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A LIMIT THEOREM FOR THE LAST EXIT TIME OVER A MOVING NONLINEAR BOUNDARY FOR A GAUSSIAN PROCESS

BY

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Abstract. We prove the convergence of the distribution of the scaled last exit time over a slowly moving nonlinear boundary for a class of Gaussian stationary processes. The limit is a double exponential (Gumbel) distribution.

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1. INTRODUCTION

Consider a stationary Gaussian process with continuous trajectories and its "last exit time over a moving boundary", i.e. the last time when the process hits a boundary $\varepsilon f(t)$, where t denotes time and $\varepsilon > 0$ is a drift (or trend) parameter. After this time, the process stays forever *under* the boundary. We are interested in the asymptotic distribution of the last exit time when the parameter ε goes to zero. In this work, we prove the convergence of the distribution of the properly centered and scaled last exit time to a double exponential (Gumbel) law.

A special case of this problem, for a particular process and a linear boundary, emerged in recent works [1, 2], that provide a mathematical study of a physical model (Brownian chain break). In [9] the same question for a wide class of processes was studied. In this work we give a natural generalization of this result to a variety of boundaries.

As far as we know, the problem setting, dealing with small trends, is new, although the last exit time is a fairly popular object in economical applications, such as studies of ruin probabilities. In risk theory, for a centered Gaussian process with continuous trajectories $Y(t)$, the process

$$R(t) = u + \varepsilon f(t) - Y(t)$$

represents company balance. For instance, when $f(t) = t$, this process can be used as the simplest model of an insurance company balance with starting balance u, fixed income per time ε and stochastic expenses Y. In this setting the values $\inf\{t : R(t) < 0\}$ and $\max\{t : R(t) \leqslant 0\}$ are called the ruin time and the ultimate recovery time respectively.

There are plenty of works dedicated to the times when a process with trend hits some level. In order to achieve a positive result one has to balance between the variety of trends and the variety of the processes considered.

For instance, the classical ruin time is well studied. The result for a locally stationary Gaussian process $Y(t)$ is given in [7].

In [5] the asymptotic behavior of the distance between the ruin time and the ultimate recovery time when u goes to infinity and $Y(t)$ has stationary increments is studied.

In [4] the Paris ruin time $\inf\{t : \forall\delta \in [0, \Delta],\ R(t - \delta) \leqslant 0\}$ for a fixed Δ is considered.

On the other hand, in [15] the ruin time and the recovery time for any smooth trend are studied for the standard Brownian motion $Y(t)$.

In those settings, however, as a rule, one considers processes with stationary increments and a fixed trend; see also [8, 12]. In this work we consider the ultimate recovery time for a stationary process without starting balance and small ratio of trend to volatility. The covariance function of the process is of Hölder type at 0 and decreasing at infinity faster than $1/\ln t$. Moreover, the class of trends is quite wide, covering, among others, the cases εt^β and $\varepsilon\exp\{t^q\}$.

2. MAIN RESULT

Throughout we use the following notation for positive functions $f(x)$ and $g(x)$:

- $f(x) \prec g(x)$ if $\exists C > 0 : f(x) < Cg(x)$ asymptotically as x approaches x_0,
- $f(x) \asymp g(x)$ if simultaneously $f(x) \succ g(x)$ and $f(x) \prec g(x)$,
- $f(x) \sim g(x)$ if $\lim_{x\to x_0} f/g = 1$.

Sometimes we consider $f_a(x)$ and $g_a(x)$ depending on a parameter a. We say that the first relation above is uniform over $a \in K$ if the constant C does not depend on a. The third relation is uniform over $a \in K$ if the convergence is uniform.

Let $Y(t)$, $t \in \mathbb{R}$, be a real-valued centered stationary Gaussian process with covariance function $\rho(t) := \mathbb{E}[Y(t)Y(0)]$. We make two assumptions on the covariance function:

- at zero (*Pickands condition*): for some $v > 0$, $Q > 0$, $\alpha \in (0, 2]$,

$$\rho(t) = v^2(1 - Q|t|^\alpha + o(|t|^\alpha)) \quad \text{as } t \to 0; \tag{2.1}$$

- at infinity (*Berman condition*):

$$\rho(t) = o((\ln t)^{-1}) \quad \text{as } t \to +\infty. \tag{2.2}$$

The Berman condition appears in the context of limit theorems for maxima of Gaussian stationary sequences and processes [3], [6, Theorem 3.8.2].

Define Y's *last exit time over a boundary* f as

$$T = T(\varepsilon) := \max\{t : Y(t) = \varepsilon f(t)\}.$$

We are interested in the asymptotic behavior of $T(\varepsilon)$ when $\varepsilon \to 0$. Therefore, if not specified otherwise, we consider all the asymptotic relations when $\varepsilon \to 0$. Clearly $T(\varepsilon) = \max\left\{t : \frac{1}{v}Y(t) = \frac{\varepsilon}{v}f(t)\right\} = T_1(\varepsilon/v)$, where T_1 is the last exit time of the scaled process $\frac{1}{v}Y(t)$. Therefore, one can study only normalized processes with $v = 1$ and then substitute ε/v for ε in the result for generality. Let us denote $\varepsilon_v = \varepsilon/v$ for convenience.

There are three conditions on the boundary function f:

- *Ultimate monotonicity*: when x tends to infinity, $f(x)$ is strictly increasing, twice differentiable and tends to infinity.

- *Restriction on growth rate*: for some $0 < \lambda \leqslant 1$ we have $f''(x)/f'(x) \asymp x^{-\lambda}$ when $x \to \infty$.

To state the third restriction we need to introduce two parameters. Take $\gamma = \gamma(\varepsilon)$ such that

$$\gamma^2 = 2\ln\left[\frac{(f^{-1})'(1/\varepsilon_v)}{\varepsilon_v}\right] + o(1) \quad \text{when } \varepsilon \to 0. \tag{2.3}$$

and $\tau_0 = \tau_0(\varepsilon)$ such that

$$f(\tau_0) = \gamma/\varepsilon_v \quad \text{when } \varepsilon \to 0. \tag{2.4}$$

They are important, because, in fact, τ_0 is the main term of $T(\varepsilon)$'s asymptotics and $\tau_0^\lambda \gamma^{-2}$ is the precision, where the stochastic part appears. Assume that f also satisfies the following condition:

- *Regularity*: for some $\kappa > 0$, β and $\tilde{\beta}$, when $\varepsilon \to 0$

(2.5)
$$(f^{-1})'(y/\varepsilon_v) \sim y^\beta (f^{-1})'(1/\varepsilon_v) \qquad \text{uniformly over } y \in [(1-\kappa)\gamma, (1+\kappa)\gamma],$$
(2.6)
$$(f^{-1})'(y/\varepsilon_v) = o\big(y^{\tilde{\beta}} (f^{-1})'(1/\varepsilon_v)\big) \quad \text{uniformly over } y \in [(1+\kappa)\gamma, \infty).$$

Then we have the following limit theorem, in which the Pickands constant $\mathcal{H}_\alpha$ appears; see [11] for more information about it.

THEOREM 2.1. *Let $Y(t)$, $t \in \mathbb{R}$, be a real-valued centered stationary Gaussian process satisfying* (2.1) *and* (2.2)*. Let f be a function satisfying the conditions above. Define $C_\alpha := \frac{Q^{1/\alpha}\mathcal{H}_\alpha}{\sqrt{2\pi}}$. Then for any $r \in \mathbb{R}$,*

$$\lim_{\varepsilon_v \to 0} \mathbb{P}\left\{\frac{T(\varepsilon) - A_\varepsilon}{B_\varepsilon} \leqslant r\right\} = \exp(-C_\alpha \exp(-r)),$$

for the shift and scaling constants defined via τ_0 and γ as follows:

$$A_\varepsilon := \tau_0 + \frac{(2/\alpha + \beta - 2)\ln\gamma}{f'(\tau_0)\varepsilon_v\gamma} + o\left(\frac{1}{f'(\tau_0)\varepsilon_v\gamma}\right),$$
$$B_\varepsilon := \frac{1 + o(1)}{f'(\tau_0)\varepsilon_v\gamma}.$$

Note that in the recent work [9] the linear boundary was considered for the same class of processes. The proof below is based on the same ideas. However, working with a much larger class of boundaries requires more technical analysis.

2.1. Examples. There are several interesting examples of boundaries for which the normalizing constants can be found in explicit form.

COROLLARY 2.1. *For a boundary function $f(x) = x^d$, $d > 0$, and a process Y, constant C_α, parameter ε_v and the moment $T(\varepsilon)$ defined as before one has*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{\frac{T(\varepsilon) - A_\varepsilon}{B_\varepsilon} \leqslant r\right\} = \exp(-C_\alpha \exp(-r)),$$

where the shift and scaling constants can be found explicitly:

$$A_\varepsilon = \frac{(-2\ln\varepsilon_v)^{\frac{1}{2d}-1}}{d^{\frac{1}{2d}}\varepsilon_v^{\frac{1}{d}}}\left(-2\ln\varepsilon_v + \left(\frac{1}{\alpha} + \frac{1}{2d} - \frac{3}{2}\right)\ln(-2\ln\varepsilon_v) - \left(\frac{1}{\alpha} + \frac{1}{2d} - \frac{1}{2}\right)\ln d\right),$$
$$B_\varepsilon = \frac{(-2\ln\varepsilon_v)^{\frac{1}{2d}-1}(1 + o(1))}{d^{\frac{1}{2d}}\varepsilon_v^{\frac{1}{d}}}.$$

The following proof works for $d \neq 1$. However, the statement remains true for $d = 1$. First of all, exactly the case of $f(x) = x$ was handled in [9]. Secondly, one can prove that for $f(x) = x^d$ the convergence in the theorem is uniform over $d \in (1/2, 1)$. Since $\mathbb{P}\big(\frac{T(\varepsilon)-A_\varepsilon}{B_\varepsilon} \leqslant r\big)$ is continuous in d, we see that convergence for $d \in (1/2, 1)$ implies the convergence for $d = 1$. Now we proceed with the proof of the corollary for $d \neq 1$.

Proof of Corollary 2.1. For $f(x)=x^d, d>0$, we get $f''(x)=(d-1)x^{-1}f'(x)$, i.e. $\lambda = 1$. Moreover, $f^{-1}(x) = x^{1/d}$ and $(f^{-1})'(x) = \frac{1}{d}x^{1/d-1}$. Therefore, we take

$$\gamma^2 = 2\ln\left(\frac{1}{d}\varepsilon_v^{-1/d}\right) + o(1) = \frac{-2\ln\varepsilon_v}{d} - 2\ln d + o(1).$$

In addition, for every y,

$$(f^{-1})'(y/\varepsilon_v) = \frac{1}{d}y^{1/d-1}\varepsilon_v^{1-1/d} = y^{1/d-1}(f^{-1})'(1/\varepsilon_v),$$

i.e. the regularity conditions (2.5), (2.6) hold with $\beta = \tilde{\beta} = 1/d - 1$.

Take

$$\tau_0 = f^{-1}(\gamma/\varepsilon_v) = (\gamma/\varepsilon_v)^{1/d}.$$

Then

$$f'(\tau_0) = d\frac{f(\tau_0)}{\tau_0} = d\frac{\gamma}{\varepsilon_v\tau_0}.$$

Hence, by substituting everything in Theorem 2.1 we obtain

$$A_\varepsilon = \tau_0 + \frac{(2/\alpha+\beta-2)\tau_0\ln\gamma}{d\gamma^2} + o\left(\frac{\tau_0}{\gamma^2}\right), \qquad B_\varepsilon = \frac{\tau_0+o(\tau_0)}{d\gamma^2}.$$

Transforming these expressions, we get

$$\begin{aligned} A_\varepsilon &= \frac{\gamma^{\frac{1}{d}}}{\varepsilon_v^{\frac{1}{d}}} + \left(\frac{2}{\alpha}+\frac{1}{d}-3\right)\frac{\gamma^{\frac{1}{d}-2}\ln\gamma}{d\varepsilon_v^{\frac{1}{d}}} + o\left(\frac{\gamma^{\frac{1}{d}-2}}{\varepsilon_v^{\frac{1}{d}}}\right) \\ &= \frac{(-2\ln\varepsilon_v)^{\frac{1}{2d}-1}}{d^{\frac{1}{2d}}\varepsilon_v^{\frac{1}{d}}}\Bigg(-2\ln\varepsilon_v + \left(\frac{1}{\alpha}+\frac{1}{2d}-\frac{3}{2}\right)\ln(-2\ln\varepsilon_v) \\ &\quad - \left(\frac{1}{\alpha}+\frac{1}{2d}-\frac{1}{2}\right)\ln d\Bigg), \\ B_\varepsilon &= \frac{\gamma^{\frac{1}{d}-2}}{d\varepsilon_v^{\frac{1}{d}}} = \frac{(-2\ln\varepsilon_v)^{\frac{1}{2d}-1}(1+o(1))}{d^{\frac{1}{2d}}\varepsilon_v^{\frac{1}{d}}}. \quad \blacksquare \end{aligned}$$

COROLLARY 2.2. *For a boundary function $f(x) = \exp\{x^q\}$, $0 < q < 1$, and a proccess $Y(t)$, constant C_α, parameter ε_v and the moment $T(\varepsilon)$ defined as before one has*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left\{\frac{T(\varepsilon)-A_\varepsilon}{B_\varepsilon} \leqslant r\right\} = \exp(-C_\alpha\exp(-r)),$$

where the shift and scaling constants can be found explicitly:

$$A_\varepsilon = \frac{(-\ln\varepsilon_v)^{\frac{1}{q}-1}}{q}\Bigg(q(-\ln\varepsilon_v) + \frac{1}{2}\ln\ln(-\ln\varepsilon_v) + \frac{1}{2}\ln\bigg(\frac{2}{q}-2\bigg)$$
$$+\bigg(\frac{1}{\alpha}-\frac{3}{2}\bigg)\frac{\ln\ln(-\ln\varepsilon_v)}{\big(\frac{2}{q}-2\big)\ln(-\ln\varepsilon_v)} + \frac{\big(\frac{1}{\alpha}-\frac{3}{2}\big)\ln\big(\frac{2}{q}-2\big)-\ln q}{\big(\frac{2}{q}-2\big)\ln(-\ln\varepsilon_v)}\Bigg),$$
$$B_\varepsilon = \frac{(-\ln\varepsilon_v)^{\frac{1}{q}-1}}{(2-2q)\ln(-\ln\varepsilon_v)}.$$

Proof. For $f(x) = \exp\{x^q\}$, $0 < q < 1$, we have

$$f'(x) = qx^{q-1}\exp\{x^q\},$$
$$f''(x) = q(q-1)x^{q-2}\exp\{x^q\} + q^2x^{2q-2}\exp\{x^q\}.$$

Therefore, the restriction on the growth rate holds with $\lambda = 1-q$, since

$$f''(x)/f'(x) \sim qx^{-(1-q)}.$$

In addition

$$(f^{-1})'(t) = \frac{(\ln t)^{\frac{1}{q}-1}}{qt},$$

Therefore,

$$\gamma^2 = \bigg(\frac{2}{q}-2\bigg)\ln(-\ln\varepsilon_v) - 2\ln q + o(1).$$

Then we get the regularity conditions (2.5), (2.6) with $\beta = -1, \tilde{\beta} = 0$, because uniformly over $y \in [0.5\gamma, 2\gamma]$, as $\varepsilon \to 0$ we have

$$(f^{-1})'(y/\varepsilon_v) = \frac{(\ln y - \ln\varepsilon_v)^{\frac{1}{q}-1}}{qy/\varepsilon_v} \sim y^{-1}(f^{-1})'(1/\varepsilon_v),$$

and uniformly over $y \in [2\gamma, \infty]$, as $\varepsilon \to 0$ we have

$$(f^{-1})'(y/\varepsilon_v) = \frac{(\ln y - \ln\varepsilon_v)^{\frac{1}{q}-1}}{qy/\varepsilon_v} = o((f^{-1})'(1/\varepsilon_v)).$$

Now we can use Theorem 2.1. Take

$$\tau_0 = f^{-1}(\gamma/\varepsilon_v) = (\ln\gamma - \ln\varepsilon_v)^{\frac{1}{q}};$$

then

$$f'(\tau_0)\varepsilon_v = q\tau_0^{q-1}f(\tau_0)\varepsilon_v = q\tau_0^{q-1}\gamma.$$

Therefore, in the formula for A_ε and B_ε the precision we are interested in is

$$o\left(\frac{1}{f'(\tau_0)\varepsilon_v\gamma}\right) = o\left(\frac{\tau_0^{1-q}}{q\gamma^2}\right) = o\left(\frac{(-\ln\varepsilon_v)^{1/q-1}}{\gamma^2}\right).$$

Consequently,

$$A_\varepsilon = \tau_0 + \frac{(2/\alpha+\beta-2)\tau_0^{1-q}\ln\gamma}{q\gamma^2} + o\left(\frac{\tau_0^{1-q}}{\gamma^2}\right), \qquad B_\varepsilon = \frac{\tau_0^{1-q}(1+o(1))}{q\gamma^2}.$$

Let us write

$$\tau_0 = (-\ln\varepsilon_v)^{\frac{1}{q}} + \frac{1}{q}(-\ln\varepsilon_v)^{\frac{1}{q}-1}\ln\gamma + o\left(\frac{(-\ln\varepsilon_v)^{\frac{1}{q}-1}}{\gamma^2}\right).$$

Additionally, $\ln\gamma = \frac{1}{2}\ln\gamma^2$ yields

$$\begin{aligned}\ln\gamma &= \frac{1}{2}\left(\ln\left(\left(\frac{2}{q}-2\right)\ln(-\ln\varepsilon_v)\right) + \ln\left(1 - \frac{\ln q}{\left(\frac{1}{q}-1\right)\ln(-\ln\varepsilon_v)}\right)\right)\\ &= \frac{1}{2}\ln\ln(-\ln\varepsilon_v) + \frac{1}{2}\ln\left(\frac{2}{q}-2\right) - \frac{\ln q}{(\frac{2}{q}-2)\ln(-\ln\varepsilon_v)} + \frac{o(1)}{\ln(-\ln\varepsilon_v)}.\end{aligned}$$

After some transformations we get

$$\begin{aligned}A_\varepsilon &= \frac{(-\ln\varepsilon_v)^{\frac{1}{q}-1}}{q}\left(q(-\ln\varepsilon_v) + \frac{1}{2}\ln\ln(-\ln\varepsilon_v) + \frac{1}{2}\ln\left(\frac{2}{q}-2\right)\right.\\ &\quad \left. + \left(\frac{1}{\alpha}-\frac{3}{2}\right)\frac{\ln\ln(-\ln\varepsilon_v)}{\left(\frac{2}{q}-2\right)\ln(-\ln\varepsilon_v)} + \frac{\left(\frac{1}{\alpha}-\frac{3}{2}\right)\ln\left(\frac{2}{q}-2\right)-\ln q}{\left(\frac{2}{q}-2\right)\ln(-\ln\varepsilon_v)}\right),\\ B_\varepsilon &= \frac{(-\ln\varepsilon_v)^{\frac{1}{q}-1}}{(2-2q)\ln(-\ln\varepsilon_v)}. \quad \blacksquare\end{aligned}$$

3. PROOF OF THE MAIN RESULT

The first restriction (2.1) on the covariance function appears in the following lemma, which serves below as one out of the two basic tools in our calculations.

LEMMA 3.1 (Pickands, Piterbarg). *Let $Y(t)$, $t \in \mathbb{R}$, be a real-valued centered stationary Gaussian process satisfying the Pickands condition* (2.1) *and such that*

$$\limsup_{t\to\infty}\rho(t) < 1.$$

Then, with $\mathcal{H}_\alpha$ being the Pickands constant (*in particular, $\mathcal{H}_1 = 1, \mathcal{H}_2 = \pi^{-1/2}$*), *we have*

$$\mathbb{P}\left\{\max_{s\in[0,t]} Y(s) \geqslant x\right\} \sim \frac{Q^{1/\alpha}\mathcal{H}_\alpha}{\sqrt{2\pi}} \cdot t \cdot (x/v)^{2/\alpha-1} e^{-x^2/(2v^2)}$$

for all x and t such that the right hand side tends to zero and $tx^{2/\alpha} \to \infty$.

A first version of this lemma with fixed t was obtained by Pickands [13], while this version with variable t (which is important for our results) is due to Piterbarg [14, Lecture 9, Theorem 9.3.1].

As was mentioned before, by scaling $Y(t) = v\widetilde{Y}(t)$ one may reduce the problem to the case $v = 1$. Hence from now on $\varepsilon_v = \varepsilon$.

Let us fix $r \in \mathbb{R}$ and let

$$\tau = \tau(\varepsilon, r) := A_\varepsilon + B_\varepsilon r.$$

The theorem's statement is equivalent to

$$\lim_{\varepsilon \to 0} \mathbb{P}\{T(\varepsilon) > \tau\} = 1 - \exp(-C_\alpha \exp(-r)).$$

The basic plan of the proof is the following. We are interested in the event of crossing the boundary $\varepsilon f(t)$ after time τ.

First of all, for appropriately large σ this crossing happens before time σ with probability close to 1. For this purpose one can pick $\sigma = A_\varepsilon + B_\varepsilon R(\varepsilon)$ with $R(\varepsilon)$ growing to infinity with any speed.

Now, to analyze the event of crossing somewhere in $[\tau, \sigma]$ we divide $[\tau, \sigma]$ into intervals and approximate the boundary $\varepsilon f(t)$ with a staircase, i.e. a function that is constant on each interval and has jumps in between. The event of crossing the staircase is much simpler and we can use Lemma 3.1 to analyze it, by studying the events of crossing on different intervals. If those events of crossing were almost independent for different intervals, we would have to work with a sum of independent random variables. The only problem here is that the events of crossing on neighboring intervals are unpredictably correlated.

To fix it, one has to choose the intervals appropriately. We divide $[\tau, \sigma]$ into alternating long and short intervals of length $\ell(\varepsilon)$ and $s(\varepsilon)$ respectively. Then we show that the crossing happens on one of the small intervals with probability close to 0. The role of the small intervals is to separate long intervals, so we can control the correlation of the process between different long intervals via the Berman condition (2.2).

Finally, to work with the crossings on different long intervals we concentrate the correlation between different intervals in one auxiliary term using the Slepian inequality (3.16). This part of the proof starts with Lemma 3.4. It allows us to pass to the sum of independent random variables and then find the probability we are interested in.

The interval lengths $\ell = \ell(\varepsilon)$, $s = s(\varepsilon)$ must satisfy the following relations:

$$\ln s \succ \gamma^2, \tag{3.1}$$

$$s/\ell \to 0, \tag{3.2}$$

$$f(\tau) f'(\tau) \varepsilon^2 \, \ell \to 0. \tag{3.3}$$

We need the first relation to get a lower bound on the correlation between different long intervals. The second relation implies that the probability of crossing on a small interval is close to 0. The third one is equivalent to $\ell = o(B_\varepsilon)$ and implies that $\ell = o(\sigma - \tau) = o(B_\varepsilon(R(\varepsilon) - r))$, i.e. the number of intervals grows to infinity and the staircase is a good enough approximation of the boundary.

The proof that one can pick ℓ and s satisfying these conditions is given in Section 3.1.

We cover the halfline $[\tau, \infty)$ with the following system of sets:

- the halfline $[\sigma, \infty)$, where $\sigma := A_\varepsilon + B_\varepsilon R$ and $R = R(\varepsilon)$ slowly tends to infinity as $\varepsilon \to 0$; the choice of R is further specified at the end of the proof but note that we assume only upper bounds on the growth rate of R,
- long intervals $L_i = [(\ell + s)i, (\ell + s)i + \ell]$, $i \in \mathbb{Z}$, of length $\ell = \ell(\varepsilon)$,
- short intervals $S_i = [(\ell + s)i + \ell, (\ell + s)(i + 1)]$, $i \in \mathbb{Z}$, of length $s = s(\varepsilon)$.

Let

$$X_i^\varepsilon := \max_{t \in L_i} Y(t), \qquad V_i^\varepsilon := \max_{t \in S_i} Y(t).$$

By using stationarity, we infer from the Pickands–Piterbarg lemma the asymptotics

$$\begin{aligned} \mathbb{P}\{X_i^\varepsilon \geqslant x\} &\sim C_\alpha \ell x^{2/\alpha - 1} \exp(-x^2/2), \\ \mathbb{P}\{V_i^\varepsilon \geqslant x\} &\sim C_\alpha s x^{2/\alpha - 1} \exp(-x^2/2), \end{aligned}$$

as soon as the respective right hand sides tend to zero and $sx^{2/\alpha} \to \infty$. We use these relations for $x \succ f(\tau)\varepsilon$ and clearly $s(f(\tau)\varepsilon)^{2/\alpha} \sim s\gamma^{2/\alpha} \to \infty$.

Define the index sets

$$\begin{aligned} I_1 &:= \{i : (\ell + s)i + \ell \geqslant \tau,\ (\ell + s)i < \sigma\}, \\ I_2 &:= \{i : (\ell + s)i \geqslant \tau,\ (\ell + s)i + \ell < \sigma\}, \\ I_3 &:= \{i : (\ell + s)(i + 1) \geqslant \tau,\ (\ell + s)i + \ell < \sigma\} \end{aligned}$$

chosen so that

$$\bigcup_{i \in I_2} L_i \subset [\tau, \sigma] \subset \Big(\bigcup_{i \in I_1} L_i\Big) \cup \Big(\bigcup_{i \in I_3} S_i\Big). \tag{3.4}$$

Let us define the events related to the exits of our process over the boundary:

$$\begin{aligned} \mathcal{E}_1 &:= \bigcup_{i \in I_1} \{X_i^\varepsilon \geqslant f((\ell + s)i)\varepsilon\}, \\ \mathcal{E}_2 &:= \bigcup_{i \in I_2} \{X_i^\varepsilon \geqslant f((\ell + s)(i + 1))\varepsilon\}, \\ \mathcal{E}_3 &:= \bigcup_{i \in I_3} \{V_i^\varepsilon \geqslant f((\ell + s)i)\varepsilon\}, \\ \mathcal{E}_4 &:= \{\exists\, t > \sigma : Y(t) \geqslant \varepsilon f(t)\}. \end{aligned}$$

By using the inclusions (3.4) and ultimate monotonicity of f, it is easy to see that

$$\mathbb{P}\{T(\varepsilon) > \tau\} = \mathbb{P}\{\exists\, t > \tau : Y(t) \geqslant \varepsilon f(t)\} \leqslant \mathbb{P}\{\mathcal{E}_1\} + \mathbb{P}\{\mathcal{E}_3\} + \mathbb{P}\{\mathcal{E}_4\},$$
$$\mathbb{P}\{T(\varepsilon) > \tau\} \geqslant \mathbb{P}\{\mathcal{E}_2\}.$$

Therefore, it is sufficient to prove that, as $\varepsilon \to 0$,

$$\mathbb{P}\{\mathcal{E}_1\}, \mathbb{P}\{\mathcal{E}_2\} \to 1 - \exp(-C_\alpha \exp(-r)), \quad \mathbb{P}\{\mathcal{E}_3\}, \mathbb{P}\{\mathcal{E}_4\} \to 0.$$

The parameter R should grow to infinity so slowly that uniformly over $\omega \in [-1, 1]$ we have

$$\tau_0 \sim \tau_0 + \omega(\sigma - \tau_0), \tag{3.5}$$
$$f'(\tau_0) \sim f'(\tau_0 + \omega(\sigma - \tau_0)), \tag{3.6}$$
$$f(\tau_0) \sim f(\tau_0 + \omega(\sigma - \tau_0)). \tag{3.7}$$

That is, f and f' do not change much in the interval around τ_0 containing σ.

The following lemma provides useful asymptotic estimates and shows that we can pick R satisfying (3.5). Since it is only technical, the proof is postponed to Section 3.1.

LEMMA 3.2. *Consider any function $\tilde{R}(\varepsilon)$ such that $\tilde{R} = o(\ln\gamma)$ and*

$$B_\varepsilon \tilde{R} = \frac{\tilde{R}}{f'(\tau_0)\varepsilon\gamma} + o\left(\frac{1}{f'(\tau_0)\varepsilon\gamma}\right)$$

Then for A_ε and B_ε described in Theorem 2.1 we have

$$f(A_\varepsilon + B_\varepsilon \tilde{R})\varepsilon = \gamma + \left(\frac{2}{\alpha} + \beta - 2\right)\frac{\ln\gamma}{\gamma} + \frac{\tilde{R}}{\gamma} + o\left(\frac{1}{\gamma}\right). \tag{3.8}$$

Moreover, for $R(\varepsilon)$ corresponding to the same restriction, uniformly over $\omega \in [-1, 1]$ one has

$$f'(\tau_0) \sim f'(\tau_0 + \omega(\sigma - \tau_0)), \quad f(\tau_0) \sim f(\tau_0 + \omega(\sigma - \tau_0)).$$

From (3.8) applied to $\tilde{R} = r$ and $\tilde{R} = R$, we obtain

$$f(\tau)\varepsilon \sim \gamma, \tag{3.9}$$
$$(f(\tau)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\tau)\varepsilon)^2/2\} \sim e^{-r-\gamma^2/2}, \tag{3.10}$$
$$(f(\sigma)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\sigma)\varepsilon)^2/2\} = o(e^{-\gamma^2/2}). \tag{3.11}$$

These relations are crucial for our choice of τ and γ, because they appear in the resulting probabilities of $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$.

For the following relations we use an analogue of the restriction on the growth rate but for f instead of f', namely the fact that

$$\frac{f'(x)}{f(x)} \asymp x^{-\lambda}, \quad x \to \infty. \tag{3.12}$$

To check this note that $(f'(x)x^{\lambda})' = f''(x)x^{\lambda} + \lambda f'(x)x^{\lambda-1}$. Therefore, since $cf'(x) < f''(x)x^{\lambda} < Cf'(x)$ and $\lambda \leqslant 1$, we can write a similar inequality but with new constants depending on λ:

$$c_{\lambda}f'(x) < (f'(x)x^{\lambda})' < C_{\lambda}f'(x) \quad \text{for large enough } x.$$

Now we integrate these inequalities to get

$$c_{\lambda}f(x) + \text{const} < f'(x)x^{\lambda} < C_{\lambda}f(x) + \text{const} \quad \text{for large enough } x.$$

Since $f(x) \to \infty$ as $x \to \infty$, we divide by $f(x)$ and conclude that $f'(x)x^{\lambda} \asymp f(x)$, i.e. $f'(x)/f(x) \asymp x^{-\lambda}$.

Notice that (3.12) and (3.9) imply that

$$f(\tau)f'(\tau)\varepsilon^2 \asymp (f(\tau)\varepsilon)^2\tau^{-\lambda} \asymp \gamma^2\tau^{-\lambda}. \tag{3.13}$$

Moreover, there is a connection between τ and γ. Due to (3.12) and the regularity condition (2.5) combined with the definition (2.3) of γ we get

$$(f^{-1}(\gamma/\varepsilon))^{\lambda} \asymp \frac{f(f^{-1}(\gamma/\varepsilon))}{f'(f^{-1}(\gamma/\varepsilon))} = \frac{\gamma}{\varepsilon}(f^{-1})'(\gamma/\varepsilon) \sim \frac{\gamma^{1+\beta}}{\varepsilon}(f^{-1})'(1/\varepsilon)\gamma^{1+\beta}e^{\gamma^2/2},$$

and since $\tau \sim \tau_0 = f^{-1}(\gamma/\varepsilon)$ due to (3.5), we get

$$\tau^{\lambda} \asymp \gamma^{1+\beta}e^{\gamma^2/2}. \tag{3.14}$$

Now, having obtained the crucial relations, we proceed with the proof. Let us first show that the probabilities of the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are almost equal; thus it is enough to find the limit of $\mathbb{P}\{\mathcal{E}_1\}$. Indeed, let $I_1 = \{m, m+1, \dots, n\}$. Then, on the one hand, $\mathbb{P}\{\mathcal{E}_2\} \leqslant \mathbb{P}\{\mathcal{E}_1\}$; on the other hand,

$$\begin{aligned}
\mathbb{P}\{\mathcal{E}_1\} &= \mathbb{P}\Big\{\bigcup_{i=m}^{n}\{X_i^{\varepsilon} \geqslant f((\ell+s)i)\varepsilon\}\Big\} \\
&\leqslant \mathbb{P}\{X_m^{\varepsilon} \geqslant f((\ell+s)m)\varepsilon\} + \mathbb{P}\{X_{m+1}^{\varepsilon} \geqslant f((\ell+s)(m+1))\varepsilon\} \\
&\quad + \mathbb{P}\Big\{\bigcup_{i=m+2}^{n}\{X_i^{\varepsilon} \geqslant f((\ell+s)i)\varepsilon\}\Big\} \\
&\leqslant 2\mathbb{P}\{X_m^{\varepsilon} \geqslant f((\ell+s)m)\varepsilon\} + \mathbb{P}\Big\{\bigcup_{j=m+1}^{n-1}\{X_{j+1}^{\varepsilon} \geqslant f((\ell+s)(j+1))\varepsilon\}\Big\} \\
&\leqslant 2\mathbb{P}\{X_m^{\varepsilon} \geqslant f((\ell+s)m)\varepsilon\} + \mathbb{P}\{\mathcal{E}_2\},
\end{aligned}$$

where in the penultimate inequality we use the stationarity of the sequence X_i^ε following from the stationarity of Y. For the remaining term we use the Pickands–Piterbarg asymptotics to obtain

$$\begin{aligned}\mathbb{P}\{X_m^\varepsilon \geqslant f((\ell+s)m)\varepsilon\} \\ \leqslant \mathbb{P}\{X_m^\varepsilon \geqslant f(\tau-\ell)\varepsilon\} \sim C_\alpha \ell (f(\tau-\ell)\varepsilon)^{2/\alpha-1}\exp\{-(f(\tau-\ell)\varepsilon)^2/2\}.\end{aligned}$$

From (3.3) we know $\ell f(\tau)f(\tau)\varepsilon^2 \to 0$ and from (3.9) and the definition of B_ε we infer that $\ell = o(\sigma-\tau)$. Moreover, due to Lagrange's theorem and (3.5), for some $\xi \in [\tau-\ell, \tau]$ one has

$$\begin{aligned}f(\tau-\ell)^2\varepsilon^2 &= (f(\tau)-\ell f'(\xi))^2\varepsilon^2 \\ &= f(\tau)^2\varepsilon^2 - 2\ell f(\tau)f'(\xi)\varepsilon^2 + \ell^2 f'(\xi)^2\varepsilon^2 = f(\tau)^2\varepsilon^2 + o(1).\end{aligned}$$

Combining this with (3.9) and (3.10) we get

$$\begin{aligned}\ell(f(\tau-\ell)\varepsilon)^{2/\alpha-1}\exp\{-(f(\tau-\ell)\varepsilon)^2/2\} \sim \ell(f(\tau)\varepsilon)^{2/\alpha-1}\exp\{-(f(\tau)\varepsilon)^2/2\} \\ \sim \ell(f(\tau)\varepsilon)^{1-\beta}e^{-r-\gamma^2/2} \sim \ell\gamma^{1-\beta}e^{-r-\gamma^2/2}.\end{aligned}$$

We know that

$$\ell f'(\tau)f(\tau)\varepsilon^2 \to 0,$$

and from (3.13), (3.14),

$$\ell f'(\tau)f(\tau)\varepsilon^2 \asymp \ell\gamma^2\tau^{-\lambda} \asymp \ell\gamma^{1-\beta}e^{-\gamma^2/2}.$$

Hence

$$\ell\gamma^{1-\beta}e^{-r-\gamma^2/2} \to 0.$$

We conclude that $\mathbb{P}\{X_m^\varepsilon \geqslant (\ell+s)m\varepsilon\} \to 0$, thus the difference between $\mathbb{P}\{\mathcal{E}_1\}$ and $\mathbb{P}\{\mathcal{E}_2\}$ is indeed negligible.

Below we repeatedly use the following technical lemma. Its proof is postponed to Section 3.1.

LEMMA 3.3. *For each $\alpha \neq 0$ and all $\theta(\varepsilon), a(\varepsilon), b(\varepsilon), c(\varepsilon)$ such that, as $\varepsilon \to 0$, one has $f'(\theta+\omega a) \sim f'(\theta)$ uniformly over $\omega \in [-1,1]$, and*

$$f(\theta)\varepsilon \sim \gamma, \quad a = o(\theta), \quad f(\theta)c\varepsilon^2 \to 0, \quad f(\theta)f'(\theta)a\varepsilon^2 \to 0,$$

it is true that

$$\begin{aligned}\sum_{i:\, ai+b\geqslant\theta}^{\infty} (f(ai+b)\varepsilon + c\varepsilon)^{2/\alpha-1}\exp\{-(f(ai+b)\varepsilon+c\varepsilon)^2/2\} \\ \sim \frac{e^{\gamma^2/2}}{a}(f(\theta)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\theta)\varepsilon)^2/2\}.\end{aligned}$$

Let us evaluate $\mathbb{P}\{\mathcal{E}_3\}$. From stationarity of the sequence V_i^ε,

$$\mathbb{P}\{\mathcal{E}_3\} \leqslant \sum_{i\in I_3} \mathbb{P}\{V_i^\varepsilon \geqslant f((\ell+s)i)\varepsilon\},$$

and from Lemma 3.1 we obtain the following upper bound for this sum, uniform in i:

$$C_\alpha s(1+o(1)) \sum_{i:\,(\ell+s)(i+1)\geqslant\tau} (f((\ell+s)i)\varepsilon)^{2/\alpha-1} \exp\{-(f((\ell+s)i)\varepsilon)^2/2\}.$$

In order to find the asymptotic behavior of this sum, we apply Lemma 3.3 with parameters $a=\ell+s$, $b=0$, $c=0$, $\theta=\tau-\ell-s$. Then, by using (3.2), (3.3) and (3.9), we have $a\sim\ell$, $\theta\sim\tau$, $(f(\theta)\varepsilon)^2=(f(\tau)\varepsilon)^2+o(1)$. Therefore, Lemma 3.3, relation (3.10) and assumption (3.2) yield

$$\begin{aligned}\frac{C_\alpha s e^{\gamma^2/2}}{a}&(f(\theta)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\theta)\varepsilon)^2/2\}\\ &\sim \frac{C_\alpha s e^{\gamma^2/2}}{\ell}(f(\tau)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\tau)\varepsilon)^2/2\} = o(1).\end{aligned}$$

Let us bound $\mathbb{P}\{\mathcal{E}_4\}$ as follows:

$$\begin{aligned}\mathbb{P}\{\mathcal{E}_4\} &\leqslant \sum_{j=0}^{\infty} \mathbb{P}\Big\{\max_{t\in[\sigma+j,\sigma+j+1]} Y(t) > f(\sigma+j)\varepsilon\Big\}\\ &\leqslant C_\alpha\,(1+o(1)) \sum_{j=0}^{\infty} (f(\sigma+j)\varepsilon)^{2/\alpha-1} \exp\{-(f(\sigma+j)\varepsilon)^2/2\}.\end{aligned}$$

Lemma 3.3 applied with parameters $a=1$, $b=\sigma$, $c=0$, $\theta=\sigma$ and relation (3.11) provide the following asymptotics for the sum:

$$C_\alpha e^{\gamma^2/2}(f(\sigma)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\sigma)\varepsilon)^2/2\} = o(1).$$

The hardest part is to show that

$$1-\mathbb{P}\{\mathcal{E}_1\} \to \exp(-C_\alpha \exp(-r)), \qquad \varepsilon\to 0. \tag{3.15}$$

Our main tool here is the following classical inequality due to Slepian (see, e.g., [10, §14], [14, Lecture 2]).

LEMMA 3.4 (Slepian). *Let* $(U_1,\dots,U_n)$ *and* $(V_1,\dots,V_n)$ *be two centered Gaussian vectors such that* $\mathbb{E}\,U_j^2=\mathbb{E}\,V_j^2$, $1\leqslant j\leqslant n$, *and* $\mathbb{E}(U_iU_j)\leqslant\mathbb{E}(V_iV_j)$, $1\leqslant i,j\leqslant n$. *Then for each* $r\in\mathbb{R}$ *one has*

$$\mathbb{P}\Big\{\max_{1\leqslant j\leqslant n} U_j\geqslant r\Big\} \geqslant \mathbb{P}\Big\{\max_{1\leqslant j\leqslant n} V_j\geqslant r\Big\}.$$

One may write this inequality in a slightly more general form (see [14, Lecture 2]): under the assumptions of the Slepian lemma, for all nonnegative $r_1, \dots, r_n$ one has

$$\mathbb{P}\{\exists j : U_j \geqslant r_j\} \geqslant \mathbb{P}\{\exists j : V_j \geqslant r_j\}.$$

This follows by applying the Slepian inequality to the vectors $(U_1/r_1, \dots, U_n/r_n)$ and $(V_1/r_1, \dots, V_n/r_n)$ and $r = 1$.

The latter inequality easily extends to Gaussian processes with continuous trajectories defined on a metric space (by the way, the processes satisfying assumption (2.1) belong to this class). Namely, let $\{U(t), t \in T\}$ and $\{V(t), t \in T\}$ be two centered Gaussian processes with continuous trajectories defined on a common metric space T. Let $\mathbb{E}\, U(t)^2 = \mathbb{E}\, V(t)^2$, $t \in T$, and $\mathbb{E}(U(t_1)U(t_2)) \leqslant \mathbb{E}(V(t_1)V(t_2))$, $t_1, t_2 \in T$. Then for all compact sets $T_1, \dots, T_n$ in T and all nonnegative $r_1, \dots, r_n$ we have

$$\mathbb{P}\Big\{\bigcup_{j=1}^{n}\Big\{\max_{t\in T_j} U(t) \geqslant r_j\Big\}\Big\} \geqslant \mathbb{P}\Big\{\bigcup_{j=1}^{n}\Big\{\max_{t\in T_j} V(t) \geqslant r_j\Big\}\Big\}. \tag{3.16}$$

Now we proceed to the proof of the remaining claim

$$1 - \mathbb{P}\{\mathcal{E}_1\} \to \exp(-C_\alpha \exp(-r)), \qquad \varepsilon \to 0. \tag{3.17}$$

We provide the corresponding upper and lower bounds. In both cases we use the Slepian inequality in the form (3.16). Since $\mathcal{E}_1$ is defined by the process on long intervals $L_i, i \in I_1$, when referring to long intervals in this part we mean $L_i, i \in I_1$.

Upper bound. Let us compare our process Y with an auxiliary process Z that is defined as follows. First, let us consider a process $\widetilde{Y}(t), t \in \bigcup_{i\in I_1} L_i$, which consists of independent copies of $Y(t)$ on the intervals L_i. Define

$$\delta^2 = \delta(\varepsilon)^2 := \sup_{t\geqslant s(\varepsilon)} |\mathbb{E}(Y(t)Y(0))|.$$

Taking into account the correlation decay assumption (2.2) and assumption (3.1) concerning the choice of s, we have, as $\varepsilon \to 0$,

$$\delta^2 = o((\ln s)^{-1}) = o(\gamma^{-2}). \tag{3.18}$$

Let ξ be an auxiliary standard normal random variable independent of the process $\widetilde{Y}$. We define the centered Gaussian process $Z(t)$, $t \in \bigcup_{i\in I_1} L_i$, by

$$Z(t) := \sqrt{1-\delta^2}\,\widetilde{Y}(t) + \delta\xi.$$

Then for all t the variances are equal: $\mathbb{E}\, Y(t)^2 = \mathbb{E}\, Z(t)^2 = 1$. For covariances we have the following inequalities:

- for t_1 and t_2 that belong to the same interval L_i we have

$$\begin{aligned}\mathbb{E}(Z(t_1)Z(t_2)) &= \mathbb{E}\big(\big(\sqrt{1-\delta^2}\,\widetilde{Y}(t_1)+\delta\xi\big)\big(\sqrt{1-\delta^2}\,\widetilde{Y}(t_2)+\delta\xi\big)\big)\\ &= (1-\delta^2)\,\mathbb{E}(Y(t_1)Y(t_2))+\delta^2 \geqslant \mathbb{E}(Y(t_1)Y(t_2)),\end{aligned}$$

  where the last inequality follows from $\mathbb{E}(Y(t_1)Y(t_2)) \leqslant \sqrt{\mathbb{E}\,Y(t_1)^2\,\mathbb{E}\,Y(t_2)^2} = 1$,

- for t_1 and t_2 that belong to different intervals L_i and L_j, by the definition of δ and by the intervals' construction we have

$$\begin{aligned}\mathbb{E}(Z(t_1)Z(t_2)) &= \mathbb{E}\big(\big(\sqrt{1-\delta^2}\,\widetilde{Y}(t_1)+\delta\xi\big)\big(\sqrt{1-\delta^2}\,\widetilde{Y}(t_2)+\delta\xi\big)\big)\\ &= \delta^2 \geqslant \mathbb{E}(Y(t_1)Y(t_2)).\end{aligned}$$

Let $\widetilde{X}_i^\varepsilon := \max_{t\in L_i} \widetilde{Y}(t)$. By applying the Slepian inequality (3.16) to the processes Y and Z, we obtain

$$\begin{aligned}\mathbb{P}\{\mathcal{E}_1\} &= \mathbb{P}\Big\{\bigcup_{i\in I_1}\{X_i^\varepsilon \geqslant f((\ell+s)i)\varepsilon\}\Big\}\\ &\geqslant \mathbb{P}\Big\{\bigcup_{i\in I_1}\{\sqrt{1-\delta^2}\,\widetilde{X}_i^\varepsilon+\delta\xi \geqslant f((\ell+s)i)\varepsilon\}\Big\}.\end{aligned}$$

Let us pass to the complementary events. For every $h = h(\varepsilon) > 0$ the following elementary bound holds:

$$\begin{aligned}1-\mathbb{P}\{\mathcal{E}_1\} &= \mathbb{P}\Big\{\bigcap_{i\in I_1}\{X_i^\varepsilon \leqslant f((\ell+s)i)\varepsilon\}\Big\}\\ &\leqslant \mathbb{P}\Big\{\bigcap_{i\in I_1}\{\sqrt{1-\delta^2}\,\widetilde{X}_i^\varepsilon+\delta\xi \leqslant f((\ell+s)i)\varepsilon\}\Big\}\\ &\leqslant \mathbb{P}\Big\{\bigcap_{i\in I_1}\{\sqrt{1-\delta^2}\,\widetilde{X}_i^\varepsilon \leqslant f((\ell+s)i)\varepsilon+h\varepsilon\}\Big\}+\mathbb{P}\{\delta\xi\leqslant -h\varepsilon\}\\ &= \prod_{i\in I_1}\mathbb{P}\Big\{X_i^\varepsilon \leqslant \frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\Big\}+\mathbb{P}\{\xi\leqslant -h\varepsilon/\delta\},\end{aligned}\tag{3.19}$$

where the last equality holds because the $\widetilde{X}_i^\varepsilon$'s are independent copies of X_i^ε. We choose the level $h = h(\delta,\varepsilon)$ so that, as $\varepsilon\to 0$,

$$h\varepsilon/\delta \to \infty, \tag{3.20}$$

$$hf(\tau)\varepsilon^2 \sim h\gamma\varepsilon \to 0, \tag{3.21}$$

which is possible under (3.18).

Under (3.20) the last term of (3.19) is negligible. Therefore, we aim to show that the product converges to $\exp(-C_\alpha \exp(-r))$ as $\varepsilon \to 0$. Taking the logarithm and passing to the complementary events, we see that we have to prove that

$$\sum_{i\in I_1} \ln\left(1-\mathbb{P}\left\{X_i^\varepsilon \geqslant \frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right\}\right) \to -C_\alpha \exp(-r).$$

The probabilities in the sum tend to 0 uniformly over $i \in I_1$, since the X_i^ε's are identically distributed. Hence

$$\sum_{i\in I_1} \ln\left(1-\mathbb{P}\left\{X_i^\varepsilon \geqslant \frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right\}\right)$$
$$= -(1+o(1)) \sum_{i\in I_1} \mathbb{P}\left\{X_i^\varepsilon \geqslant \frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right\}.$$

Let us prove that

$$\sum_{i\in I_1} \mathbb{P}\left\{X_i^\varepsilon \geqslant \frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right\} \to C_\alpha \exp(-r).$$

By Lemma 3.1 we know the exact asymptotics of each term of the sum. Moreover, due to stationarity of the sequence X_i^ε the equivalence is uniform over $i \in I_1$. Therefore the whole sum is equivalent to

$$C_\alpha \ell \sum_{i\in I_1} \left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^{2/\alpha-1} \exp\left(-\left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^2/2\right).$$

We represent this expression as the difference of two sums

(3.22)
$$C_\alpha \ell \sum_{i:(\ell+s)i+\ell\geqslant\tau} \left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^{2/\alpha-1} \exp\left(-\left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^2/2\right)$$

and

(3.23)
$$C_\alpha \ell \sum_{i:(\ell+s)i\geqslant\sigma} \left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^{2/\alpha-1} \exp\left(-\left(\frac{f((\ell+s)i)\varepsilon+h\varepsilon}{\sqrt{1-\delta^2}}\right)^2/2\right).$$

The asymptotics of the first sum follows from Lemma 3.3 applied with the variable change $\tilde\varepsilon = \varepsilon/\sqrt{1-\delta^2} \sim \varepsilon$ and parameters $a = \ell + s \sim \ell$, $b = 0$, $c = h$ and $\theta = \tau - \ell \sim \tau$. Note that because of our choice (3.21), c fits the assumptions of the lemma. Due to (3.9), (3.3) and (3.18),

$$(f(\theta)\tilde\varepsilon)^2 = (f(\tau-\ell)\tilde\varepsilon)^2 = (f(\tau)\tilde\varepsilon)^2 + o(1) = (f(\tau)\varepsilon)^2/(1-\delta^2) + o(1)$$
$$= (f(\tau)\varepsilon)^2 + O((f(\tau)\varepsilon\delta)^2) + o(1) = (f(\tau)\varepsilon)^2 + o(1).$$

Combining this with condition (3.10), Lemma 3.3 gives us the following asymptotics for (3.22):

$$C_\alpha \ell \frac{e^{\gamma^2/2}}{a}(f(\theta)\tilde{\varepsilon})^{2/\alpha+\beta-2}\exp\{-(f(\theta)\tilde{\varepsilon})^2/2\} \\ \sim C_\alpha \ell \frac{e^{\gamma^2/2}}{\ell}(f(\tau)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\tau)\varepsilon)^2/2\} \sim ce^{-r}.$$

At the same time, Lemma 3.3 with the same $\tilde{\varepsilon}$ and with $a = \ell + s \sim \ell$, $b = 0$, $c = h$ and $\theta = \sigma$ provides the asymptotics of (3.23). Since $(f(\theta)\tilde{\varepsilon})^2 = (f(\sigma)\varepsilon)^2 + o(1)$ and (3.11), we obtain

$$C_\alpha \ell \frac{e^{\gamma^2/2}}{a}(f(\theta)\tilde{\varepsilon})^{2/\alpha+\beta-2}\exp\{-f(\theta\tilde{\varepsilon})^2/2\} \\ \sim C_\alpha \ell \frac{e^{\gamma^2/2}}{\ell}(f(\sigma)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\sigma)\varepsilon)^2/2\} = o(1).$$

Subtraction of the sums' asymptotics implies the required upper bound for $1 - \mathbb{P}\{\mathcal{E}_1\}$.

Lower bound. In order to obtain an opposite bound for $1 - \mathbb{P}\{\mathcal{E}_1\}$, we introduce and compare two more auxiliary processes Y_1, $\widetilde{Y}_1$. Let ξ be an auxiliary standard normal random variable independent of the process Y. Let $Y_1(t) := Y(t) + \delta\xi$, $t \in \bigcup_{i\in I_1} L_i$. Furthermore, let us consider a sequence of independent standard Gaussian random variables ξ_i independent of $\widetilde{Y}(t)$, and let

$$\widetilde{Y}_1(t) := \widetilde{Y}(t) + \delta\xi_i, \quad t \in L_i.$$

Then for all t we have the equality of variances: $\mathbb{E}\, Y_1(t)^2 = \mathbb{E}\, \widetilde{Y}_1(t) = 1 + \delta^2$. For covariances we have the following inequalities:

- for t_1 and t_2 that belong to the same interval L_i we have

$$\mathbb{E}(Y_1(t_1)Y_1(t_2)) = \mathbb{E}(\widetilde{Y}_1(t_1)\widetilde{Y}_1(t_2)),$$

- for t_1 and t_2 that belong to different intervals L_i and L_j we have

$$\mathbb{E}(Y_1(t_1)Y_1(t_2)) = \mathbb{E}((Y(t_1) + \delta\xi)(Y(t_2) + \delta\xi)) = \mathbb{E}(Y(t_1)Y(t_2)) + \delta^2 \\ \geqslant 0 = \mathbb{E}(\widetilde{Y}_1(t_1)\widetilde{Y}_1(t_2)).$$

We choose $h = h(\delta, \varepsilon)$ as before, i.e. satisfying (3.20) and (3.21).

The Slepian inequality (3.16) yields

$$\begin{aligned}\mathbb{P}\Big\{\bigcup_{i\in I_1}\{\widetilde{X}_i^\varepsilon+\delta\xi_i\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}\\ &=\mathbb{P}\Big\{\bigcup_{i\in I_1}\{\max_{t\in L_i}\widetilde{Y}_1(t)\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}\\ &\geqslant\mathbb{P}\Big\{\bigcup_{i\in I_1}\{\max_{t\in L_i}Y_1(t)\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}\\ &=\mathbb{P}\Big\{\bigcup_{i\in I_1}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}.\end{aligned}$$

By passing to the complementary events, we obtain

$$\begin{aligned}\mathbb{P}\Big\{\bigcap_{i\in I_1}\{X_i^\varepsilon+\delta\xi\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}\\ &\geqslant\mathbb{P}\Big\{\bigcap_{i\in I_1}\{\widetilde{X}_i^\varepsilon+\delta\xi_i\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}\\ &=\prod_{i\in I_1}\mathbb{P}\{\widetilde{X}_i^\varepsilon+\delta\xi_i\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\\ &=\prod_{i\in I_1}\mathbb{P}\{X_i^\varepsilon+\delta\xi\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}.\end{aligned}$$

Further, we apply an elementary bound

$$\begin{aligned}1-\mathbb{P}\{\mathcal{E}_1\}&=\mathbb{P}\Big\{\bigcap_{i\in I_1}\{X_i^\varepsilon\leqslant f((\ell+s)i)\varepsilon\}\Big\}\\ &\geqslant\mathbb{P}\Big\{\bigcap_{i\in I_1}\{X_i^\varepsilon+\delta\xi\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\Big\}-\mathbb{P}\{\delta\xi\leqslant-h\varepsilon\}\\ &\geqslant\prod_{i\in I_1}\mathbb{P}\{X_i^\varepsilon+\delta\xi\leqslant f((\ell+s)i)\varepsilon-h\varepsilon\}-\mathbb{P}\{\delta\xi\leqslant-h\varepsilon\}.\end{aligned}$$

Under assumption (3.20), we have $\mathbb{P}\{\delta\xi\leqslant-h\varepsilon\}\to 0$ as $\varepsilon\to 0$. It remains to prove that the product is greater than $\exp(-C_\alpha\exp(-r))(1+o(1))$. Taking the logarithm and passing to the complementary events, we see that we have to prove the bound

$$\sum_{i\in I_1}\ln(1-\mathbb{P}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\})\geqslant -c\exp(-r)(1+o(1)).$$

Since all the probabilities in the sum tend to zero uniformly over $i\in I_1$, we have

$$\begin{aligned}\sum_{i\in I_1}\ln(1-\mathbb{P}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\})\\ &=-(1+o(1))\sum_{i\in I_1}\mathbb{P}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}.\end{aligned}$$

Let us prove that

$$\sum_{i\in I_1}\mathbb{P}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\leqslant C_\alpha\exp(-r)(1+o(1)).$$

We start with the estimate

$$\begin{aligned}\sum_{i\in I_1}&\mathbb{P}\{X_i^\varepsilon+\delta\xi\geqslant f((\ell+s)i)\varepsilon-h\varepsilon\}\\&\leqslant\sum_{i\in I_1}\big(\mathbb{P}\{X_i^\varepsilon\geqslant f((\ell+s)i)\varepsilon-2h\varepsilon\}+\mathbb{P}\{\delta\xi>h\varepsilon\}\big)\\&\leqslant\sum_{i\colon(\ell+s)i+\ell\geqslant\tau}\mathbb{P}\{X_i^\varepsilon\geqslant f((\ell+s)i)\varepsilon-2h\varepsilon\}+N_1\mathbb{P}\{\delta\xi>h\varepsilon\},\end{aligned}\tag{3.24}$$

where N_1 denotes the number of elements in the set I_1; it has asymptotics

$$N_1\sim\frac{\sigma-\tau}{\ell+s}=\frac{(R-r)B_\varepsilon}{\ell+s}\sim\frac{R}{f'(\tau)\varepsilon\gamma\ell}\sim\frac{R}{f(\tau)f'(\tau)\varepsilon^2\ell}.$$

For the sum in (3.24) Lemma 3.1 provides an equivalent expression

$$C_\alpha\ell\sum_{i\in I_1}(f((\ell+s)i)\varepsilon-2h\varepsilon)^{2/\alpha-1}\exp\big(-(f((\ell+s)i)\varepsilon-2h\varepsilon)^2/2\big).\tag{3.25}$$

The asymptotics for the last sum follows from Lemma 3.3 with parameters $a=\ell+s$, $b=0$, $c=-2h$, $\theta=\tau-\ell$, moreover, similarly to the upper bound, we have $a\sim\ell$, $\theta\sim\tau$, $(f(\theta)\varepsilon)^2=(f(\tau)\varepsilon)^2+o(1)$. Therefore Lemma 3.3 combined with (3.10) yields

$$\begin{aligned}&\frac{C_\alpha\ell e^{\gamma^2/2}}{a}(f(\theta)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\theta)\varepsilon)^2/2\}\\&\qquad\sim C_\alpha e^{\gamma^2/2}(f(\tau)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\tau)\varepsilon)^2/2\}\sim C_\alpha e^{-r}.\end{aligned}$$

We have obtained the asymptotics of the first term on the right hand side in (3.24):

$$\sum_{i:(\ell+s)i+\ell\geqslant\tau}\mathbb{P}\{X_i^\varepsilon\geqslant f((\ell+s)i)\varepsilon-2h\varepsilon\}=C_\alpha e^{-r}(1+o(1)).$$

It remains to estimate the second term in (3.24). To this end, we have to specify the choice of ℓ and R.

So far we have assumed a lower bound for ℓ in (3.2) and an upper bound in (3.3). We claimed that they are compatible (see Section 3.1 for details). Now we need another lower bound on ℓ, namely

$$\frac{\mathbb{P}(\delta\xi>h\varepsilon)}{\ell f(\tau)f'(\tau)\varepsilon^2}\to0.$$

This does not contradict the upper bound in (3.3), since $\mathbb{P}(\delta\xi > h\varepsilon) \to 0$, so we may choose $\ell(\varepsilon)$ satisfying this new condition too. Then $R = R(\varepsilon)$ can be chosen growing so slowly that

$$N_1 \mathbb{P}\{\delta\xi > h\varepsilon\} \sim \frac{R\mathbb{P}(\delta\xi > h\varepsilon)}{\ell f(\tau) f'(\tau)\varepsilon^2} \to 0.$$

By summing up the estimates for the terms of (3.24), we arrive at the required lower estimate for $1 - \mathbb{P}\{\mathcal{E}_1\}$.

3.1. Proofs of technical lemmas

3.1.1. *Proof of Lemma 3.2.* Define

$$\Delta = \frac{1}{f'(\tau_0)\varepsilon}\left(\left(\frac{2}{\alpha} + \beta - 2\right)\frac{\ln\gamma}{\gamma} + \frac{\tilde{R}}{\gamma}\right).$$

Then, due to (2.4) it is sufficient to show that

$$f\left(\tau_0 + \Delta + o\left(\frac{1}{f'(\tau_0)\varepsilon\gamma}\right)\right) = f(\tau_0) + \Delta f'(\tau_0) + o\left(\frac{1}{\gamma\varepsilon}\right).$$

Notice that

$$\Delta\tau_0^{-\lambda} \sim \frac{\ln\gamma}{f'(\tau_0)\gamma\varepsilon\tau_0^{\lambda}} \prec \frac{\ln\gamma}{f(\tau_0)\gamma\varepsilon} \sim \frac{\ln\gamma}{\gamma^2}.$$

In particular, $\Delta = o(\tau_0)$. Moreover, uniformly over $\omega \in [-2, 2]$ one has

$$f(\tau_0 + \omega\Delta) \sim f(\tau_0) \quad \text{and} \quad f'(\tau_0 + \omega\Delta) \sim f'(\tau_0),$$

since for $\lambda < 1$,

$$\begin{aligned}\ln\left(\frac{f(\tau_0 + \omega\Delta)}{f(\tau_0)}\right) &= \int_{\tau_0}^{\tau_0+\omega\Delta} (\ln f)'(x)\,dx \asymp \int_{\tau_0}^{\tau_0+\omega\Delta} x^{-\lambda}\,dx \\ &\asymp (\tau_0 + \omega\Delta)^{1-\lambda} - \tau_0^{1-\lambda} \asymp \omega\Delta\tau_0^{-\lambda} = o(1),\end{aligned}$$

and for $\lambda = 1$,

$$\ln\left(\frac{f(\tau_0 + \omega\Delta)}{f(\tau_0)}\right) \asymp \int_{\tau_0}^{\tau_0+\omega\Delta} x^{-1}\,dx = \ln(1 + \omega\Delta\tau_0^{-1}) = o(1).$$

A similar reasoning works for f'. This already shows the second part of the statement, since $\sigma - \tau_0$ can be represented as Δ for a right choice of $\tilde{R}$.

Uniformly over $\omega \in [0, 2]$,

$$f''(\tau_0 + \omega\Delta) \asymp f'(\tau_0 + \omega\Delta)(\tau_0 + \omega\Delta)^{-\lambda} \sim f'(\tau_0)\tau_0^{-\lambda} \asymp f''(\tau_0).$$

By the mean value theorem for some $\omega \in [0,2]$ we have

$$f\left(\tau_0 + \Delta + o\left(\frac{1}{f'(\tau_0)\varepsilon\gamma}\right)\right) - f(\tau_0) = f'(\tau_0 + \omega\Delta)\left(\Delta + o\left(\frac{1}{f'(\tau_0)\varepsilon\gamma}\right)\right),$$

since the increase in the argument is smaller than 2Δ as ε goes to 0. Notice that

$$f'(\tau_0 + \omega\Delta)\cdot o\left(\frac{1}{f'(\tau_0)\varepsilon\gamma}\right) = o\left(\frac{1}{\gamma\varepsilon}\right).$$

It remains to show that

$$(f'(\tau_0 + \omega\Delta) - f'(\tau_0))\Delta = o\left(\frac{1}{\gamma\varepsilon}\right),$$

or equivalently

$$\frac{(f'(\tau_0 + \omega\Delta) - f'(\tau_0))\ln\gamma}{f'(\tau_0)\varepsilon\gamma} = o\left(\frac{1}{\gamma\varepsilon}\right).$$

By the mean value theorem, for some $\tilde{\omega} \in [0,\omega]$,

$$f'(\tau_0 + \omega\Delta) - f'(\tau_0) = \omega\Delta f''(\tau_0 + \tilde{\omega}\Delta) \asymp \omega\Delta f''(\tau_0) \asymp \omega\Delta f'(\tau_0)\tau_0^{-\lambda}.$$

We complete the proof by noticing that $\Delta\tau_0^{-\lambda}\ln\gamma = o(1)$. ∎

3.1.2. *Choice of $\ell(\varepsilon)$ and $s(\varepsilon)$.* Recall that we are to choose ℓ and s such that (3.1)–(3.3) hold true, i.e.

$$\ln s \succ \gamma^2, \quad s/\ell \to 0, \quad f(\tau)f'(\tau)\varepsilon^2\ell \to 0.$$

We recall that due to (3.13) and (3.14),

$$f(\tau)f'(\tau)\varepsilon^2 \asymp \gamma^2\tau^{-\lambda} \asymp \gamma^{1-\beta}e^{-\gamma^2/2}.$$

Therefore we can pick $s = e^{\gamma^2/4}$ and any $\ell \succ s$ such that $\gamma^{1-\beta}e^{-\gamma^2/2}\ell \to 0$ as $\varepsilon \to 0$. Note that $\ell = e^{\gamma^2/3}$ clearly fits these conditions, thus the set of choices for ℓ is not empty.

3.1.3. *Proof of Lemma 3.3.* The monotone decay of $x \mapsto x^{2/\alpha-1}\exp\{-x^2/2\}$ for large x combined with $(f(\theta-a)+c)\varepsilon \to \infty$ allows us to squeeze the sum in the statement between two integrals:

$$\frac{1}{a}\int_{\theta\pm a}^{\infty} (f(x)\varepsilon + c\varepsilon)^{2/\alpha-1}\exp\{-(f(x)\varepsilon + c\varepsilon)^2/2\}\,dx.$$

By changing variables $y = (f(x)+c)\varepsilon$ we get $x = f^{-1}(y/\varepsilon - c)$. Then $dx = \frac{1}{\varepsilon}(f^{-1})'(y/\varepsilon - c)dy$. The integrals transform into

$$\frac{1}{a\varepsilon}\int_{(f(\theta\pm a)+c)\varepsilon}^{\infty} y^{2/\alpha-1}\exp\{-y^2/2\}(f^{-1})'(y/\varepsilon - c)\,dy.$$

To deal with these integrals we estimate $(f^{-1})'(y/\varepsilon - c)$ for large y and for small y separately via the regularity conditions (2.5) and (2.6) and the relation

$$(f(\theta \pm a) + c)\varepsilon \sim \gamma.$$

Since $f(\theta \pm a)\varepsilon \sim \gamma$ as ε goes to 0,

$$[f(\theta \pm a)\varepsilon, (1 + \kappa/2) f(\theta \pm a)\varepsilon] \subset [(1 - \kappa)\gamma, (1 + \kappa)\gamma].$$

For convenience denote $L_\pm = (f(\theta \pm a) + c)\varepsilon$. Then (2.5) gives

$$\begin{aligned}
\frac{1}{a\varepsilon} \int\limits_{L_\pm}^{(1+\kappa/2)L_\pm} & y^{2/\alpha-1} \exp\{-y^2/2\}(f^{-1})'(y/\varepsilon - c)\, dy \\
&\sim \frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} \int\limits_{L_\pm}^{(1+\kappa/2)L_\pm} y^{2/\alpha-1} \exp\{-y^2/2\}(y - c\varepsilon)^\beta\, dy \\
&\sim \frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} \int\limits_{L_\pm}^{(1+\kappa/2)L_\pm} y^{2/\alpha+\beta-1} \exp\{-y^2/2\}\, dy \\
&\sim \frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} (L_\pm)^{2/\alpha+\beta-2} \exp\{-L_\pm^2/2\} \\
&\sim \frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} (f(\theta)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\theta)\varepsilon)^2/2\}.
\end{aligned}$$

The last equivalence follows from the Lagrange theorem and our conditions, since for some $\omega \in [-1, 1]$ one has

$$\begin{aligned}
L_\pm^2 &= (f(\theta \pm a) + c)^2\varepsilon^2 \\
&= (f(\theta) + f'(\theta + \omega a)a + c)^2\varepsilon^2 \\
&= f(\theta)^2\varepsilon^2 + O\big(f(\theta)f'(\theta + \omega a)a\varepsilon^2 + f(\theta)c\varepsilon^2\big) \\
&= f(\theta)^2\varepsilon^2 + o(1).
\end{aligned}$$

Moreover, the second regularity condition (2.6) yields

$$(f^{-1})'\left(\frac{y - c\varepsilon}{\varepsilon}\right) < (y - c\varepsilon)^{\tilde{\beta}} (f^{-1})'\left(\frac{1}{\varepsilon}\right),$$

implying

$$\begin{aligned}
\frac{1}{a\varepsilon} \int\limits_{(1+\kappa/2)L_\pm}^{\infty} & y^{2/\alpha-1} \exp\{-y^2/2\}(f^{-1})'(y/\varepsilon - c) dy \\
&\prec \frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} ((1 + \kappa/2)L_\pm)^{2/\alpha+\tilde{\beta}-2} \exp\{-((1 + \kappa/2)L_\pm)^2/2\} \\
&= o\left(\frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon} (f(\theta)\varepsilon)^{2/\alpha+\beta-2} \exp\{-(f(\theta)\varepsilon)^2/2\}\right).
\end{aligned}$$

Finally, the integral is asymptotically equal to

$$\frac{(f^{-1})'(1/\varepsilon)}{a\varepsilon}(f(\theta)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\theta)\varepsilon)^2/2\},$$

which is equivalent to

$$\frac{e^{\gamma^2/2}}{a}(f(\theta)\varepsilon)^{2/\alpha+\beta-2}\exp\{-(f(\theta)\varepsilon)^2/2\}. \blacksquare$$

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