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# EXPANSIONS FOR MOMENTS OF LOGARITHMIC SKEW-NORMAL EXTREMES

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**Abstract.** Liao, Peng and Nadarajah [*J. Appl. Probab.* 50 (2013), 900–907] derived asymptotic expansions for the partial maximum of a random sample from the logarithmic skew-normal distribution. Here, we derive asymptotic expansions for moments of the partial maximum using optimal norming constants. These expansions can be used to deduce convergence rates of moments of the normalized maxima to the moments of the corresponding extreme value distribution. A numerical study is made to compare the actual values of moments with their asymptotics, which shows that the convergence is exceedingly slow, and adjustment is needed whenever we use the limits to replace moments of the partial maximum.

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**Key words and phrases:** expansions of moment, extreme value distribution, logarithmic skew-normal distribution, maximum.

## 1. INTRODUCTION

Let $\xi$ denote a random variable following the skew-normal distribution (Azzalini, 1985) with its probability density function (pdf) specified by

$$g_\lambda(x) = 2\phi(x)\Phi(\lambda x),$$

where $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\phi(x)$ denotes the standard normal pdf, and $\Phi(x)$ denotes the standard normal cumulative distribution function (cdf). Let $X = \exp(\xi)$. Then $X$ is said to have the *logarithmic skew-normal distribution*, and we write $X \sim \mathrm{LSN}(\lambda)$. The pdf of $\mathrm{LSN}(\lambda)$ is given by

$$f_\lambda(x) = \frac{2}{x}\phi(\log x)\Phi(\lambda \log x) \tag{1.1}$$

for $x > 0$. Let $F_\lambda(\cdot)$ denote the cdf corresponding to (1.1). We know that $\mathrm{LSN}(0)$ is a standard lognormal random variable.

The LSN$(\lambda)$ distribution, first introduced by Azzalini et al. (2003), has found widespread applications in modeling option price data (see Jiménez and Arunachalam (2015)), analyzing gate delay variation (Guo et al. (2019)), modeling expenditure data in tourism (Gómez-Déniz et al. (2021)), analyzing insurance claim cosls (Gómez-Déniz and Calderín-Ojeda (2020) and Bolance et al. (2008)), modeling particle size (Wang et al. (2019) and Huang and Ku (2010)), and analyzing intermittent ocean turbulence (Cael and Mashayek (2021)). As regards probabilistic properties, Lin and Stoyanov (2009) showed that LSN$(\lambda)$ distributions are heavy tailed and moment-indeterminate. Further, Liao et al. (2013) investigated the extreme value behavior of the logarithmic skew-normal distribution. They showed that $F_\lambda$ belongs to the domain of attraction of the Gumbel distribution (written $F_\lambda \in D(\Lambda)$). They also derived the following asymptotic expansion for the normalized maxima of independent logarithmic skew-normal random variables under optimal norming constants:

$$\begin{aligned}(1.2)\qquad &(\log b_n)[(\log b_n)(F_\lambda^n(a_n x + b_n) - \Lambda(x)) - \kappa(x)\Lambda(x)] \\ &\qquad \to \left[\omega(x) + \frac{\kappa^2(x)}{2}\right]\Lambda(x)\end{aligned}$$

as $n \to \infty$, where $\Lambda(x) = \exp(-\exp(-x))$ and the norming constants $a_n$, $b_n$, and the functions $\kappa(x)$, $\omega(x)$ are defined as follows;

(i) for $\lambda \geqslant 0$, $a_n$, $b_n$ are given by

$$(1.3)\qquad 1 - F_\lambda(b_n) = n^{-1}, \qquad a_n = (\log b_n)^{-1} b_n,$$

and $\kappa(x)$, $\omega(x)$ are defined by

$$\begin{aligned}\kappa(x) &= -2^{-1}x^2e^{-x},\\ \omega(x) &= -24^{-1}(3x^4 - 8x^3 - 12x^2 - 24x)e^{-x};\end{aligned}$$

(ii) for $\lambda < 0$, $a_n$, $b_n$ are given by

$$(1.4)\qquad 1 - F_\lambda(b_n) = n^{-1}, \qquad a_n = (1+\lambda^2)^{-1}(\log b_n)^{-1} b_n,$$

and $\kappa(x)$, $\omega(x)$ are defined by

$$\begin{aligned}\kappa(x) &= -2^{-1}(1+\lambda^2)^{-1}x^2e^{-x},\\ \omega(x) &= -24^{-1}(1+\lambda^2)^{-2}(3x^4 - 8x^3 - 12(1+\lambda^2)x^2 - 48(1+\lambda^2)x)e^{-x}.\end{aligned}$$

For the partial maximum $M_n$ of LSN$(\lambda)$ distributed random variables, Proposition 2.1(iii) in Resnick (1987) and (1.2) show that

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{M_n - b_n}{a_n}\right)^r = \int_{-\infty}^{\infty} x^r \, d\Lambda(x) = (-1)^r \Gamma^{(r)}(1)$$

for all nonnegative integers $r$, where $\Gamma^{(r)}(1)$ denotes the $k$th derivative of the gamma function at $x = 1$.

The objective of this paper is to establish asymptotic expansions of the quantity $E((M_n - b_n)/a_n)^r$, from which we can derive its convergence rate. A numerical study shows that the convergence is extremely slow and adjustments are needed if we want to use the limits to replace their actual moments. The asymptotics of the moments of $M_n$, the maximum of independent and identical random variables from any given cdf $F$, have been of considerable interest. McCord (1964) considered convergence of normalized moments of $M_n$ when the random variables belonged to three different classes. Pickands (1968) showed that moments of normalized extremes converge to corresponding moments of the extreme value distribution provided that the moments are finite for sufficiently large $n$ and $F$ is in the domain of attraction of an extreme value distribution. For more studies related to the asymptotic expansions for moments of $M_n$, we refer to Nair (1981) for the normal distribution, Liao et al. (2014) for the skew-normal distribution, and Pu et al. (2015) for the general error distribution.

This note is organized as follows. Section 2 gives the main result on asymptotic expansions for moments of partial maxima of independent LSN($\lambda$) random variables. Section 3 presents a numerical study. Some lemmas needed for the proof of the main result are given in Section 4. The proof of the main result is in Section 5.

## 2. MAIN RESULT

Our main result provides asymptotic expansions for moments of logarithmic skew-normal extremes. These expansions depend on the sign of $\lambda$.

THEOREM 2.1. *Let*

$$m_r(n) = \int_{-\infty}^{+\infty} x^r dF_\lambda^n(a_n x + b_n) \quad \textit{and} \quad m_r = \int_{-\infty}^{+\infty} x^r \, d\Lambda(x),$$

*where the norming constants* $a_n$ *and* $b_n$ *are given by* (1.3) *and* (1.4). *Then*

$$(2.1) \qquad (\log b_n)[(\log b_n)(m_r(n) - m_r) - 2^{-1} r m_{r+1}] \\ \to 24^{-1} r[(3r+1)m_{r+2} - 12 m_{r+1} - 24 m_r]$$

*as* $n \to \infty$ *for* $\lambda \geqslant 0$*; and*

$$(2.2) \qquad (\log b_n)[(\log b_n)(m_r(n) - m_r) - 2^{-1}(1+\lambda^2)^{-1} r m_{r+1}] \\ \to 24^{-1}(1+\lambda^2)^{-2} r[(3r+1)m_{r+2} - 12(1+\lambda^2)m_{r+1} - 48(1+\lambda^2)m_r]$$

*as* $n \to \infty$ *for* $\lambda < 0$.

REMARK 2.1. The expansions of moments of the normalized $M_n$ heavily depend on the expansions of (1.2) with the norming constants given by (1.3) and (1.4) according to the sign of $\lambda$.

REMARK 2.2. It is easy to check that $\log b_n = O((\log n)^{1/2})$ as $b_n$ satisfies (1.3) and (1.4). Theorem 2.1 shows that the pointwise convergence rates of moments of normalized $M_n$ are proportional to $1/(\log n)^{1/2}$, which is exceedingly slow. A numerical study in Section 3 supports our findings.

## 3. NUMERICAL ANALYSIS

In this section, numerical studies are presented to illustrate the accuracy of higher-order expansions of moments of $M_n$. Note that the actual values of moments of $M_n$ are $m_r(n)$. Let $E_i(r), i = 1, 2, 3$, denote the first-order, second-order and third-order asymptotics of the moments of $M_n$. By Theorem 2.1, we have

$$E_1(r) = m_r = \int_{-\infty}^{\infty} x^r \, d\Lambda(x) = (-1)^r \Gamma^{(r)}(1),$$

$$E_2(r) = \begin{cases} E_1(r) + \frac{r}{2\log b_n} m_{r+1}, & \lambda \geqslant 0, \\ E_1(r) + \frac{r}{2(1+\lambda^2)\log b_n} m_{r+1}, & \lambda < 0, \end{cases}$$

$$E_3(r) = \begin{cases} E_2(r) + \frac{r}{24(\log b_n)^2}[(3r+1)m_{r+2} \\ \quad -12m_{r+1} - 24m_r], & \lambda \geqslant 0, \\ E_2(r) + \frac{r}{24(1+\lambda^2)^2(\log b_n)^2}[(3r+1)m_{r+2} \\ \quad -12(1+\lambda^2)m_{r+1} - 24(1+\lambda^2)m_r], & \lambda < 0, \end{cases}$$

where $b_n$ satisfies $1 - F_\lambda(b_n) = n^{-1}$. Here, the second and third-order asymptotics are related to the sample size $n$. To compare the accuracies of actual values with their asymptotics, let $\Delta_i(r, n) = |m_r(n) - E_i(r)|, i = 1, 2, 3$, denote the absolute errors.

We use the **R** software to compute the asymptotics and the actual values of the moments of $M_n$. For the limit $\int_{-\infty}^{\infty} x^r \, d\Lambda(x) = (-1)^r \Gamma^{(r)}(1)$ and $r \leqslant 3$, we use **digamma** and **trigamma** in computations. In order to provide a more general method to compute the limits, we need to truncate the integral. For $x > 0$ and $r < xe^x/2$, we note that $x^r \exp(-e^x/2)$ is a decreasing function. By arguments similar to that of (4.2) below, we have

$$\begin{aligned} \int_{-\infty}^{-100} x^r \, d\Lambda(x) &\leqslant \int_{-\infty}^{-100} |x|^r e^{-x} \exp(-e^{-x}) dx \\ &= \int_{100}^{\infty} x^r e^x \exp(-e^x) dx \\ &\leqslant (100)^r \exp\left(-\frac{e^{100}}{2}\right) \int_{100}^{\infty} e^x \exp\left(-\frac{e^x}{2}\right) dx \\ &\leqslant \exp(-100) = 3.72 \times 10^{-44} \end{aligned}$$

for $r < 1000$, which shows that we may use $\int_{-100}^{\infty} x^r \, d\Lambda(x)$ to approximate the moment $\int_{-\infty}^{\infty} x^r \, d\Lambda(x)$.

First, we compute the actual values of the moments and their asymptotics with $n = 100, 1000, 10000$ and $\lambda$ and $r$ fixed. The results are documented in Tables 1–3, which show that the accuracies of the moments for all three asymptotics improve as $n$ becomes large, but the accuracies reduce as $r$ increases, and the asymptotics deviate too much from the actual values for $r > 4$.

TABLE 1. Comparison of $m_r(n)$, $E_1(r)$, $E_2(r)$ and $E_3(r)$ for $\lambda = 1$

| $n$ | $r$ | $m_r(n)$ | $E_1(r)$ | $E_2(r)$ | $E_3(r)$ | $\Delta_1(r,n)$ | $\Delta_2(r,n)$ | $\Delta_3(r,n)$ |
|---|---|---|---|---|---|---|---|---|
| 100 | 1 | 0.76482 | 0.57722 | 0.96132 | 0.86196 | 0.18760 | 0.19650 | 0.09714 |
| | 2 | 3.38088 | 3.38088 | 4.09266 | 4.74768 | 1.40277 | 0.71177 | 1.36680 |
| | 3 | 22.52394 | 5.44487 | 19.17021 | 33.59197 | 17.07907 | 3.35373 | 11.06803 |
| | 4 | $3.113\times10^2$ | 23.56147 | $1.151\times10^2$ | $2.99\times10^2$ | $2.878\times10^2$ | $1.963\times10^2$ | 12.34226 |
| 1000 | 1 | 0.75578 | 0.57722 | 0.87780 | 0.81695 | 0.17857 | 0.12201 | 0.06117 |
| | 2 | 3.30183 | 1.97811 | 3.63286 | 4.03399 | 1.32372 | 0.33103 | 0.73216 |
| | 3 | 20.16689 | 5.44487 | 16.18570 | 25.01746 | 14.72201 | 3.98119 | 4.85057 |
| | 4 | 242.6 | 23.56147 | 95.18644 | 207.8 | 219.1 | 147.4 | 4.81596 |
| 10000 | 1 | 0.74587 | 0.57722 | 0.83143 | 0.78791 | 0.16865 | 0.08556 | 0.04204 |
| | 2 | 3.20151 | 1.97811 | 3.37762 | 3.66454 | 1.22339 | 0.17611 | 0.46304 |
| | 3 | 18.23902 | 5.44487 | 14.52894 | 20.84625 | 12.79415 | 3.71008 | 2.60723 |
| | 4 | 197.1 | 23.56147 | 84.13836 | 164.7 | 173.6 | 113 | 32.43455 |

TABLE 2. Comparison of $m_r(n)$, $E_1(r)$, $E_2(r)$ and $E_3(r)$ for $\lambda = 0$

| $n$ | $r$ | $m_r(n)$ | $E_1(r)$ | $E_2(r)$ | $E_3(r)$ | $\Delta_1(r,n)$ | $\Delta_2(r,n)$ | $\Delta_3(r,n)$ |
|---|---|---|---|---|---|---|---|---|
| 100 | 1 | 0.76197 | 0.57722 | 1.00237 | 0.88064 | 0.18476 | 0.24040 | 0.11866 |
| | 2 | 3.36810 | 1.97811 | 4.31864 | 5.12115 | 1.38999 | 0.95054 | 1.75305 |
| | 3 | 23.06919 | 5.44487 | 20.63702 | 38.30596 | 17.62432 | 2.43218 | 15.23676 |
| | 4 | 334.5 | 23.56147 | 124.9 | 350.2 | 310.9 | 209.6 | 15.73402 |
| 1000 | 1 | 0.75814 | 0.57722 | 0.89727 | 0.82829 | 0.18093 | 0.13913 | 0.07014 |
| | 2 | 3.32970 | 1.97811 | 3.74007 | 4.19487 | 1.35158 | 0.41038 | 0.86518 |
| | 3 | 20.83523 | 5.44487 | 16.88162 | 26.89493 | 15.39035 | 3.95360 | 6.05970 |
| | 4 | 260.4 | 23.56147 | 99.82720 | 227.5 | 236.9 | 160.6 | 32.89478 |
| 10000 | 1 | 0.74876 | 0.57722 | 0.84316 | 0.79553 | 0.17154 | 0.09440 | 0.04677 |
| | 2 | 3.23097 | 1.97811 | 3.44217 | 3.75618 | 1.25285 | 0.21121 | 0.52522 |
| | 3 | 18.76501 | 5.44487 | 14.94798 | 21.86157 | 13.32013 | 3.81703 | 3.09656 |
| | 4 | 208.9 | 23.56147 | 86.93275 | 175.1 | 185.3 | 121.9 | 33.77110 |

TABLE 3. Comparison of $m_r(n)$, $E_1(r)$, $E_2(r)$ and $E_3(r)$ for $\lambda = -1$

| $n$ | $r$ | $m_r(n)$ | $E_1(r)$ | $E_2(r)$ | $E_3(r)$ | $\Delta_1(r,n)$ | $\Delta_2(r,n)$ | $\Delta_3(r,n)$ |
|---|---|---|---|---|---|---|---|---|
| 100 | 1 | 0.54435 | 0.57722 | 0.96310 | 0.44868 | 0.03286 | 0.41875 | 0.09568 |
| | 2 | 1.66054 | 1.97811 | 4.10244 | 2.12810 | 0.31757 | 2.44190 | 0.46755 |
| | 3 | 6.02026 | 5.44487 | 19.23371 | 20.95018 | 0.57538 | 13.21345 | 14.92992 |
| | 4 | 37.41351 | 23.56147 | 115.5 | 222.2 | 13.85204 | 78.09854 | 184.8 |
| 1000 | 1 | 0.59854 | 0.57722 | 0.84345 | 0.59857 | 0.02132 | 0.24491 | 0.00004 |
| | 2 | 2.03177 | 1.97811 | 3.44379 | 2.50394 | 0.05365 | 1.41202 | 0.47218 |
| | 3 | 7.85885 | 5.44487 | 14.95845 | 15.77554 | 2.41398 | 7.09960 | 7.91669 |
| | 4 | 51.38652 | 23.56147 | 87.00259 | $1.378\times10^2$ | 27.82505 | 35.61606 | 86.40870 |
| 10000 | 1 | 0.62023 | 0.57722 | 0.78979 | 0.63368 | 0.04301 | 0.16957 | 0.01345 |
| | 2 | 2.18624 | 1.97811 | 3.14837 | 2.54921 | 0.20813 | 0.96213 | 0.36297 |
| | 3 | 8.53681 | 5.44487 | 13.04095 | 13.56185 | 3.09194 | 4.50413 | 5.02504 |
| | 4 | 55.72164 | 23.56147 | 74.21573 | 106.6 | 32.16016 | 18.49410 | 50.87516 |

TABLE 4. Comparison of $m_r(n)$, $E_1(r)$, $E_2(r)$ and $E_3(r)$ with $n = 10000$ for $\lambda \geqslant 0$

| $\lambda$ | $r$ | $m_r(n)$ | $E_1(r)$ | $E_2(r)$ | $E_3(r)$ | $\Delta_1(r,n)$ | $\Delta_2(r,n)$ | $\Delta_3(r,n)$ |
|---|---|---|---|---|---|---|---|---|
| 0.3 | 1 | 0.75793 | 0.57722 | 0.83330 | 0.78914 | 0.18072 | 0.07537 | 0.03121 |
| | 2 | 3.30642 | 1.97811 | 3.38791 | 3.67907 | 1.32831 | 0.08148 | 0.37265 |
| | 3 | 19.19587 | 5.44487 | 14.59572 | 21.00627 | 13.75099 | 4.60015 | 1.81040 |
| 0.7 | 1 | 0.74632 | 0.57722 | 0.83147 | 0.78793 | 0.16910 | 0.08515 | 0.04162 |
| | 2 | 3.20594 | 1.97811 | 3.37780 | 3.66480 | 1.22783 | 0.17185 | 0.45885 |
| | 3 | 18.27072 | 5.44487 | 14.53011 | 20.84905 | 12.82584 | 3.74061 | 2.57833 |
| 3 | 1 | 0.74582 | 0.57722 | 0.83143 | 0.78791 | 0.16861 | 0.08561 | 0.04209 |
| | 2 | 3.20133 | 1.97811 | 3.37761 | 3.66454 | 1.22322 | 0.17628 | 0.46320 |
| | 3 | 18.23787 | 5.44487 | 14.52890 | 20.84617 | 12.79299 | 3.70896 | 2.60830 |
| 7 | 1 | 0.74582 | 0.57722 | 0.83143 | 0.78791 | 0.16860 | 0.08562 | 0.04209 |
| | 2 | 3.20132 | 1.97811 | 3.37761 | 3.66454 | 1.22321 | 0.17629 | 0.46322 |
| | 3 | 18.23775 | 5.44487 | 14.52890 | 20.84616 | 12.79287 | 3.70885 | 2.60841 |
| 15 | 1 | 0.74581 | 0.57722 | 0.83143 | 0.78791 | 0.16859 | 0.08562 | 0.04210 |
| | 2 | 3.20130 | 1.97811 | 3.37761 | 3.66453 | 1.22319 | 0.17631 | 0.46324 |
| | 3 | 18.23761 | 5.44487 | 14.52889 | 20.84615 | 12.79274 | 3.70872 | 2.60854 |

In order to further analyze the behavior of the three asymptotics, we compute the actual values and their asymptotics with $n = 10000$ and $\lambda$ fixed. Tables 4 and 5 show the results for $\lambda \geqslant 0$ and $\lambda < 0$ respectively. For $\lambda \geqslant 0$, Table 4 shows that the second-order asymptotics of the second moments of $M_n$ are closer to the actual values. Table 5 shows that the second-order asymptotics always deviate from the

TABLE 5. Comparison of $m_r(n)$, $E_1(r)$, $E_2(r)$ and $E_3(r)$ with $n = 10000$ for $\lambda < 0$

| $\lambda$ | $r$ | $m_r(n)$ | $E_1(r)$ | $E_2(r)$ | $E_3(r)$ | $\Delta_1(r,n)$ | $\Delta_2(r,n)$ | $\Delta_3(r,n)$ |
|---|---|---|---|---|---|---|---|---|
| $-0.3$ | 1 | 0.70776 | 0.57722 | 0.84565 | 0.74039 | 0.13055 | 0.13788 | 0.03263 |
| | 2 | 2.87695 | 1.97811 | 3.45585 | 3.39580 | 0.89884 | 0.57890 | 0.51885 |
| | 3 | 15.33378 | 5.44487 | 15.03677 | 20.42611 | 9.88890 | 0.29701 | 5.09233 |
| $-0.7$ | 1 | 0.65551 | 0.57722 | 0.81752 | 0.68256 | 0.07829 | 0.16202 | 0.02705 |
| | 2 | 2.44662 | 1.97811 | 3.30103 | 2.93750 | 0.46851 | 0.85441 | 0.49087 |
| | 3 | 10.90578 | 5.44487 | 14.03182 | 16.74508 | 5.46091 | 3.12604 | 5.83930 |
| $-3$ | 1 | 0.50849 | 0.57722 | 0.68138 | 0.45368 | 0.068729 | 0.1729 | 0.054803 |
| | 2 | 1.51949 | 1.97811 | 2.55157 | 1.22237 | 0.458617 | 1.0321 | 0.297127 |
| | 3 | 3.86689 | 5.44487 | 9.16715 | 3.25690 | 1.577983 | 5.3003 | 0.609997 |
| $-7$ | 1 | 0.45475 | 0.57722 | 0.62795 | 0.34833 | 0.12247 | 0.17320 | 0.10642 |
| | 2 | 1.26639 | 1.97811 | 2.25741 | 0.53623 | 0.71173 | 0.99103 | 0.73016 |
| | 3 | 2.62151 | 5.44487 | 7.25779 | $-1.30252$ | 2.82336 | 4.63628 | 3.92402 |
| $-15$ | 1 | 0.42210 | 0.57722 | 0.60332 | 0.26654 | 0.15511 | 0.18121 | 0.15556 |
| | 2 | 1.12406 | 1.97811 | 2.12181 | 0.02878 | 0.85406 | 0.99776 | 1.09528 |
| | 3 | 2.06089 | 5.44487 | 6.37761 | $-4.22564$ | 3.38399 | 4.31673 | 6.28652 |

actual values for $\lambda < 0$, and the third-order asymptotics are always closer to the actual values. For $r \geqslant 4$, the asymptotics deviate too much from the actual values; this case is omitted in Tables 4–5. The numerical analysis suggests that we need exceedingly large $n$ for the asymptotics to approach the actual values.

## 4. AUXILIARY LEMMAS

In order to derive our main result, we need to divide the integration region $(-\infty, \infty)$ into three parts: $(-\infty, -d \log\log b_n)$, $[-d\log\log b_n, c\log b_n]$ and $(c\log b_n, +\infty)$ with $0 < d, c < 1$. Lemmas 4.1 and 4.2 estimate the given integrands in $(-\infty, -d\log\log b_n)$ and $(c\log b_n, +\infty)$, respectively. In order to use the dominated convergence theorem, Lemmas 4.3 and 4.4 help us to find suitable integrable functions controlling the integrand on $[-d\log\log b_n, c\log b_n]$.

LEMMA 4.1. *For* $0 < d, c < 1$ *and any nonnegative integers* $i$ *and* $j$, *we have*

$$\lim_{n\to\infty} \int_{-\infty}^{-d\log\log b_n} (\log b_n)^i |x|^j \Lambda(x)\, dx = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \int_{-\infty}^{-d\log\log b_n} (\log b_n)^i |x|^j \, d\Lambda(x) = 0, \tag{4.2}$$

$$\lim_{n\to\infty} \int_{c\log b_n}^{+\infty} (\log b_n)^i x^j \, d\Lambda(x) = 0. \tag{4.3}$$

*Proof.* Note that $1 - F_\lambda(b_n) = n^{-1}$ implies that $b_n \to \infty$ as $n \to \infty$. Note that (4.3) follows from

$$\int_{c\log b_n}^{+\infty} (\log b_n)^i x^j \, d\Lambda(x) < \frac{1}{c^i} \int_{c\log b_n}^{+\infty} x^{i+j} \, d\Lambda(x) \to 0$$

as $n \to \infty$ since $\int_{-\infty}^{+\infty} x^r \, d\Lambda(x) = (-1)^r \Gamma^{(r)}(1)$ for all positive integers $r$ (see Proposition 2.1 in Resnick (1987)).

For $0 < d < 1$, we have

$$\begin{aligned}
&\int_{-\infty}^{-d\log\log b_n} (\log b_n)^i |x|^j \Lambda(x) \, dx \\
&\qquad \leqslant (\log b_n)^i \exp(-(\log b_n)^d/2) \int_{-\infty}^{-1} |x|^j \exp(-e^{-x}/2) \, dx \to 0
\end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned}
&\int_{-\infty}^{-\log\log b_n} (\log b_n)^i |x|^j \, d\Lambda(x) \\
&\qquad \leqslant (\log b_n)^i \exp(-(\log b_n)^d/2) \int_{-\infty}^{-1} |x|^j e^{-x} \exp(-e^{-x}/2) \, dx \\
&\qquad = (\log b_n)^i \exp(-(\log b_n)^d/2) \int_{1}^{+\infty} x^j e^{x} \exp(-e^{x}/2) \, dx \to 0
\end{aligned}$$

as $n \to \infty$ since $\int_1^{+\infty} x^j e^x \exp(-e^x/2) \, dx < 2 \sum_{i=0}^{j} \binom{j}{i} (\log 2)^{j-i} \Gamma^{(i)}(1)$. So, (4.1) and (4.2) follow. The proof is complete. ∎

LEMMA 4.2. *For $0 < d, c < 1$ and any integers $i, j \geqslant 0$, we have*

$$\lim_{n\to\infty} \int_{c\log b_n}^{+\infty} (\log b_n)^i x^j \, dF_\lambda^n(a_n x + b_n) = 0$$

*and*

$$\lim_{n\to\infty} \int_{-\infty}^{-d\log\log b_n} (\log b_n)^i |x|^j F_\lambda^n(a_n x + b_n) \, dx = 0. \tag{4.4}$$

*Proof.* Obviously, by the definition of $f_\lambda(x)$, we have $\int_{-\infty}^{0} |x|^r f_\lambda(x) \, dx = 0 < \infty$ for all positive integers $r$. So, for even positive numbers $k > i + j$,

$$\lim_{n\to\infty} m_k(n) = \lim_{n\to\infty} \mathrm{E}\left(\frac{M_n - b_n}{a_n}\right)^k = m_k = \int_{-\infty}^{+\infty} x^k \, d\Lambda(x) = \Gamma^{(k)}(1)$$

by Proposition 2.1(iii) in Resnick (1987), which implies

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{M_n - b_n}{a_n}\right)^k \mathrm{I}_{\{|\frac{M_n-b_n}{a_n}|\geqslant L\}} = \int\limits_{|x|\geqslant L} x^k \, d\Lambda(x)$$

for arbitrary $L > 0$. Hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \int\limits_{c\log b_n}^{+\infty} (\log b_n)^i x^j \, dF_\lambda^n(a_n x + b_n) \\
&\qquad \leqslant \limsup_{n\to\infty} \frac{1}{c^i} \int\limits_{c\log b_n}^{+\infty} x^{i+j} \, dF_\lambda^n(a_n x + b_n) \\
&\qquad \leqslant \lim_{n\to\infty} \frac{1}{c^i} \int\limits_{L}^{+\infty} x^{i+j} \, dF_\lambda^n(a_n x + b_n) \leqslant \frac{1}{c^i} \int\limits_{|x|\geqslant L} x^{i+j} \, d\Lambda(x).
\end{aligned}$$

The desired result follows by letting $L \to \infty$.

For (4.4), we only handle the case of $\lambda > 0$; the case $\lambda \leqslant 0$ is similar. First note that by Proposition 1 in Liao et al. (2013),

$$\begin{aligned}
&(\log b_n)^k F_\lambda^n\left(b_n - \frac{db_n \log\log b_n}{\log b_n}\right) \\
&< (\log b_n)^k \exp\left(-n\left(1 - F_\lambda\left(b_n - \frac{db_n \log\log b_n}{\log b_n}\right)\right)\right) \\
&< \exp\Bigg[-\frac{\Phi\big(\lambda\big(\log b_n + \log\big(1 - \frac{d\log\log b_n}{\log b_n}\big)\big)\big)\big(1 - \frac{d\log\log b_n}{\log b_n}\big)\big(1 - \frac{\phi(\lambda\log b_n)}{\lambda\log b_n}\big)\log b_n}{\big(\log b_n + \log\big(1 - \frac{d\log\log b_n}{\log b_n}\big)\big)\big(1 + \big(\log b_n + \log\big(1 - \frac{d\log\log b_n}{\log b_n}\big)\big)^{-2}\big)} \\
&\quad \cdot \exp\left(-(\log b_n)\log\left(1 - \frac{d\log\log b_n}{\log b_n}\right) - \frac{\big(\log\big(1 - \frac{d\log\log b_n}{\log b_n}\big)\big)^2}{2}\right) + k\log\log b_n\Bigg] \\
&= (\log b_n)^k \exp(-(1 + o(1))(\log b_n)^d) \to 0
\end{aligned}$$

as $n \to \infty$. Hence, for $k \geqslant i + j + 1$,

$$\begin{aligned}
&\int\limits_{-\infty}^{-d\log\log b_n} (\log b_n)^i |x|^j F_\lambda^n\left(b_n + \frac{b_n}{\log b_n} x\right) dx \\
&\qquad = \int\limits_0^{b_n - db_n(\log\log b_n)/\log b_n} \frac{(\log b_n)^{i+j+1}}{(b_n)^{j+1}} |t - b_n|^j F_\lambda^n(t) \, dt \\
&\qquad = \int\limits_0^{1 - d(\log\log b_n)/\log b_n} (\log b_n)^{i+j+1} |z - 1|^j F_\lambda^n(b_n z) \, dz \\
&\qquad < (\log b_n)^{i+j+1} F_\lambda^n\left(b_n - \frac{db_n \log\log b_n}{\log b_n}\right) \int\limits_0^1 |z - 1|^j \, dz \to 0
\end{aligned}$$

as $n \to \infty$. The desired result follows. ∎

In order to use the dominated convergence theorem, the following lemmas prove that $x^r(\log b_n)\left[(\log b_n)(F_\lambda^n(a_nx+b_n))-\kappa(x)\Lambda(x)\right]$ is bounded by integrable functions on $[-d\log\log b_n, c\log b_n]$.

LEMMA 4.3. *Let* $q_\lambda(b_n,x)=n\log F_\lambda(a_nx+b_n)+e^{-x}$ *with norming constants* $a_n$ *and* $b_n$ *given by* (1.3) *and* (1.4). *If* $-d\log\log b_n<x<c\log b_n$ *with* $0<d,c<1$, *then*

$$|q_\lambda(b_n,x)|<3 \quad \textit{for sufficiently large } n.$$

*Proof.* We only consider the case of $\lambda<0$; the case $\lambda\geqslant 0$ is similar. By Proposition 3 in Liao et al. (2013), we have

$$\begin{aligned}
(4.5)\qquad & 1-F_\lambda(x)\\
&=\frac{\exp\left(-\frac{1+\lambda^2}{2}(\log x)^2\right)}{(-\lambda)\pi(1+\lambda^2)(\log x)^2}-r_\lambda(x)\\
&=\frac{\exp\left(-\frac{1+\lambda^2}{2}(\log x)^2\right)}{(-\lambda)\pi(1+\lambda^2)(\log x)^2}\left(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log x)^{-2}\right)+s_\lambda(x)
\end{aligned}$$

for large $x$, where

$$0<r_\lambda(x)<\frac{(1+3\lambda^2)\exp\left(-\frac{1+\lambda^2}{2}(\log x)^2\right)}{(-\lambda)\pi\lambda^2(1+\lambda^2)^2(\log x)^4} \tag{4.6}$$

for $s_\lambda(x)>0$.

Let $\Psi_n(x)=1-F_\lambda(a_nx+b_n)$. Then

$$n\log F_\lambda(a_nx+b_n)=-n\Psi_n(x)-R_n(x)$$

with

$$0<R_n(x)<\frac{n(\Psi_n(x))^2}{2(1-\Psi_n(x))}$$

since $-x-\frac{x^2}{2(1-x)}<\log(1-x)<-x$ for $0<x<1$.

For $-d\log\log b_n<x<c\log b_n$, we have

$$0<\Psi_n(x)<1-F_\lambda\left(b_n\left(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n}\right)\right)<c_1<1$$

for large $n$, and by (4.5)–(4.6),

$$\begin{aligned}
(4.7)\qquad 0<R_n(x)&<\frac{n(\Psi_n(x))^2}{2(1-\Psi_n(x))}<\frac{\left(1-F_\lambda\left(b_n\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right)\right)^2}{2(1-c_1)(1-F_\lambda(b_n))}\\
&<\frac{(\log b_n)^2\exp\left(-\frac{1+\lambda^2}{2}(\log b_n)^2-2(1+\lambda^2)(\log b_n)\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right)}{2(1-c_1)(-\lambda)\pi(1+\lambda^2)\left(\log b_n+\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right)^4\left(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{(\log b_n)^2 \exp\bigl(-\frac{1+\lambda^2}{2}(\log b_n)^2+\frac{2c^2}{1+\lambda^2}\log b_n - x\bigr)}{2(1-c_1)(-\lambda)\pi(1+\lambda^2)\bigl(\log b_n+\log\bigl(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n}\bigr)\bigr)^4\bigl(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\bigr)}\\
&< \frac{(\log b_n)^2 \exp(-\frac{1+\lambda^2}{2}(\log b_n)^2+\frac{2c^2}{1+\lambda^2}\log b_n + d\log\log b_n)}{2(1-c_1)(-\lambda)\pi(1+\lambda^2)\bigl(\log b_n+\log\bigl(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n}\bigr)\bigr)^4\bigl(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\bigr)}\\
&< 1
\end{aligned}
$$

for large $n$.

Note that

$$|q_\lambda(b_n,x)| < |-n\Psi_n(x)+e^{-x}| + R_n(x), \tag{4.8}$$

and, for $x \geqslant 0$,

$$|-n\Psi_n(x)+e^{-x}| < n\Psi_n(x)+e^{-x} \leqslant 2. \tag{4.9}$$

So, by (4.7)–(4.9), the desired result holds for $0 \leqslant x < c\log b_n$.

It remains to check that (4.9) also holds for $-d\log\log b_n < x < 0$. By (4.5) and (4.6), we have

$$
\begin{aligned}
-n\Psi_n(x)+e^{-x} &= -\frac{1-F_\lambda\bigl(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\bigr)}{1-F_\lambda(b_n)}+e^{-x}\\
&= -\frac{1-\mu\bigl(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\bigr)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}))^{-2}}{\bigl(1+\frac{\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)}{\log b_n}\bigr)^2\bigl(1-\mu(b_n)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\bigr)}\\
&\qquad \cdot\exp\biggl[-(1+\lambda^2)(\log b_n)\log\biggl(1+\frac{x}{(1+\lambda^2)\log b_n}\biggr)\\
&\qquad\qquad -\frac{1+\lambda^2}{2}\log^2\biggl(1+\frac{x}{(1+\lambda^2)\log b_n}\biggr)\biggr]+e^{-x}\\
&= e^{-x}\left(1+\frac{\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)}{\log b_n}\right)^{-2} E_n(x),
\end{aligned}
$$

where $0 < \mu(x) < 1$ and

$$
\begin{aligned}
E_n(x) &= \left(1+\frac{\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)}{\log b_n}\right)^2\\
&\quad - \frac{1-\mu\bigl(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\bigr)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}\bigl(\log\bigl(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\bigr)\bigr)^{-2}}{1-\mu(b_n)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}}\\
&\cdot\exp\biggl[x-(1+\lambda^2)(\log b_n)\log\biggl(1+\frac{x}{(1+\lambda^2)\log b_n}\biggr)-\frac{(1+\lambda^2)\bigl(\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)\bigr)^2}{2}\biggr].
\end{aligned}
$$

For $-d\log\log b_n < x < 0$, we have

$$
\begin{aligned}
E_n(x) &< \left(1+\frac{\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)}{\log b_n}\right)^2 \\
&\qquad - \left(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}\left(\log b_n+\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right)^{-2}\right) \\
&\qquad\qquad \cdot\left[1+x-(1+\lambda^2)(\log b_n)\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right. \\
&\qquad\qquad\qquad \left. -\frac{(1+\lambda^2)(\log(1+\frac{x}{(1+\lambda^2)\log b_n}))^2}{2}\right] \\
&< \frac{2x}{(1+\lambda^2)(\log b_n)^2} \\
&\quad + \left(\frac{1}{(\log b_n)^2}+\frac{1+\lambda^2}{2}\right)\frac{x^2}{(1+\lambda^2)^2(\log b_n)^2}\left(1-\frac{x}{2((1+\lambda^2)\log b_n+x)}\right)^2 \\
&\quad + \left(1-x+\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2} \\
&\qquad\qquad \cdot\left(1+\frac{\log(1+\frac{x}{(1+\lambda^2)\log b_n})}{\log b_n}\right)^{-2}
\end{aligned}
$$

and

$$
\begin{aligned}
E_n(x) &> \left(1+\frac{\log\bigl(1+\frac{x}{(1+\lambda^2)\log b_n}\bigr)}{\log b_n}\right)^2 \\
&\quad - \frac{1}{1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}} \\
&\qquad \cdot\left(1+\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right)\right) \\
&> \frac{2x}{(1+\lambda^2)(\log b_n)^2}-\frac{x^2}{(1+\lambda^2)(\log b_n)^2((1+\lambda^2)\log b_n+x)} \\
&\quad - \frac{(1+3\lambda^2)(\log b_n)^{-2}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}} \\
&\quad - \frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}) \\
&\quad - \frac{(1+3\lambda^2)(\log b_n)^{-2}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}} \\
&\qquad\qquad \cdot\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right),
\end{aligned}
$$

where the first inequality follows from

$$\exp(x-(1+\lambda^2)(\log b_n)$$
$$\cdot \log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)-\frac{(1+\lambda^2)\left(\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)\right)^2}{2}\Bigg)$$
$$< \exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right)$$
$$< 1+\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right)$$

because $e^x < 1+xe^x$ for $x>0$. Therefore,

$$\begin{aligned}
&|E_n(x)| \\
&< \frac{2|x|}{(1+\lambda^2)(\log b_n)^2} \\
&\quad + \left(\frac{1}{(\log b_n)^2}+\frac{1+\lambda^2}{2}\right)\frac{x^2}{(1+\lambda^2)^2(\log b_n)^2}\left(1-\frac{x}{2((1+\lambda^2)\log b_n+x)}\right)^2 \\
&\quad + \left(1+|x|+\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)(\log b_n)^2}\Big(1+\frac{\log(1+\frac{x}{\left(1+\lambda^2\right)\log b_n})}{\log b_n}\Big)^{-2} \\
&\quad + \frac{x^2}{(1+\lambda^2)(\log b_n)^2((1+\lambda^2)\log b_n+x)}+\frac{(1+3\lambda^2)(\log b_n)^{-2}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}} \\
&\quad + \frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right) \\
&\quad + \frac{(1+3\lambda^2)(\log b_n)^{-2}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}}\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\exp\left(\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\right) \\
&< \frac{1}{\log b_n}\Bigg[\frac{2d\log\log b_n}{(1+\lambda^2)\log b_n}+\frac{x^2}{2\left(1+\lambda^2-\frac{d\log\log b_n}{\log b_n}\right)}\exp\left(\frac{d^2(\log\log b_n)^2}{2((1+\lambda^2)\log b_n-d\log\log b_n)}\right) \\
&\quad + \left(\frac{1}{(\log b_n)^2}+\frac{1+\lambda^2}{2}\right)\frac{d^2(\log\log b_n)^2}{(1+\lambda^2)^2\log b_n}\left(1+\frac{d\log\log b_n}{2((1+\lambda^2)\log b_n-d\log\log b_n)}\right)^2 \\
&\quad + \left(1+d\log\log b_n+\frac{d^2(\log\log b_n)^2}{2((1+\lambda^2)\log b_n-d\log\log b_n)}\right)\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)\log b_n} \\
&\quad \cdot\left(1+\frac{\log\left(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n}\right)}{\log b_n}\right)^{-2}+\frac{d^2(\log\log b_n)^2}{(1+\lambda^2)(\log b_n)((1+\lambda^2)\log b_n-d\log\log b_n)} \\
&\quad + \frac{(1+3\lambda^2)(\log b_n)^{-1}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}}+\frac{(1+3\lambda^2)(\log b_n)^{-1}}{\lambda^2(1+\lambda^2)-(1+3\lambda^2)(\log b_n)^{-2}} \\
&\quad \cdot\frac{d^2(\log\log b_n)^2}{2((1+\lambda^2)\log b_n-d\log\log b_n)}\exp\left(\frac{d^2(\log\log b_n)^2}{2((1+\lambda^2)\log b_n-d\log\log b_n)}\right)\Bigg] \\
&< \frac{1}{\log b_n}\left(1+\frac{x^2}{1+\lambda^2}\right)
\end{aligned}$$

for large $n$ and $-d\log\log b_n < x < 0$. So, we have

$$|-n\Psi_n(x)+e^x| = e^{-x}\left(1+\frac{\log\left(1+\frac{x}{(1+\lambda^2)\log b_n}\right)}{\log b_n}\right)^{-2}|E_n(x)|$$
$$< \left(1+\frac{\log\left(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n}\right)}{\log b_n}\right)^{-2}\frac{1}{(\log b_n)^{1-d}}\left(1+\frac{d^2(\log\log b_n)^2}{1+\lambda^2}\right) < 2$$

for large $n$, where $0 < d < 1$ and $-d\log\log b_n < x < 0$, which is just the claimed (4.9) for $-d\log\log b_n < x < 0$. The desired result follows for $\lambda < 0$. ∎

LEMMA 4.4. *For $r > 0$, $0 < d, c < 1$ and large $n$, the quantity $x^r(\log b_n)\big[(\log b_n)(F_\lambda^n(a_nx+b_n)-\Lambda(x))-\kappa(x)\Lambda(x)\big]$ is bounded for $-d\log\log b_n < x < c\log b_n$ by integrable functions independent of $n$, where $\kappa(x)$ is defined by $\kappa(x) = -2^{-1}x^2e^{-x}$ for $\lambda \geqslant 0$, and $\kappa(x) = -2^{-1}(1+\lambda^2)^{-1}x^2e^{-x}$ for $\lambda < 0$.*

*Proof.* By Lemma 4.3, we have

$$\begin{aligned}
&\big|(\log b_n)\big[(\log b_n)\big(F_\lambda^n(a_nx+b_n)-\Lambda(x)\big)-\kappa(x)\Lambda(x)\big]\big| \\
&\quad< |(\log b_n)[(\log b_n)q_\lambda(b_n,x)-\kappa(x)]\Lambda(x)| \\
&\qquad+(\log b_n)^2q_\lambda^2(b_n,x)\big(\tfrac{1}{2}+\exp(|q_\lambda(b_n,x)|)\big)\Lambda(x) \\
&\quad< |(\log b_n)[(\log b_n)q_\lambda(b_n,x)-\kappa(x)]\Lambda(x)|+(\log b_n)^2q_\lambda^2(b_n,x)\big(\tfrac{1}{2}+e^3\big)\Lambda(x)
\end{aligned}$$

for large $n$, where $q_\lambda(b_n,x) = n\log F_\lambda(a_nx+b_n)+e^{-x}$.

Note that if $0 < t \leqslant 1$ and $k$ is a nonnegative integer then

$$\int_{-\infty}^{+\infty} x^ke^{-tx}\exp(-e^{-x})\,dx = (-1)^k\Gamma^{(k)}(t).$$

It remains to show that $|(\log b_n)\,[(\log b_n)q_\lambda(b_n,x)-\kappa(x)]|$ and $|(\log b_n)q_\lambda(b_n,x)|$ are bounded above by $s(x)e^{-tx}$, where $s(x)$ is a polynomial of $x$.

We only prove that $|(\log b_n)((\log b_n)q_\lambda(b_n,x)-\kappa(x))|$ is bounded above by $s(x)e^{-x}$ for $\lambda < 0$; the arguments for the other case are similar. Rewrite

$$\begin{aligned}
(4.10)\qquad &(\log b_n)[(\log b_n)q_\lambda(b_n,x)-\kappa(x)] \\
&\qquad= (\log b_n)^2(-n\Psi_n(x)+e^{-x})-(\log b_n)\kappa(x)-(\log b_n)^2R_n(x),
\end{aligned}$$

where

$$n\log F_\lambda(a_nx+b_n) = -n\Psi_n(x)-R_n(x) = -n(1-F_\lambda(a_nx+b_n))-R_n(x).$$

By using (4.7), we have

$$(\log b_n)^2 R_n(x) < e^{-x} \tag{4.11}$$

for large $n$ and $-d\log\log b_n < x < c\log b_n$.

By (4.5)–(4.6), we have

$$\begin{aligned}\frac{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1-F_\lambda(b_n)} &< \frac{\exp\big(-\frac{1+\lambda^2}{2}\big(\log b_n+\log\big(1+\frac{x}{(1+\lambda^2)\log b_n}\big)\big)^2\big)}{\big(\log b_n+\log\big(1+\frac{x}{(1+\lambda^2)\log b_n}\big)\big)^2} \\ &\quad\cdot\frac{(\log b_n)^2}{\big(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\big)\exp\big(-\frac{1+\lambda^2}{2}(\log b_n)^2\big)} \\ &< \frac{\exp\big(-(1+\lambda^2)(\log b_n)\log\big(1+\frac{x}{(1+\lambda^2)\log b_n}\big)\big)}{\big(1+\frac{\log(1+\frac{x}{(1+\lambda^2)\log b_n})}{\log b_n}\big)^2\big(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\big)}\end{aligned}$$

for large $n$. So, if $-d\log\log b_n < x < 0$, we have

$$\begin{aligned}\frac{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1-F_\lambda(b_n)} &< \frac{\exp\big(-x+\frac{x^2}{2((1+\lambda^2)\log b_n+x)}\big)}{\big(1+\frac{\log(1+\frac{x}{(1+\lambda^2)\log b_n})}{\log b_n}\big)^2\big(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\big)} \\ &< \frac{\exp\big(\frac{d^2(\log\log b_n)^2}{2((1+\lambda^2)\log b_n-d\log\log b_n)}-x\big)}{\big(1+\frac{\log(1-\frac{d\log\log b_n}{(1+\lambda^2)\log b_n})}{\log b_n}\big)^2\big(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\big)} < 2e^{-x}\end{aligned} \tag{4.12}$$

for large $n$. If $0 < x < c\log b_n$ we have

$$\begin{aligned}\frac{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1-F_\lambda(b_n)} &< \frac{\exp\big(-x+\frac{x^2}{(1+\lambda^2)\log b_n+x}\big)}{\big(1+\frac{\log(1+\frac{x}{(1+\lambda^2)\log b_n})}{\log b_n}\big)^2\big(1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}\big)} \\ &< \frac{\exp(-x/2)}{1-\frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2}} < 2e^{-x/2}\end{aligned} \tag{4.13}$$

for large $n$.

By Proposition 3 in Liao et al. (2013),

$$\frac{1-F_\lambda(b_n)}{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}e^{-x} = A(b_n)\exp\Big(\int_0^x (g_1(t)+g_2(t))\,dt\Big),$$

where

$$g_1(t) = \frac{-t + (1+\lambda^2)\log\big(1 + \frac{t}{(1+\lambda^2)\log b_n}\big)}{(1+\lambda^2)\log b_n + t},$$

$$g_2(t) = 2\left[((1+\lambda^2)\log b_n + t)\left(\log b_n + \log\left(1 + \frac{t}{(1+\lambda^2)\log b_n}\right)\right)\right]^{-1}$$

and

$$\begin{aligned}
&A(b_n)\\
&= \left(1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}(\log b_n)^{-2} + \frac{15\lambda^4+10\lambda^2+3}{\lambda^4(1+\lambda^2)^2}(\log b_n)^{-4} + O((\log b_n)^{-6})\right)\\
&\quad\cdot\left[1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}\left(\log b_n + \log\left(1 + \frac{x}{(1+\lambda^2)\log b_n}\right)\right)^{-2}\right.\\
&\left. + \frac{15\lambda^4+10\lambda^2+3}{\lambda^4(1+\lambda^2)^2}\left(\log b_n + \log\left(1 + \frac{x}{(1+\lambda^2)\log b_n}\right)\right)^{-4} + O((\log b_n)^{-6})\right]^{-1}
\end{aligned}$$

with $\lim_{n\to\infty} A(b_n) = 1$. Therefore,

$$\begin{aligned}
(4.14)\qquad &(\log b_n)^2(-n\Psi_n(x) + e^{-x}) - (\log b_n)\kappa(x)\\
&= \frac{1 - F_\lambda\big(b_n + \frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1 - F_\lambda(b_n)}[B(b_n) + C(b_n) + D(b_n) + H(b_n)],
\end{aligned}$$

where

$$B(b_n) = (\log b_n)^2(A(b_n) - 1),$$

$$C(b_n) = (\log b_n)^2 A(b_n)\left[\int_0^x (g_1(t) + g_2(t))\,dt + \frac{x^2}{2(1+\lambda^2)\log b_n}\right],$$

$$D(b_n) = (\log b_n)^2 A(b_n)\frac{x^2}{2(1+\lambda^2)\log b_n}\int_0^x (g_1(t) + g_2(t))\,dt,$$

$$H(b_n) = (\log b_n)^2 A(b_n)\left(1 + \frac{x^2}{2(1+\lambda^2)\log b_n}\right)\sum_{i=2}^{\infty}\frac{\left(\int_0^x (g_1(t)+g_2(t))\,dt\right)^i}{i!}.$$

For $-d\log\log b_n < x < c\log b_n$ and for large $n$, we have

$$\begin{aligned}
(4.15)\qquad &|B(b_n)|\\
&< 2\left[\frac{15\lambda^4+10\lambda^2+3}{\lambda^4(1+\lambda^2)^2}(\log b_n)^{-2}\left(1 + \left(1 + \frac{\log\big(1 + \frac{x}{(1+\lambda^2)\log b_n}\big)}{\log b_n}\right)^{-4}\right)\right.\\
&\quad\left. + \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}\left(1 + \left(1 + \frac{\log\big(1 + \frac{x}{(1+\lambda^2)\log b_n}\big)}{\log b_n}\right)^{-2}\right) + O((\log b_n)^{-4})\right]\\
&< 2\left(\frac{3(1+3\lambda^2)}{\lambda^2(1+\lambda^2)} + \frac{30\lambda^4+20\lambda^2+6}{\lambda^4(1+\lambda^2)^2}\right),
\end{aligned}$$

$$(4.16)\quad |C(b_n)|$$
$$= A(b_n)\left|\int_0^x \left(\frac{t^2+(1+\lambda^2)^2(\log b_n)\log\big(1+\frac{t}{(1+\lambda^2)\log b_n}\big)}{(1+\lambda^2)\big((1+\lambda^2)+\frac{t}{\log b_n}\big)}+(\log b_n)^2 g_2(t)\right)dt\right|$$
$$< \frac{2|x|^3}{(1+\lambda^2)^2}+\frac{x^2}{1+\lambda^2}+\frac{8|x|}{1+\lambda^2}$$

and

$$|D(b_n)| < \frac{|x|^3}{3(1+\lambda^2)^2}+\frac{3x^2}{1+\lambda^2}+\frac{4|x|}{1+\lambda^2}. \tag{4.17}$$

Meanwhile, we can see that

$$(4.18)\quad \left|\frac{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1-F_\lambda(b_n)}H(b_n)\right|$$
$$< \left(1+\frac{x^2}{2(1+\lambda^2)}\right)\left(\frac{|x|^3}{6(1+\lambda^2)^2}+\frac{3x^2}{2(1+\lambda^2)}+\frac{2|x|}{1+\lambda^2}\right)^2 e^{-x}$$

for large $n$ and $-d\log\log b_n < x < 0$; and

$$(4.19)\quad \left|\frac{1-F_\lambda\big(b_n+\frac{b_n x}{(1+\lambda^2)\log b_n}\big)}{1-F_\lambda(b_n)}H(b_n)\right|$$
$$< \left(1+\frac{x^2}{2(1+\lambda^2)}\right)\left(\frac{|x|^3}{6(1+\lambda^2)^2}+\frac{3x^2}{2(1+\lambda^2)}+\frac{2|x|}{1+\lambda^2}\right)^2 \exp\left(-\frac{1-c+\lambda^2}{1+c+\lambda^2}x\right)$$

for large $n$ and $0 < x < c\log b_n$.

Combining (4.11)–(4.19), we obtain the desired result. ■

## 5. PROOF OF THE MAIN RESULT

With the above four lemmas, this section gives the proof of Theorem 2.1.

*Proof of Theorem 2.1.* By Proposition 4 in Liao et al. (2013), we have

$$1-F_\lambda(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,dt\right),$$

where $c(x)\to c>0$ and $g(x)\to 1$ as $x\to\infty$, and the auxiliary function $f$ is given by $f(x)=x/\log x$ for $\lambda \geqslant 0$, and $f(x)=x/((1+\lambda^2)\log x)$ for $\lambda<0$. Recall that

$$1-F_\lambda(b_n)=n^{-1},\qquad a_n=f(b_n)$$

(see (1.3) and (1.4)). For an arbitrary integer $r > 0$, $\int_{-\infty}^{0} |x|^r dF_\lambda(x) = 0$, so $m_r(n) \to m_r$ as $n \to \infty$ by Proposition 2.1(iii) in Resnick (1987), and $m_r(n)$ is finite for large $n$. So,

$$
\begin{aligned}
m_r(n) - m_r &= \int_{-\infty}^{+\infty} x^r (F_\lambda(a_n x + b_n) - \Lambda(x))' \, dx \\
&= \int_{-\infty}^{+\infty} x^r d(F_\lambda(a_n x + b_n) - \Lambda(x)).
\end{aligned}
$$

By arguments similar to Lemma 2.2(a) in Resnick (1987), we have

$$
1 - F_\lambda^n(a_n x + b_n) \leqslant (1+\varepsilon)^2 (1+\varepsilon x)^{-\varepsilon^{-1}+1}
$$

for $x > 0$, arbitrary $\varepsilon > 0$ and large $n$. So,

$$
\begin{aligned}
0 &\leqslant \limsup_{n\to\infty} (\log b_n)^{r+2} (1 - F_\lambda^n(a_n \log b_n + b_n)) \\
&\leqslant \lim_{n\to\infty} (\log b_n)^{r+2} (1+\varepsilon)^2 (1+\varepsilon \log b_n)^{-\varepsilon^{-1}+1} = 0
\end{aligned}
\tag{5.1}
$$

for $0 < \varepsilon < \frac{1}{r+3}$.

Note that

$$
\int_{-\infty}^{+\infty} x^k e^{-2x} \Lambda(x) \, dx = \int_{-\infty}^{+\infty} x^k e^{-x} \, d\Lambda(x) = -k m_{k-1} + m_k
$$

and $F_\lambda(a_n x + b_n) = 0$ for $x \leqslant -b_n/a_n$ by the definition of LSN($\lambda$). Note also that $F_\lambda(a_n x + b_n) = 0$ for $x \leqslant -(1+\lambda^2)\log b_n$ and $\lambda < 0$. So, by Theorem 1 in Liao et al. (2013), (5.1), Lemmas 4.1–4.4 and the dominated convergence theorem, we have, for $\lambda < 0$,

$$
\begin{aligned}
&(\log b_n)[(\log b_n)(m_r(n) - m_r) - 2^{-1}(1+\lambda^2)^{-1} r m_{r+1}] \\
&= \int_{-(1+\lambda^2)\log b_n}^{c \log b_n} (\log b_n)^2 x^r d(F_\lambda^n(a_n x + b_n) - \Lambda(x)) \\
&\quad - 2^{-1}(1+\lambda^2)^{-1} r (\log b_n) \int_{-d \log\log b_n}^{c \log b_n} x^{r+1} \, d\Lambda(x) \\
&\quad + \int_{c \log b_n}^{+\infty} (\log b_n)^2 x^r \, d(F_\lambda^n(a_n x + b_n) - \Lambda(x)) \\
&\quad - 2^{-1}(1+\lambda^2)^{-1} r (\log b_n) \int_{c \log b_n}^{+\infty} x^{r+1} \, d\Lambda(x)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{-\infty}^{-(1+\lambda^2)\log b_n} (\log b_n)^2 x^r \, d\Lambda(x) \\
&- 2^{-1}(1+\lambda^2)^{-1} r(\log b_n) \int_{-\infty}^{-d\log\log b_n} x^{r+1} \, d\Lambda(x) \\
= {}& c^r (\log b_n)^{r+2} \Big(F_\lambda^n \Big( \Big(1 + \frac{c}{1+\lambda^2}\Big) b_n \Big) - \Lambda(c \log b_n) \Big) \\
&+ (-1)^r (1+\lambda^2)^r (\log b_n)^{r+2} \Lambda(-(1+\lambda^2)\log b_n) \\
&- \int_{-(1+\lambda^2)\log b_n}^{c\log b_n} r x^{r-1} (\log b_n)^2 (F_\lambda^n(a_n x + b_n) - \Lambda(x)) \, dx \\
&- 2^{-1}(1+\lambda^2)^{-1} r(\log b_n) \int_{-d\log\log b_n}^{c\log b_n} x^{r+1} \, d\Lambda(x) \\
&+ \int_{c\log b_n}^{+\infty} (\log b_n)^2 x^r d(F_\lambda^n(a_n x + b_n) - \Lambda(x)) \\
&- 2^{-1}(1+\lambda^2)^{-1} r(\log b_n) \int_{c\log b_n}^{+\infty} x^{r+1} \, d\Lambda(x) \\
&- \int_{-\infty}^{-(1+\lambda^2)\log b_n} (\log b_n)^2 x^r \, d\Lambda(x) \\
&- 2^{-1}(1+\lambda^2)^{-1} r(\log b_n) \int_{-\infty}^{-d\log\log b_n} x^{r+1} \, d\Lambda(x) \\
= {}& c^r (\log b_n)^{r+2} (1 - \Lambda(c\log b_n)) \\
&- c^r (\log b_n)^{r+2} \Big(1 - F_\lambda^n \Big( \Big(1 + \frac{c}{1+\lambda^2}\Big) b_n \Big) \Big) \\
&+ (-1)^r (1+\lambda^2)^r (\log b_n)^{r+2} \Lambda(-(1+\lambda^2)\log b_n) \\
&- \int_{-d\log\log b_n}^{c\log b_n} r x^{r-1} (\log b_n)((\log b_n)(F_\lambda^n(a_n x + b_n) - \Lambda(x)) - \kappa(x)\Lambda(x)) \, dx \\
&- \int_{-(1+\lambda^2)\log b_n}^{-d\log\log b_n} r x^{r-1} (\log b_n)^2 (F_\lambda^n(a_n x + b_n) - \Lambda(x)) \, dx \\
&+ \int_{c\log b_n}^{+\infty} (\log b_n)^2 x^r d(F_\lambda^n(a_n x + b_n) - \Lambda(x)) \\
&- 2^{-1}(1+\lambda^2)^{-1} r(\log b_n) \int_{c\log b_n}^{+\infty} x^{r+1} \, d\Lambda(x)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{-\infty}^{-(1+\lambda^2)\log b_n} (\log b_n)^2 x^r \, d\Lambda(x) \\
&- 2^{-1}(1+\lambda^2)^{-1} r (\log b_n) \int_{-\infty}^{-d\log\log b_n} x^{r+1} \, d\Lambda(x) \\
\to & -r \int_{-\infty}^{+\infty} x^{r-1} \left( \omega(x) + \frac{\kappa^2(x)}{2} \right) \Lambda(x) \, dx \\
= & \, 24^{-1}(1+\lambda^2)^{-2} r [(3r+1) m_{r+2} - 12(1+\lambda^2) m_{r+1} - 48(1+\lambda^2) m_r]
\end{aligned}
$$

as $n \to \infty$, which is (2.2).

Similarly, we can derive (2.1) for $\lambda \geqslant 0$. The proof is complete. ■

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