)})}, \quad t > 0, \tag{4.3}$$

where m is a finite Borel measure such that $m(\{0\}) = \sigma^2$. The integrand in (4.3) is equal to $\frac{1}{2^{k+1}} \frac{1}{it}$ at zero. Here $\operatorname{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}$, $|z| < 1$, (analytically continued over $\mathbb{C}$) is the polylogarithmic function.

Proof. For $h(s) = e^{-s}, r_k(s) = s^{k+1}/(k+1)!$ and $(a,b] = (0,\infty)$, using Theorem 1, we get $\mathbf{c} = 1, \mathbf{d} = 2^{-k-1}$ and

$$\mathbf{g}_+(z) = \int_0^\infty \frac{e^{-s}}{1+ze^{-s}} \frac{s^k}{k!}\, ds = -z^{-1} \operatorname{Li}_{k+1}(-z),$$

by Wolframalpha.com. It follows that $\mathbf{g}_+(ix/t) = -t/(ix) \operatorname{Li}_{k+1}(-ix/t) = it/xL_{k+1}(x/it)$. Inserting this, together with $\mathbf{c} = 1$ and $\mathbf{d} = 2^{-k-1}$, into (1.1) in Theorem 1 we get

$$V_{\tilde{\rho}}(it) = a + \frac{\sigma^2}{it} 2^{-k-1} + \int_{\mathbb{R}\setminus(0)} \left[it \operatorname{Li}_{k+1}(x/it) - \frac{x}{1+x^2} \right] M(dx), \quad t > 0. \tag{4.4}$$

Since $m(dx) := \log^{k+1}(1+|x|^{2/(k+1)})M(dx)$ is a finite measure on $\mathbb{R}\setminus\{0\}$ (see Jurek and Mason [21, Proposition 1.8.13]) and adding an atom $m(\{0\}) := \sigma^2$, we complete the proof. ■

REMARK 4. Since $\operatorname{Li}_1(\frac{x}{it}) \equiv \operatorname{PolyLog}[1, \frac{x}{it}] = -\log(1 - \frac{x}{it})$, taking $k = 0$ in Proposition 3 above we retrieve Proposition 2 of [18].

Here is a relation between the consecutive classes $(L_k, \boxplus)$.

COROLLARY 3. *Let* $Df(t) := f(t) - t\frac{d}{dt}f(t)$. *Then for* $k \geqslant 0$,

$$D : (L_k, \boxplus) \to (L_{k-1}, \boxplus), \quad \textit{where} \quad (L_{-1}, \boxplus) \equiv (\mathrm{ID}, \boxplus).$$

Hence $D^{k+1} : (L_k, \boxplus) \to (\mathrm{ID}, \boxplus)$.

Proof. Let $V_{\tilde{\nu}}(it) = a + \frac{1}{2^{k+1}} \frac{\sigma^2}{it} \in (L_k, \boxplus)$. Then

$$D(V_{\tilde{\nu}}(it)) = a + \frac{1}{2^{k+1}} \frac{\sigma^2}{it} - t\frac{\sigma^2}{2^{k+1}}(-1)i(it)^{-2} = a + \frac{1}{2^k} \frac{\sigma^2}{it} \in (L_{k-1}, \boxplus).$$

Keeping in mind that $(d/dz) \operatorname{Li}_{k+1}(z) = z^{-1} \operatorname{Li}_k(z)$ for the Poissonian part of (4.2) we have

$$\begin{aligned}
D&\left[\int_{\mathbb{R}\setminus(0)} \left(it \operatorname{Li}_{k+1}\left(\frac{x}{it}\right) - \frac{x}{1+x^2} \right) M(dx) \right] \\
&= \int_{\mathbb{R}\setminus(0)} \left[\left(it \operatorname{Li}_{k+1}\left(\frac{x}{it}\right) - \frac{x}{1+x^2} \right) - t\frac{d}{dt}\left(it \operatorname{Li}_{k+1}\left(\frac{x}{it}\right) \right) \right] M(dx) \\
&= \int_{\mathbb{R}\setminus(0)} \left[-\frac{x}{1+x^2} - t\left(it \frac{d}{dt}\left(\operatorname{Li}_{k+1}\left(\frac{x}{it}\right) \right) \right) \right] M(dx) \\
&= \int_{\mathbb{R}\setminus(0)} \left[-\frac{x}{1+x^2} - it^2\left(\frac{it}{x} \operatorname{Li}_k\left(\frac{x}{it}\right)(-ix)(it)^{-2} \right) \right] M(dx) \\
&= \int_{\mathbb{R}\setminus(0)} \left[-\frac{x}{1+x^2} + it \operatorname{Li}_k\left(\frac{x}{it}\right) \right] M(dx) \in (L_{k-1}, \boxplus),
\end{aligned}$$

which completes the proof. ■

5. RELATIONS BETWEEN $(L_k, \boxplus)$ AND $(\mathcal{U}^{<k>}, \boxplus)$

Since $(L_k, *) \subset (\mathcal{U}^{\langle k\rangle}, *)$ for $k = 0, 1, \ldots$ (see [15, Corollaries 2 and 7]), the injection (0.4) between the classical and free infinite-divisibility implies that

$$(L_k, \boxplus) \subset (\mathcal{U}^{\langle k\rangle}, \boxplus), \quad k = 0, 1, 2, \ldots. \tag{5.1}$$

The classes $(L_k, *)$ and $(\mathcal{U}^{\langle k\rangle}, *)$ were introduced via linear and non-linear scaling, respectively; here is another relation, besides (5.1), between their free-probability counterparts.

For the notational simplicity, as in [12], set

$$\mathcal{I}(\nu) \equiv I^{e^{-t},t}_{(0,\infty)}(\nu), \quad \nu \in ID_{\log}, \quad \mathcal{J}(\rho) \equiv I^{t,t}_{(0,1]}(\rho), \quad \rho \in \mathrm{ID}.$$

Then we have

$$(L_0, *) = \mathcal{I}(\mathrm{ID}_{log}), \quad L_{k+1} = \mathcal{I}(I^{e^{-t},r_k(t)}_{(0,\infty)}(\mathrm{ID}_{\log^{k+2}}), \quad r_k(t) = \frac{1}{(k+1)!}t^{k+1};$$

$$\mathcal{U}^{\langle 0\rangle} = \mathcal{J}(\mathrm{ID}), \quad \mathcal{U}^{\langle k+1\rangle} = \mathcal{J}(I^{t,\tau_k(t)}_{(0,1]}(\mathrm{ID})), \quad \tau_k(t) = \int_0^t (-\log x)^{k-1}dx;$$

that is, those classes correspond to the compositions of $k+1$ mappings $\mathcal{I}$ and $\mathcal{J}$, respectively; see [19] for the general theory of compositions of random integral mappings.

THEOREM 2. *For $k = 0, 1, \ldots$, a measure $\tilde{\rho} \in (\mathcal{U}^{\langle k\rangle}, \boxplus)$ is in $(L_k, \boxplus)$ if and only if there exists $\omega \in (\mathrm{ID}_{\log^{k+1}}, *)$ such that $\tilde{\rho} = \widetilde{\mathcal{I}(\omega)} \boxplus \tilde{\omega}$.*

Proof. Let $k = 0$ and $\tilde{\rho}$ be the free counterpart of $\rho \in (\mathcal{U}^{\langle 0\rangle}, *) \cap (L_0, *)$. Then there exist $\nu \in \mathrm{ID}$ and $\mu \in \mathrm{ID}_{\log}$ such that $\rho = \mathcal{J}(\nu) = \mathcal{I}(\mu)$. For such equality to hold it is necessary and sufficient that $\nu = \mu * \mathcal{I}(\mu)$; see [12, Theorem 4.5]. Equivalently $\rho = \mathcal{J}(\nu) = \mathcal{J}(\mu) * \mathcal{I}(\mathcal{J}(\mu))$. Taking $\omega := \mathcal{J}(\mu)$ we have $\omega \in \mathrm{ID}_{\log}$ as $\mu \in \mathrm{ID}_{\log}$ and finally $\rho = \omega * \mathcal{I}(\omega)$. Hence $\tilde{\rho} = (\widetilde{\mathcal{I}(\omega) * \omega}) = \widetilde{\mathcal{I}(\omega)} \boxplus \tilde{\omega}$, which proves Theorem 2 for $k = 0$.

Assume that the conclusion is true for the classes with indices $0 \leqslant j \leqslant k$ and let $\tilde{\rho} \in \mathcal{U}^{\langle k+1\rangle}, \boxplus) \cap (L_{k+1}, \boxplus)$ be the counterpart of $\rho \in \mathcal{U}^{\langle k+1\rangle}, *) \cap (L_{k+1}, *)$. Then there exist $\nu \in \mathrm{ID}$ and $\mu \in \mathrm{ID}_{\log^{k+2}}$ such that

$$\rho = I^{t,\tau_{k+1}(t)}_{(0,1)}(\nu) = \mathcal{J}(I^{t,\tau_k(t)}_{(0,1)}(\nu)) \text{ and } \rho = I^{e^{-t},r_{k+1}(t)}_{(0,\infty)}(\mu) = \mathcal{I}(I^{e^{-t},r_k(t)}_{(0,\infty)}(\mu)),$$

and by putting $\nu_1 := I^{t,\tau_k(t)}_{(0,1)}(\nu) \in \mathcal{U}^{\langle k\rangle}$ and $\mu_1 := \mathcal{I}(I^{e^{-t},r_k(t)}_{(0,\infty)}(\mu) \in L_k$, we have $\rho = \mathcal{J}(\nu_1) = \mathcal{I}(\mu_1)$. From this (as in the case $k = 0$) we get $\nu_1 = \mu_1 * \mathcal{I}(\mu_1)$ and

$\rho = \mathcal{J}(\mu_1) * \mathcal{I}(\mathcal{J}(\mu_1))$. Taking $\omega := \mathcal{J}(\mu_1) \in \mathrm{ID}_{\log^{k+1}}$ we get $\tilde{\rho} = \tilde{\omega} \boxplus \widetilde{\mathcal{I}(\omega)}$, which completes the proof. ■

Since there is no random integral representation for the class $(L_\infty, *)$, there is no direct application of Theorem 1. Nevertheless we have

PROPOSITION 4. *(i) A measure $\tilde{\nu}$ is a free-probability counterpart of $\nu \in (L_\infty, *)$, that is, $\tilde{\nu} \in (L_\infty, \boxplus)$, if and only if*

$$V_{\tilde{\nu}}(it) = c - \int_{(-2,2]\setminus\{0\}} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{t^{|x|-1}(1-|x|)}\, G(dx), \tag{5.2}$$

where $c \in \mathbb{R}$, G is a finite Borel measure and the integrand is equal to $i\pi/2 \mp \gamma$ at ± 1, respectively.

(ii) A measure $\tilde{\nu}$ is a free-probability counterpart of $\nu \in (\mathcal{U}^{\langle\infty\rangle}, *)$, that is, $\tilde{\nu} \in (\mathcal{U}^{\langle\infty\rangle}, \boxplus)$, if and only if $V_{\tilde{\nu}}(it)$ is of the form (5.2) *above.*

Proof. From [24, Theorem 2] or [25, Theorem 2] we know that $\nu \in (L_\infty, *)$ iff

$$\phi_\nu(t) = \exp\bigg(iat - \int_{(-2,2]\setminus\{0\}} \bigg[|t|^{|x|}\bigg(\cos\Big(\frac{\pi x}{2}\Big) - i\frac{t}{|t|}\sin\Big(\frac{\pi x}{2}\Big)\bigg) + itx\bigg]\frac{G(dx)}{1-|x|}\bigg)$$

where $a \in \mathbb{R}$ and G is a finite Borel measure on $(-2,0) \cup (0,2]$.

Let us use the identification (0.4). Then

$$\begin{aligned}
V_{\tilde{\nu}}(it) &= it^2 \int_0^\infty \log \phi_\nu(-u) e^{-tu}\, du \\
&= it^2 \int_0^\infty \bigg(-iau - \int_{(-2,2]\setminus\{0\}} \bigg[|u|^{|x|}\bigg(\cos\Big(\frac{\pi x}{2}\Big) + i\frac{u}{|u|}\sin\Big(\frac{\pi x}{2}\Big)\bigg) - iux\bigg]\frac{G(dx)}{1-|x|}\bigg) \\
&\qquad\qquad \times e^{-tu}\, du \\
&= a - \int_{(-2,2]\setminus\{0\}} it^2 \int_0^\infty \bigg[u^{|x|}\bigg(\cos\Big(\frac{\pi x}{2}\Big) + i\frac{u}{|u|}\sin\Big(\frac{\pi x}{2}\Big)\bigg) - iux\bigg] e^{-tu}\, du\, \frac{G(dx)}{1-|x|} \\
&= a - \int_{(-2,2]\setminus\{0\}} \bigg[it^2\Gamma(1+|x|)t^{-(1+|x|)}\bigg(\cos\Big(\frac{\pi x}{2}\Big) + i\sin\Big(\frac{\pi x}{2}\Big)\bigg) + x\bigg]\frac{G(dx)}{1-|x|} \\
&= a - \int_{(-2,2]\setminus\{0\}} \Big[\Gamma(1+|x|)t^{1-|x|}\, ie^{i\pi x/2} + x\Big]\frac{G(dx)}{1-|x|}
\end{aligned}$$

where

$$\lim_{x\to 1} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{1-|x|} = i\pi/2 - \gamma,$$

$$\lim_{x\to -1} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{1-|x|} = i\pi/2 + \gamma,$$

and for $x > 0$,

$$\lim_{x\to 1}\frac{d}{dx}\Gamma(x+1) = \lim_{x\to 1}\frac{d}{dx}\int_0^\infty u^x e^{-u}\,du = \lim_{x\to 1}\int_0^\infty \log(u)u^x e^{-u}\,du$$
$$= \int_0^\infty u\log(u)e^{-u}\,du = 1-\gamma \quad (\gamma = \text{Euler's constant}),$$

which gives part (i). Part (ii) follows from the identity $(L_\infty, *) = (\mathcal{U}^{\langle\infty\rangle}, *)$; see [15, Corollary 7]. ■

Because of the special role of ± 1 in Proposition 4, let us consider the following example:

EXAMPLE 1. Let $G(dx) := \frac{1}{2}\delta_{-1}(dx) + \frac{1}{2}\delta_1(dx)$ (Rademacher distribution) in Proposition 4. Then

$$V_{\tilde{\nu}}(it) = c - i\pi/2 = c + \frac{1}{2}\int_{\mathbb{R}} \frac{1+itx}{it-x}\,\frac{dx}{1+x^2}, \qquad t > 0,$$

which is the classical example of Pick function (Voiculescu representation of a free-infinitely divisible $\tilde{\nu}$) ($\int_{\mathbb{R}} \frac{1+itx}{it-x}\frac{dx}{1+x^2} = -i\pi$).

Acknowledgments. The author would like to thank Krzysztof Kępczyński for his help in setting the paper in the style required by the Editors.

REFERENCES

[1] O. E. Barndorff-Nielsen and S. Thorbjorsen, *Classical and free infinite divisibility and Lévy processes*, in: Lecture Notes in Math. 1866, Springer, 2006, 33–159.

[2] H. Bercovici and D. V. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. 42 (1993), 733–773.

[3] R. C. Bradley and Z. J. Jurek, *The strong mixing and the selfdecomposability*, Statist. Probab. Lett. 84 (2014), 67–71.

[4] W. Feller, *An Introduction to Probability Theory and its Applications, Vol. II*, Wiley, New York, 1966.

[5] B. V. Gnedenko and A. N. Kolomogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, MA, 1954.

[6] I. S. Gradshteyn and M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., Academic Press, New York, 1994.

[7] L. Jankowski and Z. J. Jurek, *Remarks on restricted Nevanlinna transforms*, Demonstratio Math. 45 (2012), 297–307.

[8] Z. J. Jurek, *Limit distributions for sums of shrunken random variables*, in: Second Vilnius Conf. Probab. Theor. Math. Statistics, Abstracts of Communications 3 (1977), 95–96.

[9] Z. J. Jurek, *Limit distributions for sums of shrunken random variables*, Dissertationes Math. 185 (1981), 46 pp.

[10] Z. J. Jurek, *Limit distributions and one-parameter groups of linear operators on Banach spaces*, J. Multivariate Anal. 13 (1983), 578–604.

[11] Z. J. Jurek, *The classes $L_m(Q)$ of probability measures on Banach spaces*, Bull. Polish Acad. Sci. Math. 31 (1983), 51–62.

[12] Z. J. Jurek, *Relations between the s-selfdecomposable and selfdecomposable measures*, Ann. Probab. 13 (1985), 592–608; see also www.math.uni.wroc.pl/~zjjurek/Conjecture.pdf.

[13] Z. J. Jurek, *Random integral representations for classes of limit distributions similar to Lévy class* L_0, Probab. Theory Related Fields 78 (1988), 473–490.

[14] Z. J. Jurek, *Random integral representations for classes of limit distributions similar to Lévy class* L_0*. II*, Nagoya Math. J. 114 (1989), 53–64.

[15] Z. J. Jurek, *Random integral representation hypothesis revisited: new classes of s-selfdecomposable laws*, in: Abstract and Applied Analysis (Hanoi, 2002), World Sci., Singapore, 2004, 479–498; see also www.math.uni.wroc.pl/~zjjurek/Hanoi2002.pdf.

[16] Z. J. Jurek, *Cauchy transforms of measures viewed as some functionals of Fourier transforms*, Probab. Math. Statist. 26 (2006), 187–200.

[17] Z. J. Jurek, *Random integral representations for free-infinitely divisible and tempered stable distributions*, Statist. Probab. Lett. 77 (2007), 417–425.

[18] Z. J. Jurek, *On a method of introducing free-infinitely divisible probability measures*, Demonstratio Math. 49 (2016), 235–251.

[19] Z. J. Jurek, *Remarks on compositions of some random integral mappings*, Statist. Probab. Lett. 137 (2018), 277–282.

[20] Z. J. Jurek, *On a relation between classical and free infinitely divisible transforms*, Probab. Math. Statist. 40 (2020), 349–367.

[21] Z. J. Jurek and J. D. Mason, *Operator-Limit Distributions in Probability Theory*, Wiley, New York, 1993.

[22] Z. J. Jurek and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrsch. Verw. Gebiete 62 (1983), 247–262.

[23] M. Loeve, *Probability Theory*, 3rd ed., Van Nostrand, Princeton, 1963.

[24] K. Urbanik, *Slowly varying sequences of random variables*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), 679–682.

[25] K. Urbanik, *Limit laws for sequences of normed sums satisfying some stability conditions*, in: Proc. 3rd Internat. Sympos. on Multivariate Analysis (Dayton, OH, 1972), Academic Press, 1973; see also www.math.uni.wroc.pl/~zjjurek/urb-limitLawsOhio1973.pdf.

[26] D. V. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. 66 (1986), 323–346.

Zbigniew J. Jurek
Institute of Mathematics
Wroclaw University
Pl. Grunwaldzki 2/4
50-384 Wroclaw, Poland
E-mail: zjjurek@math.uni.wroc.pl

Received 24.2.2021;
accepted 7.10.2021