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# CONTINUOUS-STATE BRANCHING PROCESSES WITH SPECTRALLY POSITIVE MIGRATION*

BY

MATIJA VIDMAR (LJUBLJANA)

**Abstract.** Continuous-state branching processes (CSBPs) with immigration (CBIs), stopped on hitting zero, are generalized by allowing the process governing immigration to be any Lévy process without negative jumps. Unlike CBIs, these newly introduced processes do not appear to satisfy any natural affine property on the level of the Laplace transforms of the semigroups. Basic properties of these processes are described. Explicit formulae (on neighborhoods of infinity) for the Laplace transforms of the first passage times downwards and of the explosion time are derived.

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## 1. INTRODUCTION

**1.1. Motivation and agenda.** CSBPs (resp. CBIs) are the continuous analogues and scaling limits of the basic, but fundamental Bienaymé–Galton–Watson branching processes (resp. with independent constant-rate immigration). In [21] there was added to the latter (so in discrete space) the phenomenon of "culling", that is, emigration (killing) of individuals at constant rate, but never more than one at a time. One spoke of continuous-time Bienaymé–Galton–Watson processes with immigration and culling. It was noted [21, Remark 2.1] that these in turn should also allow for a continuous-space version.

In this article we construct said continuous-space analogues, and christen them continuous-state branching processes with spectrally positive migration (CBMs). Unlike the discrete-space case, it is no longer possible (in general) to separate immigration and culling, so it is no longer appropriate to speak about them separately.

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We derive the basic properties of the new processes and we study their first-passage times downwards and explosion times (on the level of Laplace transforms).

The main results are as follows: Theorem 2.1 (construction and basic properties); Proposition 2.1 (Lamperti-style representation); Theorem 3.1 (first passage times) and Theorem 3.2 (explosions).

**1.2. General notation.** For a probability measure $\mathbb{P}$ and a random element $Y$,

(i) $Y_\star\mathbb{P}$ is the law of $Y$ under $\mathbb{P}$, i.e. the probability measure $(A \mapsto \mathbb{P}(Y \in A))$, its domain being understood from context;

(ii) when $Y$ is numerical, $\mathbb{P}[Y] := \int Y \mathrm{d}\mathbb{P}$ is its expectation under $\mathbb{P}$;

(iii) when further $\mathcal{G}$ is a sub-$\sigma$-field, $\mathbb{P}[Y|\mathcal{G}] := \mathbb{E}_\mathbb{P}[Y|\mathcal{G}]$ is the conditional expectation of $Y$ given $\mathcal{G}$ under $\mathbb{P}$.

The symbols $\uparrow$ and $\downarrow$ mean nondecreasing and nonincreasing respectively.

## 2. CONSTRUCTION OF CBMS AND FIRST PROPERTIES

We are given two Laplace exponents of Lévy processes having no negative jumps:

$$\Psi_b(x) := \frac{\sigma_b^2}{2}x^2 - \gamma_b x + \int \left(e^{-xh} - 1 + xh\mathbb{1}_{(0,1]}(h)\right) \pi_b(\mathrm{d}h), \qquad x \in [0,\infty),$$

where $\pi_b$ is a measure on $(0,\infty)$ satisfying $\int (1 \wedge h^2)\, \pi_b(\mathrm{d}h) < \infty$, $\sigma_b \in [0,\infty)$, and $\gamma_b \in \mathbb{R}$; and

$$\Psi_m(x) := \frac{\sigma_m^2}{2}x^2 - \gamma_m x + \int \left(e^{-xh} - 1 + xh\mathbb{1}_{(0,1]}(h)\right) \pi_m(\mathrm{d}h), \qquad x \in [0,\infty),$$

with the analogous qualifications on $(\pi_m, \sigma_m, \gamma_m)$. The subscripts $b$ and $m$ stand for branching and migration, respectively. The corresponding generators are given by

$$\mathcal{L}^{\Psi_b} f(z) := \frac{\sigma_b^2}{2} f''(z) + \gamma_b f'(z) + \int_0^\infty \left(f(z+h) - f(z) - hf'(z)\mathbb{1}_{(0,1]}(h)\right) \pi_b(\mathrm{d}h),$$

$$\mathcal{L}^{\Psi_m} f(z) := \frac{\sigma_m^2}{2} f''(z) + \gamma_m f'(z) + \int_0^\infty \left(f(z+h) - f(z) - hf'(z)\mathbb{1}_{(0,1]}(h)\right) \pi_m(\mathrm{d}h),$$

for $f \in C_0^2(\mathbb{R})$ (i.e. for twice continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ with $f, f', f''$ all vanishing at infinity) and $z \in \mathbb{R}$; more generally, $\mathcal{L}^{\Psi_b} f(z)$ and $\mathcal{L}^{\Psi_m} f(z)$ are defined by the right-hand sides above whenever the relevant expressions are defined. Let also $\mu$ be a Borel probability measure on $[0,\infty)$, to be thought of as the initial distribution of the CBM.

On a filtered probability space $(\Omega, \mathcal{H}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ satisfying the usual assumptions we consider the following independent processes: an $\mathcal{F}$-Poisson random measure $\mathcal{M}_b(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h)$ on $[0,\infty)^3$ with intensity $\mathrm{d}s\,\mathrm{d}v\,\pi_b(\mathrm{d}h)$ (for Poisson random measures, time will always be the first coordinate; also, when speaking of adaptedness to $\mathcal{F}$, independent increments relative to $\mathcal{F}$, etc., the measures are canonically identified with the associated point processes); two independent standard $\mathcal{F}$-Brownian motions $W$ and $B$; and an $\mathcal{F}$-Poisson random measure $\mathcal{N}_m(\mathrm{d}s, \mathrm{d}h)$ on $[0,\infty)^2$ with intensity $\mathrm{d}s\,\pi_m(\mathrm{d}h)$. Let $\bar{\mathcal{N}}_m$ and $\bar{\mathcal{M}}_b$ denote the compensated versions of $\mathcal{N}_m$ and $\mathcal{M}_b$, respectively. There is also an $\mathcal{F}_0$-measurable random variable $X_0$ satisfying $X_{0\star}\mathbb{P} = \mu$. Clearly, constellations as just described exist for any given triplet $(\mu, \pi_b, \pi_m)$. Note also that automatically $(\mathcal{M}_b, W, B, \mathcal{N}_m)$ has in fact $\mathcal{F}$-independent increments (jointly, and not just each component separately) [8, Theorem II.6.3, (II.6.12)].

We combine $X_0$, $\mathcal{N}_m$ and $B$ according to Lévy–Itô into a Lévy process $X$ having no negative jumps,

$$X_t = X_0 + \sigma_m B_t + \gamma_m t + \int_{(0,t]\times[0,1]} h\,\bar{\mathcal{N}}_m(\mathrm{d}s, \mathrm{d}h) + \int_{(0,t]\times(1,\infty)} h\,\mathcal{N}_m(\mathrm{d}s, \mathrm{d}h), \quad t \in [0,\infty), \text{ a.s.-}\mathbb{P},$$

in the filtration $\mathcal{F}$, with Laplace exponent $\Psi_b$ and starting law $\mu$. For $f$ and $z$ for which the right-hand side is defined we also set

$$\mathcal{A}f(z) := \mathcal{L}^{\Psi_m} f(z) + z\mathcal{L}^{\Psi_b} f(z). \tag{2.1}$$

Now the stage is set for the construction of CBMs.

THEOREM 2.1 (SDE construction of CBMs). *There is a $\mathbb{P}$-a.s. unique càdlàg, nonnegative real, $\mathcal{F}$-adapted process with lifetime $\zeta$, which we denote $Y = (Y_t)_{t\in[0,\zeta)}$, having $0$ as an absorbing state, no negative jumps, and such that with $\tau_0$ the first entrance time into $\{0\}$ by $Y$, a.s.-$\mathbb{P}$,*

$$Y_t = X_{t\wedge\tau_0} + \int_{(0,t]\times[0,Y_{s-}]\times[0,1]} h\,\bar{\mathcal{M}}_b(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h) + \int_{(0,t]\times[0,Y_{s-}]\times(1,\infty)} h\,\mathcal{M}_b(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h) + \sigma_b \int_0^t \sqrt{Y_s}\,\mathrm{d}W_s + \gamma_b \int_0^t Y_s\,\mathrm{d}s \tag{2.2}$$

*for $t \in [0,\zeta)$, and $\sup_{[0,\zeta)} Y = \infty$ a.s.-$\mathbb{P}$ on $\{\zeta < \infty\}$ (implicitly, necessarily $\zeta > 0$ a.s.-$\mathbb{P}$). The process $Y$ further enjoys the following properties:*

(i) *it is adapted to the $\mathbb{P}$-augmented natural filtration generated by $X_0$, $B$, $W$, $\mathcal{N}_m$, $\mathcal{M}_b$ (we mean of course $X_0$ as a constant process here; $Y$ is viewed as* *a process on* $[0,\infty)$ *by transferring it to the cemetery* (*natural:* $\infty$) *after* $\zeta$) *and its* $\mathbb{P}$*-law,* $\mathbb{P}_\mu := Y_\star\mathbb{P}$ (*completion implied*)*, is uniquely determined by the triplet* $(\mu, \Psi_b, \Psi_m)$*;*

(ii) *it is quasi-left-continuous on* $[0,\zeta)$ *in the filtration* $\mathcal{F}$*, in the sense that for any sequence* $(S_n)_{n\in\mathbb{N}}$ *of* $\mathcal{F}$*-stopping times with* $S_n \uparrow S$ *a.s.-*$\mathbb{P}$ *for some* $S$ *one has* $\lim_{n\to\infty} Y(S_n) = Y(S)$ *a.s.-*$\mathbb{P}$ *on* $\{S<\zeta\}$*;*

(iii) *it is strong Markov on* $[0,\zeta)$ *in the filtration* $\mathcal{F}$*, in the sense that for any* $\mathcal{F}$*-stopping time* $S$*,* $\mathcal{F}_S$ *is* $\mathbb{P}$*-independent of* $Y_{S+\cdot}$ *given* $Y_S$ *on* $\{S<\zeta\}$*, furthermore, for any nonnegative measurable map* $G$*,* $\mathbb{P}[G(Y_{S+\cdot}) \mid S<\zeta] = \mathbb{P}_\nu[G]$*, where* $\nu$ *is the* $\mathbb{P}$*-law of* $Y_S$ *conditionally on* $\{S<\zeta\}$ (*assuming of course* $\mathbb{P}(S<\zeta)>0$)*;*

(iv) $\lim_{\zeta-} Y = \infty$ *a.s.-*$\mathbb{P}$ *on* $\{0<\zeta<\infty\}$*;*

(v) (I) *for any* $\{\alpha,\bar\alpha\}\subset[0,\infty)$ *and any* $f\in C_c^2([0,\infty))$ (*i.e. compactly supported and admitting a* $C^2$ *extension to a neighborhood of* $[0,\infty)$) *satisfying* $\mathcal{L}^{\Psi_m}f(0)=0$*, the process* $M$ *given by, for* $t\in[0,\infty)$*,*

$$M_t := f(Y_t)e^{-\alpha t-\bar\alpha\int_0^t Y_s\,\mathrm{d}s}\mathbb{1}_{\{t<\zeta\}} - f(Y_0) - \int_0^{t\wedge\zeta} e^{-\alpha s-\bar\alpha\int_0^s Y_u\,\mathrm{d}u}\big(\mathcal{A}f(Y_s)-\alpha f(Y_s)-\bar\alpha Y_s f(Y_s)\big)\,\mathrm{d}s$$

*is an* $\mathcal{F}$*-martingale under* $\mathbb{P}$*, vanishing at zero, and bounded up to every finite deterministic time, so that in particular* $\mathcal{A}$ *is the generator of* $Y$ *on the set* $\{g\in C_c^2([0,\infty)) : \mathcal{L}^{\Psi_m}g(0)=0\}$*;*

(II) *if* $f\in C^2([0,\infty))$ *is merely bounded and* $\mathcal{L}^{\Psi_m}f(0)=0$*, then the same process* $M$ (*restricted to* $[0,\zeta)$) *is a local martingale on* $[0,\zeta)$ *in* $\mathcal{F}$ *under* $\mathbb{P}$*, in the sense that there exists a sequence* $(S_n)_{n\in\mathbb{N}}$ *of* $\mathcal{F}$*-stopping times that is* $\uparrow\zeta$*, and such that for each* $n\in\mathbb{N}$*,* $S_n<\zeta$ *a.s.-*$\mathbb{P}$ *on* $\{\zeta<\infty\}$*, and* $M^{S_n} = M_{S_n\wedge\cdot}$ *is an* $(\mathcal{F},\mathbb{P})$*-martingale.*

When $X$ is a subordinator (resp. the zero process), we recognize in (2.2) the stochastic equation solved by a CBI process *stopped on hitting zero* with branching mechanism $\Psi_b$ and immigration mechanism $-\Psi_m$ (resp. a CSBP with branching mechanism $\Psi_b$). We thus add here a further dynamics in that we allow $X$ to be any Lévy process without negative jumps. Oscillations or negative drift represent the possibility of culling, so that immigration is counterbalanced by killing an individual independently of the population size at a constant rate in time. We stress that because of the presence of culling it is only natural, *in general*, to stop the process $Y$ on hitting zero (unlike when $X$ is a (nonzero) subordinator), at least as long as we insist on the state space being $[0,\infty)$ (this is clearly visible if one considers the special case when $\gamma_b$, $\gamma_m$ are both $<0$, $\pi_c$ and $\pi_b$ are nonzero and carried by $(1,\infty)$, $\sigma_b = \sigma_m = 0$). This, however, does not a priori preclude the possibility that for other, but necessarily non-generic, constellations of input data the $Y$ of (2.2) could not be naturally prolongated after $\tau_0$ as a nonnegative process (as happens for CBIs).

We may also mention here that squared Bessel processes of negative dimension [7, Definition 3] (stopped on hitting zero) are instances of CBMs. The SDE technique is not uncommon in the branching literature; see e.g. [17, 4, 6] and the references therein.

*Proof of Theorem 2.1. Existence.* Fix, for the time being, $n \in \mathbb{N}$. We claim that thanks to [15, Theorems 3.2 and 5.1] there exists a pathwise unique strong (global, no lifetimes) solution $Y^n$ to the stochastic integral equation

$$\begin{aligned} Y_t^n &= Y_0^n + \int_0^t \big(\gamma_m + \gamma_b((Y_{s-}^n \vee 0) \wedge n)\big)\,\mathrm{d}s \\ &\quad + \int_0^t \sigma_b \sqrt{(Y_{s-}^n \vee 0) \wedge n}\,\mathrm{d}W_s + \int_0^t \sigma_m\,\mathrm{d}B_s \\ &\quad + \int_{(0,t]\times U_0} g_0(Y_{s-}^n, u_0)\,\overline{N}_0(\mathrm{d}s, \mathrm{d}u_0) \\ &\quad + \int_{(0,t]\times U_1} g_1(Y_{s-}^n, u_1)\,N_1(\mathrm{d}s, \mathrm{d}u_1), \qquad t \in [0,\infty), \\ Y_0^n &= X_0, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} U_0 &:= ([0,n]\times[0,1]) \cup [0,1], \\ U_1 &:= ([0,n]\times(1,\infty)) \cup (1,\infty), \\ N_0(\mathrm{d}s,\mathrm{d}h) &:= \mathcal{N}_m(\mathrm{d}s,\mathrm{d}h) \quad \text{for } h \in [0,1], \\ N_0(\mathrm{d}s,\mathrm{d}u_0) &:= \mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) \quad \text{for } u_0 = (v,h) \in [0,n]\times[0,1], \\ \overline{N}_0 &:= \text{ the compensated measure of } N_0, \\ N_1(\mathrm{d}s,\mathrm{d}h) &:= \mathcal{N}_m(\mathrm{d}s,\mathrm{d}h) \quad \text{for } h \in (1,\infty), \\ N_1(\mathrm{d}s,\mathrm{d}u_1) &:= \mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) \quad \text{for } u_1 = (v,h) \in [0,n]\times(1,\infty), \\ g_0(x,h) &:= h \quad \text{for } (x,h) \in \mathbb{R}\times[0,1], \\ g_0(x,u_0) &:= \mathbb{1}_{[v,\infty)}(x)h \quad \text{for } (x,u_0) = (x,(v,h)) \in \mathbb{R}\times([0,n]\times[0,1]), \\ g_1(x,h) &:= h \quad \text{for } (x,h) \in \mathbb{R}\times(1,\infty), \\ g_1(x,u_1) &:= \mathbb{1}_{[v,\infty)}(x)h \quad \text{for } (x,u_1) = (x,(v,h)) \in \mathbb{R}\times([0,n]\times(1,\infty)). \end{aligned}$$

Indeed, (2.3) is just (2.2) except that one allows the process $Y^n$ to evolve after it hits negative values (without branching, just following $X$) and that the branching is also "truncated" at level $n$ (to preclude explosion), vis-à-vis the process $Y$. In fact, we may write more succinctly

$$
\begin{aligned}
(2.4)\qquad Y_t^n &= X_t + \int\limits_{(0,t]\times[0,(Y_{s-}^n\vee 0)\wedge n]\times[0,1]} h\,\bar{\mathcal{M}}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) \\
&\quad + \int\limits_{(0,t]\times[0,(Y_{s-}^n\vee 0)\wedge n]\times(1,\infty)} h\,\mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) \\
&\quad + \sigma_b\int_0^t \sqrt{(Y_s^n\vee 0)\wedge n}\,\mathrm{d}W_s + \gamma_b\int_0^t (Y_s^n\vee 0)\wedge n\,\mathrm{d}s, \\
Y_0^n &= X_0.
\end{aligned}
$$

However, it is the form (2.3) given above, not (2.4), that relates directly to [15, (2.1)].

Now, strictly speaking the stochastic equation for $Y^n$ does not fall under [15] because of the following minor point: in [15, (2.1)] the Brownian motion is one-dimensional, while here it is two-variate. However, the generalization of the results of [15] to a setting allowing a multidimensional Brownian motion is straightforward, especially if one of the two Brownian motions is just integrated against a constant (which is the present case). Furthermore, as far as the question of the existence of a strong pathwise unique solution is concerned, the integral $\int_{[0,t]\times U_1}$ may be ignored [15, Proposition 2.1]. Once this has been noted it is easy to check that all the conditions of [15] required to establish the strong pathwise unique solution $Y^n$ of the above stochastic equation (trivialy adjusted to allow a two-variate Brownian motion) are in fact met.

Let next $\zeta^n$ be the first time the stopped process $(Y^n)^{\tau_0^n}$ exits the interval $[0,n]$; here $\tau_0^n$ is the first entrance time of $Y^n$ into $\{0\}$. Then $\zeta^{n+1}\geqslant\zeta^n$ and $Y^{n+1}=Y^n$ on $[0,\zeta^n)$ a.s.-$\mathbb{P}$. Set $\zeta:=\lim_{n\to\infty}\zeta^n$ and $Y:=\lim_{n\to\infty}Y^n$ on $[0,\zeta)$ a.s.-$\mathbb{P}$. We get all the properties stipulated for $Y$ in the "unique existence" part of the proposition. Thus existence is proved.

*Uniqueness.* Let $Y^I=(Y_t^I)_{t\in[0,\zeta^I)}$ and $Y^{II}=(Y_t^{II})_{t\in[0,\zeta^{II})}$ both have the properties listed for $Y$ in the "unique existence" part of the proposition. Let $\tau_n^i$ (resp. $\tau_0^i$) be the first exit time from $[0,n]$ (resp. first entrance time into $\{0\}$) of $Y^i$, $n\in\mathbb{N}$, $i\in\{I,II\}$. Then, for each $i\in\{I,II\}$, $n\in\mathbb{N}$, the stopped process $(Y^i)^{\tau_n^i\wedge\tau_0^i}$ satisfies, on $[0,\tau_n^i\wedge\tau_0^i]\cap[0,\zeta^i)$, the stochastic equation given above for $Y^n$. By the pathwise uniqueness of these solutions we obtain $Y^I=Y^{II}$ a.s.-$\mathbb{P}$ on $[0,\tau_n^I\wedge\tau_n^{II}\wedge\tau_0^I\wedge\tau_0^{II}]\cap[0,\zeta^I\wedge\zeta^{II})$. Since $\sup_{[0,\zeta^i)}Y^i=\infty$ a.s.-$\mathbb{P}$ on $\{\zeta^i<\infty\}$ and since $Y^i$ is càdlàg, therefore locally bounded on $[0,\zeta^i)$, we see that $\tau_n^i\uparrow\zeta^i$ a.s.-$\mathbb{P}$ as $n\to\infty$, $i\in\{I,II\}$. Therefore, letting $n\to\infty$, $Y^I=Y^{II}$ a.s.-$\mathbb{P}$ on $[0,\tau_0^I\wedge\tau_0^{II}]\cap[0,\zeta^I\wedge\zeta^{II})$. Because 0 is absorbing for $Y^i$, $i\in\{I,II\}$, it follows further that $Y^I=Y^{II}$ a.s.-$\mathbb{P}$ on $[0,\zeta^I\wedge\zeta^{II})$. Next, write $I':=II$ and $II':=I$; because again $\sup_{[0,\zeta^i)}Y^i=\infty$ a.s.-$\mathbb{P}$ on $\{\zeta^i<\infty\}$ and because

$Y^{i'}$ admits left limits on $[0,\zeta^{i'})$ we get $\zeta^i \geqslant \zeta^{i'}$ a.s.-$\mathbb{P}$ for each $i \in \{I, II\}$. In conclusion, $\zeta^I = \zeta^{II}$ a.s.-$\mathbb{P}$ and finally $Y^1 = Y^2$ a.s.-$\mathbb{P}$. Therefore there is a.s.-$\mathbb{P}$ unique existence.

We now establish the other properties of $Y$.

(i) Adaptedness to the augmented natural filtration is by construction, since each $Y^n$, $n \in \mathbb{N}$, is a strong solution in its own right. That $\mathbb{P}_\mu$ is uniquely determined by the triplet $(\mu, \Psi_b, \Psi_m)$ is because pathwise uniqueness implies uniqueness in law by a well-known general argument (as in [18, proof of Theorem IX.1.7]).

(ii) Quasi-left-continuity is immediate from (2.2).

(iii) Let us prove the strong Markov property. On $\{S < \zeta\}$ introduce

- $W' := (W_{S+t} - W_S)_{t\in[0,\infty)}$, the increments of $W$ after time $S$;
- $B' := (B_{S+t} - B_S)_{t\in[0,\infty)}$, the increments of $B$ after time $S$;
- $X' := Y_S + (X_{S+t} - X_S)_{t\in[0,\infty)}$, the increments of $X$ after $S$ offset by $Y_S$;
- $\mathcal{M}_b'(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h) := \mathcal{M}_b((\mathrm{d}s - S) \cap (S,\infty), \mathrm{d}v, \mathrm{d}h)$, the shifted Poisson random measure $\mathcal{M}_b$;
- $\mathcal{N}_m'(\mathrm{d}s, \mathrm{d}h) := \mathcal{N}_m((\mathrm{d}s - S) \cap (S,\infty), \mathrm{d}h)$, the shifted Poisson random measure $\mathcal{N}_m$;
- $Y' := Y_{S+\cdot}$ with lifetime $\zeta' := \zeta - S$, the shifted process $Y$.

Then we have, with $\tau_0'$ the first entrance time of $Y'$ into $\{0\}$, a.s.-$\mathbb{P}$,

$$\begin{aligned} Y_t' = X'_{t\wedge\tau_0'} &+ \int\limits_{(0,t]\times[0,Y'_{s-}]\times[0,1]} h\, \bar{\mathcal{M}}_b'(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h) \\ &+ \int\limits_{(0,t]\times[0,Y'_{s-}]\times(1,\infty)} h\, \mathcal{M}_b'(\mathrm{d}s, \mathrm{d}v, \mathrm{d}h) \\ &+ \sigma_b \int_0^t \sqrt{Y_s'}\, \mathrm{d}W_s' + \gamma_b \int_0^t Y_s'\, \mathrm{d}s, \qquad t \in [0,\zeta'), \end{aligned}$$

on $\{S < \zeta\}$ (the bar in $\bar{\mathcal{M}}_b'$ designates the compensated measure); also, 0 is absorbing for $Y'$, $Y'$ has no negative jumps and $\sup_{[0,\zeta')} Y' = \infty$ a.s.-$\mathbb{P}$ on $\{\zeta' < \infty\}$. By the strong Markov property of $(\mathcal{M}_b, W, B, \mathcal{N}_m)$ in $\mathcal{F}$ (applied to the stopping time $S' = S\mathbb{1}_{\{S<\zeta\}} + \infty\mathbb{1}_{\{\zeta\leqslant S\}}$; note $S' = S$ on $\{S' < \infty\} = \{S < \zeta\}$) and the very construction of $Y$ this means two things: first, $Y'$, being a measurable function of $(Y_S, \mathcal{M}_b', B', W', \mathcal{N}_m')$, is $\mathbb{P}$-independent of $\mathcal{F}_S$ given $Y_S$ on $\{S < \zeta\}$; second, $Y'_\star\mathbb{P}(\cdot \,|\, S < \zeta) = \mathbb{P}_\nu$ for $\nu := (Y_S)_\star\mathbb{P}(\cdot \,|\, S < \zeta)$. But this is precisely what is required for the strong Markov property.

(iv) If it is not the case that $\lim_{\zeta-} Y = \infty$ a.s.-$\mathbb{P}$ on $\{0<\zeta<\infty\}$, then, for some $c\in(0,\infty)$, with positive $\mathbb{P}$-probability, $Y$ hits the level $c$ after having first gone above the level $c+1$, and does so consecutively infinitely many times in finite time, which is in contradiction with the strong Markov property (coupled with the downwards skip-free property) and the strong law of large numbers.

(v) The first martingale claim is a consequence of the second by bounded convergence. For the second martingale claim we appeal, for each $n\in\mathbb{N}$, to Itô's formula [8, Theorem II.5.1] for the stopped $(\mathcal{F},\mathbb{P})$-vector semimartingale

$$\Big(\Big(Y_t^n, t, \int_0^t Y_s^n \,\mathrm{d}s\Big)_{t\in[0,\infty)}\Big)^{\zeta^n\wedge\tau_0^n}$$

to obtain $\mathbb{P}$-a.s., for $t\in[0,\zeta^n\wedge\tau_0^n]\cap[0,\infty)$,

$$\begin{aligned}
f(Y_t)e^{-\alpha t-\bar{\alpha}\int_0^t Y_s\,\mathrm{d}s} - f(Y_0) &= \int_0^t e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u} f'(Y_s)(\sigma_m\,\mathrm{d}B_s+\sigma_b\sqrt{Y_s}\,\mathrm{d}W_s)\\
&+\int_0^t e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}\big(f'(Y_s)(\gamma_m+\gamma_b Y_s)-\alpha f(Y_s)-\bar{\alpha}Y_s f(Y_s)\big)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^t e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u} f''(Y_s)(\sigma_m^2\,\mathrm{d}s+\sigma_b^2 Y_s\,\mathrm{d}s)\\
&+\int_{(0,t]\times(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}(f(Y_{s-}+h)-f(Y_{s-}))\,\bar{\mathcal{N}}_m(\mathrm{d}s,\mathrm{d}h)\\
&+\int_0^t\int_{(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}\big(f(Y_s+h)-f(Y_s)-hf'(Y_s)\mathbb{1}_{(0,1]}(h)\big)\,\pi_m(\mathrm{d}h)\,\mathrm{d}s\\
&+\int_{(0,t]\times[0,Y_{s-}]\times(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}(f(Y_{s-}+h)-f(Y_{s-}))\,\bar{\mathcal{M}}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h)\\
&+\int_0^t\int_{(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}\big(f(Y_s+h)-f(Y_s)-hf'(Y_s)\mathbb{1}_{(0,1]}(h)\big)Y_s\,\pi_b(\mathrm{d}h)\,\mathrm{d}s\\
&=\int_0^{t\wedge\zeta} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}\big(\mathcal{A}f(Y_s)-\alpha f(Y_s)-\bar{\alpha}Y_s f(Y_s)\big)\,\mathrm{d}s\\
&+\int_0^t e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u} f'(Y_s)(\sigma_m\,\mathrm{d}B_s+\sigma_b\sqrt{Y_s}\,\mathrm{d}W_s)\\
&+\int_{(0,t]\times(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}(f(Y_{s-}+h)-f(Y_{s-}))\,\bar{\mathcal{N}}_m(\mathrm{d}s,\mathrm{d}h)\\
&+\int_{(0,t]\times[0,Y_{s-}]\times(0,\infty)} e^{-\alpha s-\bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}(f(Y_{s-}+h)-f(Y_{s-}))\,\bar{\mathcal{M}}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h).
\end{aligned}$$

On the r.h.s. above we recognize local martingales in the last three lines (once we have stopped them at $\zeta^n\wedge\tau_0^n$). We deduce that $M^{\zeta_n\wedge\tau_0^n}=M^{\zeta_n}$ (the equality thanks to $\mathcal{L}^{\Psi_m} f(0) = 0$, which renders $\mathcal{A} f(0) = 0$) is an $(\mathcal{F}, \mathbb{P})$-local martingale for each $n \in \mathbb{N}$, so $M$ is "locally a local martingale on $[0, \zeta)$". By the usual trick (basically of [9, Lemma 1.35], mutatis mutandis to handle the lifetime $\zeta$) it follows that $M$ is a local martingale on $[0, \zeta)$ in the sense stipulated. ∎

We call the (law of the) process $Y$ as rendered in the preceding theorem a *continuous-state branching process with spectrally positive migration* (CBM), *branching* mechanism $\Psi_b$, *migration* mechanism $\Psi_m$, and initial law $\mu$. If needed, to emphasize $\mu$, we write $Y^\mu$ and/or $\mathbb{P}^\mu$ in lieu of $Y$ and/or $\mathbb{P}$, respectively (which should not be confused with $\mathbb{P}_\mu = Y_\star \mathbb{P} = (Y^\mu)_\star(\mathbb{P}^\mu)$). For $a \in [0, \infty)$ we let $\tau_a$ be the first entrance time of $Y$ into $[0, a]$.

REMARK 2.1. By a standard general argument (cf. [8, Theorem IV.1.1]) one shows that the map $[0, \infty) \ni x \mapsto \mathbb{P}_x(A)$ is universally measurable for each measurable $A$, that $\mathbb{P}[G(Y_{S+\cdot}) \,|\, \mathcal{F}_S] = \mathbb{P}_{Y_S}[G]$ holds a.s.-$\mathbb{P}$ on $\{S < \zeta\}$ in Theorem 2.1(iii), and that $\mathbb{P}_\mu = \int \mathbb{P}_z \, \overline{\mu}(\mathrm{d}z)$. Here we have written (and will continue to write) $\mathbb{P}_x := \mathbb{P}_{\delta_x}$, $x \in [0, \infty)$, for short.

REMARK 2.2. In Theorem 2.1(v), if for some $a \in [0, \infty)$ the initial law $\mu$ is carried by $[a, \infty)$ and if, ceteris paribus, the process $M$ is stopped at $\tau_a$, then one can drop the assumption $\mathcal{L}^{\Psi_m} f(0) = 0$ and it is enough for the $C^2$ and boundedness property to prevail on $[a, \infty)$, and still the same martingale claims hold true.

In complete analogy with the discrete-space case [21, Remark 2.1], the process $Y$ satisfies a random time-change integral equation involving two Lévy processes having no negative jumps, one of which is $X$.

PROPOSITION 2.1 (Lamperti transform for CBMs). *On an extension of the underlying probability space there exists a Lévy process $L$ having no negative jumps, vanishing at zero a.s., with Laplace exponent $\Psi_b$, and such that*

$$Y_t = X_{t \wedge \tau_0} + L_{\int_0^t Y_s \, \mathrm{d}s}, \quad t \in [0, \zeta), \text{ a.s.}, \tag{2.5}$$

*and also* [*in what follows,* $^{-1}$ *means the left-continuous inverse*]

(a) *$L$ has independent increments relative to the augmented natural filtration of the pair of processes $(L, (\int_0^{\cdot \wedge \zeta} Y_s \, \mathrm{d}s)^{-1})$ initially enlarged by $\sigma(X)$, in particular $L$ is independent of $X$, while*

(b) *$X$ has independent increments relative to the augmented join of the natural filtration of $(L, (\int_0^{\cdot \wedge \zeta} Y_s \, \mathrm{d}s)^{-1})$, time-changed by $\int_0^{\cdot \wedge \zeta} Y_s \, \mathrm{d}s$, and the natural filtration of $X$.*

*Proof.* We may and do assume that $\mathcal{F}$ is the augmented natural filtration of $(X_0, W, \mathcal{M}_b, B, \mathcal{N}_m)$. By first extending (if necesary) the probability space by an independent factor supporting a standard Brownian motion $H$ and a Poisson random measure $N(\mathrm{d}u, \mathrm{d}h)$ on $[0,\infty)^2$ with intensity $\mathrm{d}u\, \pi_b(\mathrm{d}h)$, we may and do also assume the latter were there to begin with.

Put $\gamma_t := \int_0^{t\wedge\zeta} Y_s\,\mathrm{d}s$ for $t \in [0,\infty)$; then $\gamma|_{[0,\zeta\wedge\tau_0)}$ is strictly increasing, vanishing at zero and continuous. Let $\gamma^{-1}$ be its inverse, defined, strictly increasing, vanishing at zero and continuous on $[0,\rho)$, where $\rho := \int_0^\zeta Y_s\,\mathrm{d}s$. Additionally put $\gamma^{-1} := \zeta\wedge\tau_0 = \lim_{\rho-}\gamma^{-1}$ on $[\rho,\infty)$. Thus $\gamma^{-1}$ is just the left-continuous inverse of $\gamma$ and $(\gamma^{-1}(u))_{u\in[0,\infty)}$ is a continuous nondecreasing family of $\mathcal{F}$-stopping times, a time-change. Define the filtration $\mathcal{G}$ as the augmented join of the time-changed filtration $\mathcal{F}_{\gamma^{-1}}$, of the natural filtration of the pair of processes $(H,N)$, and of $\sigma(B,\mathcal{N}_m,X_0)$ (initial enlargement). Since $\{\rho \leqslant u\} = \{\gamma^{-1}(u) = \infty\}$ for all $u \in [0,\infty)$, we see that $\rho$ is a $\mathcal{G}$-stopping time. For $u\in[0,\rho)$ set $\tilde{W}_u := \int_0^{\gamma^{-1}(u)} \sqrt{Y_s}\,\mathrm{d}W_s$ and

$$\tilde{\mathcal{N}_b}([0,u]\times A) := \int_0^{\gamma^{-1}(u)} \int_0^{Y_{s-}} \int_A \mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h), \qquad A \in \mathcal{B}_{[0,\infty)}$$

(it is easy to check that this uniquely determines a random measure on $[0,\rho)\times[0,\infty)$).

By time-change (optional sampling) and independent enlargement we see that the process $W' = \int_0^{\gamma^{-1}(\cdot)} \sqrt{Y}_s\,\mathrm{d}W_s$, which agrees with $\tilde{W}$ on $[0,\rho)$, is a continuous local martingale in the filtration $\mathcal{G}$ vanishing at zero that is stopped at $\rho$ with terminal value $\tilde{W}_\rho := \int_0^\zeta \sqrt{Y}_s\,\mathrm{d}W_s$ a.s. on $\{\rho<\infty\}$. Its quadratic variation process is given by $\langle W'\rangle_u = \int_0^{\gamma^{-1}(u)} Y_s\,\mathrm{d}W_s = u\wedge\rho$ a.s. for $u\in[0,\infty)$. Now define $\tilde{W}_u := \tilde{W}_\rho + H(u) - H(\rho)$ for $u\in[\rho,\infty)$. Then $\tilde{W} = W' + \mathbb{1}_{[[\rho,\infty))}(H - H(\rho))$ is a $\mathcal{G}$-continuous local martingale vanishing at zero with increasing process given by $\langle\tilde{W}\rangle_u = u$ a.s. for $u\in[0,\infty)$. By Lévy's martingale characterization of Brownian motion [8, Theorem II.6.1] it follows that $\tilde{W}$ is a $\mathcal{G}$-Brownian motion.

Similarly, again by time-change (optional sampling) and independent enlargement, we see that for each Borel set $A \subseteq [0,\infty)$ of finite $\pi_b$-measure the process $\int_0^{\gamma^{-1}(\cdot)} \int_0^{Y_{s-}} \int_A \mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) - \pi_b(A)(\cdot\wedge\rho)$ is a $\mathcal{G}$-martingale that is stopped at $\rho$ with terminal value

$$\int_0^{\zeta} \int_0^{Y_{s-}} \int_A \mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h) - \pi_b(A)\rho =: \tilde{\mathcal{N}_b}([0,\rho]\times A) - \pi_b(A)\rho$$

a.s. on $\{\rho<\infty\}$. Then define the random measure $\tilde{\mathcal{N}_b}$ on $[0,\infty)^2$ unambiguously by specifying further that $\tilde{\mathcal{N}_b}([0,u]\times A) := \tilde{\mathcal{N}_b}([0,\rho]\times A) + N((\rho,u]\times A)$ for $u\in[\rho,\infty)$ and $A\in\mathcal{B}_{[0,\infty)}$. We see that for each Borel set $A\subset[0,\infty)$ of finite $\pi_b$-measure the process $\tilde{\mathcal{N}_b}([0,\cdot]\times A) - \pi_b(A)\cdot$ is a $\mathcal{G}$-martingale. It follows from the martingale characterization of Poisson point processes [8, Theorem II.6.2] that $\mathcal{N}_b(\mathrm{d}w, \mathrm{d}h)$ is a $\mathcal{G}$-Poisson random measure with intensity $\pi_b(\mathrm{d}h)\,\mathrm{d}w$.

Further, it is well-known that in a common filtration a Poisson point process and a Brownian motion are automatically independent [8, Theorem II.6.3]; what is more, they have jointly independent increments in that filtration, which is in fact proved in [8, (II.6.12)]. Therefore, setting (as usual a bar indicates the compensated measure)

$$L_u := \sigma_b \tilde{W}_u + \gamma_b u + \int_{[0,u]\times[0,1]} \bar{\tilde{\mathcal{N}}}_b(\mathrm{d}w, \mathrm{d}h) + \int_{[0,u]\times(1,\infty)} \tilde{\mathcal{N}}_b(\mathrm{d}w, \mathrm{d}h), \quad u \in [0,\infty), \text{ a.s.},$$

we get a $\mathcal{G}$-Lévy process $L$ having no negative jumps, with Laplace exponent $\Psi_b$. Moreover, from (2.2) we obtain exactly (2.5). The fact that $L$ is a Lévy process in the filtration $\mathcal{G}$ gives (a), just because $\mathcal{G}$ contains the augmented natural filtration of the pair of processes $(L, \gamma^{-1})$ initially enlarged by $\sigma(X)$. On the other hand, $X$ has independent increments relative to $\mathcal{F}$ initially enlarged by $\sigma(H, N)$ (and augmented), which contains the natural filtration of $X$ but also that of $(L, \gamma^{-1})$ time-changed by $\gamma$, since the latter is contained in

$$(\mathcal{F}_{\gamma^{-1}} \vee \sigma(H,N))_\gamma = (\mathcal{F}_{\gamma^{-1}})_\gamma \vee \sigma(H,N) = \mathcal{F}_{\cdot \wedge \tau_0 \wedge \zeta} \vee \sigma(H,N) \subset \mathcal{F} \vee \sigma(H,N)$$

(for the second equality see [18, Exercise 1.12]; the first follows easily because $\sigma(H,N)$ is independent of $\mathcal{F}_\infty$). ■

Some historical comments on the preceding are in order. When $X = 0$, the time-change delineated above is originally due to Lamperti [13], which explains the name "Lamperti transform"; see also [2]. In fact, the Lamperti transform for CSBPs works also in the other direction, when constructing $Y$ from $L$. Paper [1] generalizes the transform to CBIs (without stopping on hitting zero), albeit only in the latter direction, starting from the pair $(X, L)$ to obtain $Y$. Paper [16] handles the Lamperti transform of CSBPs with competition (both ways) in the SDE setting; many of the ideas of the proofs of Theorem 2.1 and of Proposition 2.1 come from that source.

It appears that obtaining the converse to Proposition 2.1 is more involved for CBMs, vis-à-vis CBIs or CSBPs with competition, and this is left as an open problem. The main difficulty lies in establishing that (2.5) has a unique solution for $Y$ given $(X, L)$, in the appropriate precise sense (if indeed it can be made precise in any reasonable way). Note that, on the one hand, the approach of [1, p. 1603, proof of Theorem 1, uniqueness, esp. last display] for CBIs relies heavily on the monotonicity of $X$, which is absent in the setting of CBMs. On the other hand, the line of attack of [16, proof of Theorem 2.2] (to relate uniqueness of (2.5) to the uniqueness of (2.2)) is hindered by the fact that, roughly speaking, one has to work simultaneously with the filtrations of $X$ and $L$ which "run on different time-scales" (it is not enough to just shift between time-changed filtrations, cf. items (a) and (b) of Proposition 2.1).

We might also mention that a further Lamperti-style transform of a CBM process leads to "CSBPs with collisions" (see e.g. [14, Section 2, the process $R$ when $g$ is quadratic] for a special case); but this connection will not be explored here.

COROLLARY 2.1. *Assume the CSBP with branching mechanism $\Psi_b$ is nonexplosive. Then $\mathbb{P}(\zeta < \infty) = 0$, i.e. the CBM is nonexplosive as well.*

This result will be refined to an equivalence in Corollary 3.4 (under assumption (3.1)).

*Proof of Corollary 2.1. Coupling argument.* Suppose per absurdum that $\mathbb{P}(\zeta < \infty) > 0$. We may and do assume that $\mu = \delta_z$ for some $z \in (0,\infty)$. Let $\overline{X} = (\overline{X}_t)_{t\in[0,\infty)}$ be the running supremum of $X$: $\overline{X}_t := \sup_{s\in[0,t]} X_s$ for $t \in [0,\infty)$. By continuity from below, $\mathbb{P}(\zeta < \infty, \overline{X}_\zeta < m) > 0$ for some $m \in (z,\infty)$. Let $\tilde{Y}$ be the solution to (2.5) with $X \equiv m$ (for a constant $X$ we know that (2.5) has an a.s. unique solution). So $\tilde{Y}$ is a CSBP with starting point $m$ and branching mechanism $\Psi_b$. All the corresponding quantities get a tilde. We claim that $\tilde{\zeta} \leqslant \zeta$ a.s. on $\{\zeta < \infty, \overline{X}_\zeta < m\}$ (which implies that $\tilde{\zeta} < \infty$ with positive probability, contradicting the nonexplosivity of $\tilde{Y}$ and completing the proof). Suppose $\zeta < \tilde{\zeta}$ with positive probability on $\{\zeta < \infty, \overline{X}_\zeta < m\}$. Then we cannot have $\int_0^t Y_s\,\mathrm{d}s \leqslant \int_0^t \tilde{Y}_s\,\mathrm{d}s$ for all $t \in [0,\zeta)$ a.s. on $\{\zeta < \tilde{\zeta}, \overline{X}_\zeta < m\}$, since otherwise, letting $t \uparrow \zeta$ yields (by the Lamperti transform for $Y$) $\infty = \int_0^\zeta \tilde{Y}_s\,\mathrm{d}s$ a.s. on the event $\{\zeta < \tilde{\zeta}, \overline{X}_\zeta < m\}$ of positive probability, contradicting the local boundedness of $\tilde{Y}$ on $[0,\tilde{\zeta})$. So with positive probability on $\{\zeta < \tilde{\zeta}, \overline{X}_\zeta < m\}$, we have $\int_0^t Y_s\,\mathrm{d}s > \int_0^t \tilde{Y}_s\,\mathrm{d}s$ for some $t \in [0,\zeta)$. Let

$$\delta := \inf\Big\{t \in [0,\zeta\wedge\tilde{\zeta}) : \int_0^t Y_s\,\mathrm{d}s > \int_0^t \tilde{Y}_s\,\mathrm{d}s\Big\}.$$

A.s. on $\{\delta < \zeta < \tilde{\zeta}, \overline{X}_\zeta < m\}$, an event of positive probability, we have by the Lamperti transform

$$Y_\delta = X_\delta + L_{\int_0^\delta Y_s\,\mathrm{d}s} < m + L_{\int_0^\delta Y_s\,\mathrm{d}s} = m + L_{\int_0^\delta \tilde{Y}_s\,\mathrm{d}s} = \tilde{Y}_\delta,$$

contradicting the fact that $\int_0^\cdot Y_s\,\mathrm{d}s > \int_0^\cdot \tilde{Y}_s\,\mathrm{d}s$ immediately after $\delta$. ∎

Let us conclude this section by emphasizing, at least on an informal level, the fundamental difference between CBI and CBM processes. The former are such that for independent $X_1, X_2, L_1, L_2, L$, with $L_1$ and $L_2$ having the same law as $L$, the process associated (via the Lamperti transform) to the pair $(X_1 + X_2, L)$ has the same law as the sum of the processes associated to $(X_1, L_1)$ and $(X_2, L_2)$. Put more succinctly, CBIs can be superposed.

This property fails for CBMs. Analytically it is a manifestation of the "affine" property of the Laplace transform of a CBI process [11, (1.1)], which cannot hold for the CBM class [11, Theorem 1.1]. In this connection, one should emphasize that the exponential functions $e_\alpha := e^{-\alpha\cdot}$, $\alpha \in (0,\infty)$, do not actually fall under Theorem 2.1(v) (not even the local martingale part, just because $\mathcal{L}^{\Psi_m} e_\alpha(0)$ need not be 0), at least not generically. So even though as a matter of analytical fact

$$\begin{aligned}\mathcal{A}_z e_\alpha(z) &= \Psi_m(\alpha)e_\alpha(z) + z\Psi_b(\alpha)e_\alpha(z)\\ &= \left(\Psi_m(\alpha) - \Psi_b(\alpha)\frac{\partial}{\partial\alpha}\right)e_\alpha(z) =: \mathcal{B}_\alpha e_\alpha(z),\end{aligned}$$

and even if $\Psi_m \geqslant 0$ (so that $\Psi_m$ may be interpreted as the instantaneous rate of killing in the operator $\mathcal{B}$), one is not able to simply "integrate" this duality on the level of (sic) generators to a Laplace duality on the level of semigroups [10, Proposition 1.2] (as is the case for CBI). Nevertheless:

PROPOSITION 2.2. *Suppose two CBMs $Y^1$ and $Y^2$ have been defined according to* (2.2) *using independent Brownian and Poisson drivers in the common filtration $\mathcal{F}$ under a common probability $\mathbb{P}$. All the quantities pertaining to $Y^i$ get a superscript $i \in \{1,2\}$. Suppose $\Psi_b^1 = \Psi_b^2 =: \Psi_b$. Then there exists* (*still on the same probability space, in the same filtration*) *a CBM process $Y = (Y_t)_{t\in[0,\zeta)}$ with initial distribution $\mu := \mu^1 \star \mu^2$, branching mechanism $\Psi_b$, and migration mechanism $\Psi_m := \Psi_m^1 + \Psi_m^2$, such that, a.s.-$\mathbb{P}$, $\zeta \wedge \tau_0^1 \wedge \tau_0^2 = \zeta^1 \wedge \zeta^2 \wedge \tau_0^1 \wedge \tau_0^2$ and $Y = Y^1 + Y^2$ on $[0, \tau_0^1 \wedge \tau_0^2 \wedge \zeta)$. If $\Psi_m^2 = 0$ then we may further insist that, on the event $\{\tau_0^2 < \tau_0^1 \wedge \zeta\}$, a.s.-$\mathbb{P}$, $\zeta = \zeta^1$ and $Y = Y^1$ on $[\tau_0^2, \zeta)$.*

*Proof.* Define the random measure

$$\begin{aligned}\mathcal{M}_b(A) := &\int_A \mathbb{1}_{[0,Y^1_{s-})}(v)\,\mathcal{M}_b^1(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h)\\ &+\int_A \mathbb{1}_{[Y^1_{s-},\infty)}(v)\mathbb{1}_A(s, v-Y^1_{s-}, h)\,\mathcal{M}_b^2(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h), \qquad A \in \mathcal{B}_{[0,\infty)^3},\end{aligned}$$

where we understand $Y^1 = \infty$ on $[\zeta^1,\infty)$. Then by [8, Theorem II.6.2] we find that $\mathcal{M}_b(\mathrm{d}s,\mathrm{d}v,\mathrm{d}h)$ is an $\mathcal{F}$-Poisson random measure with intensity $\mathrm{d}s\,\mathrm{d}v\,\pi_b(\mathrm{d}h)$. More trivially, $\mathcal{N}_m := \mathcal{N}_m^1 + \mathcal{N}_m^2$ is an $\mathcal{F}$-Poisson random measure and the intensity of $\mathcal{N}_m(\mathrm{d}s,\mathrm{d}h)$ is $\mathrm{d}s\,\pi_m(\mathrm{d}h)$ with $\pi_m := \pi_m^1 + \pi_m^2$. Besides, $\mathbb{P}$-a.s. $\mathcal{M}_b$ has no jumps in common with $\mathcal{N}_m$.

Similarly, the process $W := (Y^1+Y^2)^{-1/2}\cdot(\sqrt{Y^1}\cdot W^1 + \sqrt{Y^2}\cdot W^2)$, defined on $[0, \zeta^1 \wedge \zeta^2 \wedge \tau_0^1 \wedge \tau_0^2)$ and extended by the increments of $W^1$ thereafter, is a standard $\mathcal{F}$-Brownian motion. Again more trivially,

$$B := ((\sigma_m^1)^2 + (\sigma_m^2)^2)^{-1}(\sigma_m^1 B^1 + \sigma_m^2 B^2)$$

(or just $B := B^1$, say, if $\sigma_m^1 = \sigma_m^2 = 0$) is a standard $\mathcal{F}$-Brownian motion. In addition, $\mathbb{P}$-a.s. the covariation process of $W$ and $B$ vanishes.

From the preceding it follows [8, Theorem II.6.3] that the processes $\mathcal{M}_b$, $\mathcal{N}_m$, $B$ and $W$ are independent. Let $Y$ be the CBM corresponding to the initial value $X_0 := X_0^1 + X_0^2$ and these drivers according to (2.2). Its branching mechanism is $\Psi_b$, its migration mechanism is $\Psi_m$ and its initial value is $\mu$. Taking the sum of (2.2) corresponding to $Y^1$ and $Y^2$ we get the remainder of the claim (by the uniqueness of $Y$ as a solution to (2.2)). ■

COROLLARY 2.2. *CBMs are stochastically monotone in the starting law, with the* (*natural*) *convention that the coffin state is set equal to* $\infty$, $[0,\infty]$ *having the usual order: if* $\mu'$ *is another law on* $\mathcal{B}_{[0,\infty)}$ *with* $\mu \leqslant \mu'$ (*in first-order stochastic dominance*) *then* $\mathbb{P}_\mu[G] \leqslant \mathbb{P}_{\mu'}[G]$ *for any measurable map* $G : [0,\infty]^{[0,\infty)} \to [-\infty,\infty]$ *satisfying*

$$\omega \leqslant \omega' \Rightarrow G(\omega) \leqslant G(\omega')$$

*for* $\{\omega,\omega'\} \subset [0,\infty]^{[0,\infty)}$ ($G$ *is nondecreasing relative to the natural partial order induced on* $[0,\infty]^{[0,\infty)}$ *by the linear order* $\leqslant$ *on* $[0,\infty]$) *and* $\mathbb{P}_\mu[G^-] < \infty$.

*Proof. Coupling.* In the context of Proposition 2.2 take $X^2$ constant and equal to $X_0^2$ (so $\Psi_m^2 = 0$). The processes $Y$ and $Y^1$ have the same branching and migration mechanisms but $Y$ starts above $Y^1$ a.s.-$\mathbb{P}$. Also, a.s.-$\mathbb{P}$, $\zeta \wedge \tau_0^1 \wedge \tau_0^2 = \zeta^1 \wedge \zeta^2 \wedge \tau_0^1 \wedge \tau_0^2$ and $Y = Y^1 + Y^2$ on $[0, \tau_0^1 \wedge \tau_0^2 \wedge \zeta)$; furthermore, on the event $\{\tau_0^2 < \tau_0^1 \wedge \zeta\}$, a.s.-$\mathbb{P}$, $\zeta = \zeta^1$ and $Y = Y^1$ on $[\tau_0^2, \zeta)$. It follows that $Y \geqslant Y^1$ everywhere a.s.-$\mathbb{P}$ [with the convention that the coffin state is set equal to $\infty$, $[0,\infty]$ having the usual order]. ■

## 3. FIRST PASSAGE TIMES AND EXPLOSIONS

To avoid the analysis of some special cases, which are perhaps not of much interest in the present context, we assume for the remainder of this text that (in the notation of the Lamperti transform, Proposition 2.1)

$$\boxed{\text{neither } X \text{ nor } L \text{ have a.s. nondecreasing paths.}} \tag{3.1}$$

We call such CBMs *nondegenerate*. In other words, $\Psi_b$ and $\Psi_m$ are Laplace exponents of spectrally positive Lévy processes (SPLPs) [in the narrow sense] or of strictly negative drifts. Recall that the case when $X$ is a subordinator (resp. the zero process) corresponds to a CBI (resp. a CSBP) process and this has been studied in [3] (at least as far as first passage times are concerned). So it is only the case when $L$ is a subordinator that is being left (completely) untreated.

Let $\xi$ be the canonical process on the space of càdlàg nonnegative real paths with lifetime; let $\mathsf{l}$ be the lifetime of $\xi$ and set $\sigma_a := \inf\{t \in [0,\mathsf{l}) : \xi_t \leqslant a\}$ for

$a \in [0, \infty)$. We believe that no confusion can arise from the notation $\sigma_a$ vis-à-vis the diffusion coefficients $\sigma_b$ and $\sigma_m$ (we will never use $b$ and $m$ for the first passage level).

Denote by $\Psi_b^{-1} : [0, \infty) \to [0, \infty)$ and $\Psi_m^{-1} : [0, \infty) \to [0, \infty)$ the so-called right-continuous inverses of $\Psi_b$ and $\Psi_m$, respectively:

$$\Psi_w^{-1}(t) := \inf\{z \in [0, \infty) : \Psi_w(z) > t\}, \quad t \in [0, \infty),\ w \in \{b, m\}$$

(finite, due to (3.1)). In the next theorem the case $\bar{\alpha} = 0$ is mainly of interest, but the inclusion of $\bar{\alpha} > 0$ does not really make the proof any longer or more difficult, and it allows us to procure some information on the "cumulative lifetime-to-date" process $\int_0^\cdot \xi_s \, \mathrm{d}s$.

THEOREM 3.1 (First passage times of CBMs). *Let* $\alpha, \bar{\alpha} \in [0, \infty)$. *Suppose* $\Psi_m(\Psi_b^{-1}(\bar{\alpha})) < \alpha$ (*i.e.* $\Psi_b^{-1}(\bar{\alpha}) < \Psi_m^{-1}(\alpha)$ *and* $[\alpha > 0$ *or* $\Psi_b^{-1}(\bar{\alpha}) > 0]$), *or else suppose that* $\Phi := \Psi_b^{-1}(\bar{\alpha}) = \Psi_m^{-1}(\alpha) > 0$. *For* $x \in [0, \infty)$ *put*

$$\Phi_{\alpha,\bar{\alpha}}(x) := \int_{\Psi_b^{-1}(\bar{\alpha})}^{\infty} \frac{\mathrm{d}z}{\Psi_b(z) - \bar{\alpha}} \exp\left(-xz - \int_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u) - \alpha}{\Psi_b(u) - \bar{\alpha}} \, \mathrm{d}u\right) \tag{3.2}$$

*or*

$$\Phi_{\alpha,\bar{\alpha}}(x) := e^{-\Phi x}$$

*according as* $\Psi_m(\Psi_b^{-1}(\bar{\alpha})) < \alpha$ *or* $\Psi_b^{-1}(\bar{\alpha}) = \Psi_m^{-1}(\alpha) > 0$. *Then for* $a \leqslant x$ *from* $[0, \infty)$,

$$\mathbb{P}_x[e^{-\alpha\sigma_a - \bar{\alpha}\int_0^{\sigma_a} \xi_s \mathrm{d}s}; \sigma_a < \mathsf{I}] = \frac{\Phi_{\alpha,\bar{\alpha}}(x)}{\Phi_{\alpha,\bar{\alpha}}(a)}. \tag{3.3}$$

Before we proceed to the proof, some comments are in order.

REMARK 3.1. Any $\theta \in (\Psi_b^{-1}(\bar{\alpha}), \infty)$ may replace $\Psi_m^{-1}(\alpha)$ in (3.2): it then changes it only by a multiplicative constant, which is immaterial. The delimiter $\Psi_m^{-1}(\alpha)$ seems most natural because it precisely separates the area of positivity and negativity of the integrand. Equation (3.2) may be compared with the CBI case [3, (11)]. It is somewhat agreeable that it actually attains a more "symmetric" form when viewed through the lens of Laplace exponents of SPLPs (as opposed to one SPLP and one subordinator).

REMARK 3.2. We see from (3.2)–(3.3), by dominated convergence, that for all $a \in [0, \infty)$, $\lim_{b \to a} \tau_b = \tau_a$ a.s.-$\mathbb{P}$. This may also be gleaned from the general properties of CBMs as follows. On the one hand, Proposition 2.1 and the regularity downwards of SPLPs [12, p. 232] imply that $Y$ is also regular downwards at all levels from $(0, \infty)$, which renders $\downarrow$-$\lim_{b\uparrow a}\tau_b = \tau_a$ a.s.-$\mathbb{P}$ for all $a \in (0, \infty)$. On the other hand, quasi-left-continuity, coupled with the property $\lim_{\zeta-} Y = \infty$ a.s.-$\mathbb{P}$ on $\{0 < \zeta < \infty\}$, yields $\uparrow$-$\lim_{b\downarrow a}\tau_b = \tau_a$ a.s.-$\mathbb{P}$ for all $a \in [0, \infty)$.

As a check:

EXAMPLE 3.1. Let $\{\mathsf{b},\mathsf{m}\}\subset(0,\infty)$ and $\Psi_b=\mathsf{b}\cdot\mathrm{id}_{[0,\infty)}$, $\Psi_m=\mathsf{m}\cdot\mathrm{id}_{[0,\infty)}$. Then for $x\in[0,\infty)$ and $\alpha\in(0,\infty)$,

$$\Phi_{\alpha,0}(x)=\frac{\Gamma\big(\frac{\alpha}{\mathsf{b}}\big)}{\mathsf{b}\big(\frac{\alpha}{\mathsf{b}}+\frac{\alpha}{\mathsf{m}}x\big)^{\frac{\alpha}{\mathsf{b}}}},\quad \text{hence}\quad \mathbb{P}_x[e^{-\alpha\sigma_0};\sigma_0<\mathsf{l}]=\left(1+\frac{\mathsf{b}}{\mathsf{m}}x\right)^{-\frac{\alpha}{\mathsf{b}}};$$

therefore $\sigma_{0\star}\mathbb{P}_x=\delta_{\mathsf{b}^{-1}\log(1+\frac{\mathsf{b}}{\mathsf{m}}x)}$, as it should be (e.g. from (2.5)).

Formulas (3.2) above, and (3.6) to follow, may seem at first sight to appear "out of the blue". Of course they do not. First, they may be guessed from the discrete counterparts [21, Corollary 4.14, Theorem 4.2], using [21, Remark 4.16], as was actually the case. Second, they could be obtained by solving the relevant o.d.e. problems (though this still involves "guessing" the "Laplace transform/completely monotone" forms of (3.2) & (3.6), cf. the discrete case [21, p. 9]). The discrete analogs being available, the first option seems decidedly preferable (and faster).

*Proof of Theorem 3.1.* We focus on the case $\Psi_m(\Psi_b^{-1}(\bar\alpha))<\alpha$, the version with $\Psi_b^{-1}(\bar\alpha)=\Psi_m^{-1}(\alpha)>0$ being similar (and easier) and left to the reader.

First, $\Phi_{\alpha,\bar\alpha}:[0,\infty)\to(0,\infty)$ is well-defined, finite, strictly decreasing, continuous and vanishing at infinity, which is easy to see using the assumption $\Psi_m(\Psi_b^{-1}(\bar\alpha))<\alpha$ by recalling that $\Psi_m-\alpha$ is continuous with $\lim_\infty\Psi_m=\infty$, also strictly negative on $(0,\Psi_m^{-1}(\alpha))$ and strictly positive on $(\Psi_m^{-1}(\alpha),\infty)$, with the analogous observation being true for $\Psi_b$. Differentiating under the integral sign we also see that $\Phi_{\alpha,\bar\alpha}$ is $C^2$ (indeed $C^\infty$) on $(0,\infty)$ (maybe also at zero, but we do not need it).

We let $\mu=\delta_x$ be the initial distribution of the CBM $Y$, thus $\mathbb{P}_x$ is the $\mathbb{P}=\mathbb{P}^\mu$-law of $Y=Y^\mu$. If necessary, taking the limit as $\alpha\downarrow0$, we see by monotone (or bounded) convergence on the l.h.s. and by monotone and dominated convergence on the r.h.s. of (3.3) that we may (and do) assume $\alpha>0$ (indeed, for $\alpha=0$, one can first replace the delimiter $\Psi_m^{-1}(\alpha)$ with $\Psi_m^{-1}(0)$ in the expression for $\Phi_{\alpha,\bar\alpha}$, then pass to the limit $\alpha\downarrow0$ by monotone convergence on $z\in(\Psi_b^{-1}(\bar\alpha),\Psi_m^{-1}(0))$ and by dominated convergence on $z\in(\Psi_m^{-1}(0),\infty)$). Because $Y$ is quasi-left-continuous, due to the continuity of $\Phi_{\alpha,\bar\alpha}$ and because now $\alpha>0$, we further infer that it suffices to establish the Laplace transform formula for $a>0$ (one can pass to the limit $a\downarrow0$ by bounded convergence on the l.h.s. and by continuity of $\Phi_{\alpha,\bar\alpha}$ on the r.h.s. of (3.3)). Thus we may and do assume $a>0$.

Consider now the process $M$ of Theorem 2.1(v)(II) with $f=\Phi_{\alpha,\bar\alpha}$, but stopped at $\tau_a$, i.e., modulo its deterministic initial value, the process

$$\Bigg(\Phi_{\alpha,\bar\alpha}(Y_{t\wedge\tau_a})e^{-\alpha(t\wedge\tau_a)-\bar\alpha\int_0^{t\wedge\tau_a}Y_s\,\mathrm{d}s}\mathbb{1}_{\{t\wedge\tau_a<\zeta\}}\\ -\int_0^{t\wedge\tau_a\wedge\zeta}e^{-\alpha s-\bar\alpha\int_0^sY_u\,\mathrm{d}u}(\mathcal{A}\Phi_{\alpha,\bar\alpha}-\alpha\Phi_{\alpha,\bar\alpha}-\bar\alpha\,\mathrm{id}_{[0,\infty)}\Phi_{\alpha,\bar\alpha})(Y_s)\,\mathrm{d}s\Bigg)_{t\in[0,\infty)}.$$

Notice that $Y_{\tau_a} = a$ on $\{\tau_a < \zeta\} = \{\tau_a < \infty\}$ (since $Y$ has no negative jumps), while $\lim_{t\uparrow\zeta} \Phi_{\alpha,\bar{\alpha}}(Y_t)e^{-\alpha t - \bar{\alpha}\int_0^t Y_s\,\mathrm{d}s} = 0$ a.s.-$\mathbb{P}$ on $\{\tau_a = \infty\}$ (since $\alpha > 0$ and $\Phi_{\alpha,\bar{\alpha}}$ vanishes at infinity). Therefore, by Theorem 2.1(v)(II) & Remark 2.2, and because martingales have a constant expectation, the Laplace transform formula (3.3) reduces further to establishing that $\Phi_{\alpha,\bar{\alpha}}$ is $C^2$, bounded on $[a,\infty)$ and satisfies

$$\mathcal{A}\Phi_{\alpha,\bar{\alpha}}(x) = \alpha\Phi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Phi_{\alpha,\bar{\alpha}}(x), \qquad x \in [a,\infty).$$

This however is a straightforward computation made easy by the fact that we are working on restriction to $x \in [a,\infty) \subset (0,\infty)$ (justifying differentiation under the integral sign):

$$\begin{aligned}
\mathcal{A}\Phi_{\alpha,\bar{\alpha}}(x) &= \mathcal{L}^{\Psi_m}\Phi_{\alpha,\bar{\alpha}}(x) + x\mathcal{L}^{\Psi_b}\Phi_{\alpha,\bar{\alpha}}(x)\\
&= \int\limits_{\Psi_b^{-1}(\bar{\alpha})}^{\infty} \frac{\mathrm{d}z}{\Psi_b(z)-\bar{\alpha}} \exp\Bigg(-xz - \int\limits_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u)-\alpha}{\Psi_b(u)-\bar{\alpha}}\,\mathrm{d}u\Bigg)\\
&\qquad\qquad \times (\Psi_m(z) + x\Psi_b(z))\\
&= \int\limits_{\Psi_b^{-1}(\bar{\alpha})}^{\infty} \frac{\mathrm{d}z}{\Psi_b(z)-\bar{\alpha}} \exp\Bigg(-xz - \int\limits_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u)-\alpha}{\Psi_b(u)-\bar{\alpha}}\,\mathrm{d}u\Bigg)\\
&\qquad\qquad \times \big(\Psi_m(z) - \alpha + \alpha + x(\Psi_b(z) - \bar{\alpha} + \bar{\alpha})\big)\\
&= \alpha\Phi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Phi_{\alpha,\bar{\alpha}}(x)\\
&\quad + \int\limits_{\Psi_b^{-1}(\bar{\alpha})}^{\infty} \mathrm{d}z\, \frac{\Psi_m(z)-\alpha}{\Psi_b(z)-\bar{\alpha}} \exp\Bigg(-xz - \int\limits_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u)-\alpha}{\Psi_b(u)-\bar{\alpha}}\,\mathrm{d}u\Bigg)\\
&\quad + x \int\limits_{\Psi_b^{-1}(\bar{\alpha})}^{\infty} \mathrm{d}z \exp\Bigg(-xz - \int\limits_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u)-\alpha}{\Psi_b(u)-\bar{\alpha}}\,\mathrm{d}u\Bigg)\\
&= \alpha\Phi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Phi_{\alpha,\bar{\alpha}}(x)\\
&\quad - \Bigg[\exp\Bigg(-xz - \int\limits_{\Psi_m^{-1}(\alpha)}^{z} \frac{\Psi_m(u)-\alpha}{\Psi_b(u)-\bar{\alpha}}\,\mathrm{d}u\Bigg)\Bigg|_{z=\Psi_b^{-1}(\bar{\alpha})}^{\infty}\Bigg]\\
&= \alpha\Phi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Phi_{\alpha,\bar{\alpha}}(x),
\end{aligned}$$

where the penultimate equality is integration by parts and the last equality is by elementary estimation using $\Psi_m(\Psi_b^{-1}(\bar{\alpha})) < \alpha$ (at $z = \infty$ it is trivial, at $z = \Psi_b^{-1}(\bar{\alpha})+$ one gets a divergent integral of the form $\sim \int_{0+} \frac{\mathrm{d}u}{u}$). ■

COROLLARY 3.1. *We have* $\mathbb{P}_x(\sigma_0 < \infty) > 0$ *for all* $x \in [0,\infty)$ *and* $\lim_{|-} \xi = \infty$ *a.s. on* $\{\sigma_0 = \infty\}$*, in particular there is no phenomenon of extinguishing* (*i.e. the event* $\{\lim_\infty \xi = 0\} \cap \{\sigma_0 = \infty\}$ *is negligible*).

REMARK 3.3. We recall that CSBPs can become extinguished [12, p. 343].

*Proof of Corollary 3.1.* It is clear from the strict positivity of the scale function of (3.2) that $\mathbb{P}_x(\sigma_0 < \infty) > 0$ for all $x \in [0, \infty)$.

Suppose per absurdum that $\lim_{\zeta-} Y \neq \infty$ with positive $\mathbb{P}$-probability on $\{\tau_0 = \infty\}$. Then for some $N \in [0, \infty)$, on an event $A$ of positive $\mathbb{P}$-probability, the process $Y$ will be $\leqslant N$ at arbitrarily large times, but never hit zero.

Fix some $\alpha \in (0, \infty)$. Consider the sequence $(S_k)_{k \in \mathbb{N}}$ of random times defined as follows. First, if needed, enlarge the probability space to gain access to $e_\alpha^{(k)}$, $k \in \mathbb{N}$, independent exponentially with rate $\alpha$ distributed $(0, \infty)$-valued random variables, independent of $Y$. Second, put, inductively,

$$S_k := \inf \{t \in [S_{k-1}, \zeta) : Y_t \in [0, N]\} + e_\alpha^{(k)}, \qquad k \in \mathbb{N},$$

with the convention $S_0 := 0$. Thus, in plainer tongue,

$$S_1 = (\text{the first time } Y \text{ enters } [0, N]) + e_\alpha^{(1)},$$
$$S_2 = (\text{the first time } Y \text{ enters } [0, N] \text{ after } S_1) + e_\alpha^{(2)},$$

and so on. Perhaps $S_k = \infty$ at some $k \in \mathbb{N}$, in which case $S_{l+1} = \infty$ for all $l \in \mathbb{N}$, $l \geqslant k$. But anyway $A \subset \{S_k < \infty \text{ for all } k \in \mathbb{N}\}$. Now, $Y$ always has a strictly positive chance $\beta > 0$ to hit zero before an independent exponential random time of rate $\alpha$ has elapsed, no matter where in $[0, N]$ it starts. On $A$ it must fail to do so infinitely many times. By the strong Markov property, this is impossible. (Of course, a sufficiently large deterministic time could also be used in lieu of the $e_\alpha^{(k)}$, $k \in \mathbb{N}$.) ∎

COROLLARY 3.2. *Suppose* $(\Psi_b)'(0+) \geqslant 0$ (*i.e.* $\Psi_b^{-1}(0) = 0$) *and fix a* $\theta \in (0, \infty)$. *Then* $\mathbb{P}_x(\sigma_0 < \infty) = 1$ *for all* $x \in [0, \infty)$ *iff*

$$\int_0^\theta \frac{\mathrm{d}z}{\Psi_b(z)} \exp\left(\int_z^\theta \frac{\Psi_m(u)}{\Psi_b(u)} \mathrm{d}u\right) = \infty \tag{3.4}$$

(*which is true if* $(\Psi_m)'(0+) > -\infty$ *and* $(\Psi_b)'(0+) > 0$)*; when* (3.4) *fails, then*

$$\mathbb{P}_x(\sigma_0 < \infty) = \frac{\int_0^\infty \frac{\mathrm{d}z}{\Psi_b(z)} e^{-xz - \int_\theta^z \frac{\Psi_m(u)}{\Psi_b(u)} \mathrm{d}u}}{\int_0^\infty \frac{\mathrm{d}z}{\Psi_b(z)} e^{-\int_\theta^z \frac{\Psi_m(u)}{\Psi_b(u)} \mathrm{d}u}} < 1 \quad \text{for all } x \in (0, \infty). \tag{3.5}$$

Thus when $(\Psi_b)'(0+) > 0$ and $(\Psi_m)'(0+) > -\infty$, migrations cannot offset the branching to turn the process from one that a.s. becomes extinct or extinguished to one for which this would not be the case, while for $(\Psi_b)'(0+) > 0$ but $(\Psi_m)'(0+) = -\infty$ the situation appears to be more delicate:

EXAMPLE 3.2. Let $\Psi_b = \mathrm{id}_{[0,\infty)}$ (so $\Psi_b'(0+) > 0$), $\sigma_m^2 > 0$, $\gamma_m = 0$, $\pi_m$ vanishing on $(0,2]$. If $\pi_m((z,\infty)) = 2/\log(z)$ for $z \in [2,\infty)$ (so $(\Psi_m)'(0+) = -\infty$; such a situation *can* occur), then it is elementary (if tedious) to check using the abelian theorem for Laplace transforms [5, Theorem XIII.5.2] that the integral of (3.4) converges. On the other hand, if $\pi_m((z,\infty)) = z^{-1/2}$ for $z \in [2,\infty)$ (so $(\Psi_m)'(0+) = -\infty$; and again such a situation *can* occur), then one checks similarly that the integral of (3.4) diverges.

Condition (3.4) may also be compared with the recurrence/transience condition of nonsupercritical CBIs [3, Theorem 3(a)]. Examples in which $\Psi_b'(0+) = 0$, $\Psi_m'(0+) > -\infty$ and when (3.4) fails or obtains can be reconstructed from those of [3, Corollary 4] (say by adding a Brownian component to the $\Phi$ featuring there).

*Proof of Corollary 3.2.* Take $\bar{\alpha} = 0$, $\alpha > 0$, $a = 0$ in (3.3) and pass to the limit $\alpha \downarrow 0$ (with $\theta$ as per Remark 3.1) splitting the integral of $\Phi_{\alpha,0}$ into two parts: monotone convergence applies on $(0,\theta)$, and dominated convergence on $(\theta,\infty)$. If (3.4) fails, (3.5) follows at once; otherwise we get $\mathbb{P}_x(\sigma_0 < \infty) \geqslant e^{-\theta x}$, and since $\theta \in (0,\infty)$ is arbitrary, on letting $\theta \downarrow 0$, $\mathbb{P}_x(\sigma_0 < \infty) = 1$. ∎

COROLLARY 3.3. *Assume* $\Psi_m^{-1}(0) \geqslant \Psi_b^{-1}(0) > 0$. *Then* $\mathbb{P}_x(\sigma_0 < \infty) < 1$ *for all* $x \in (0,\infty)$.

*Proof.* Take $\alpha = \bar{\alpha} = a = 0$ in (3.3). ∎

REMARK 3.4. In the context of Theorem 3.1 let us restrict to $\bar{\alpha} = 0$ and $\alpha > 0$. (3.2)–(3.3) were given under the condition $\Psi_m(\Psi_b^{-1}(0)) \leqslant \alpha$, which is a little mysterious. To make it somewhat less so, we shall argue that when $\Psi_m(\Psi_b^{-1}(0)) > 0$, i.e. $\Psi_m^{-1}(0) < \Psi_b^{-1}(0)$, then for $\alpha \in (0, \Psi_m(\Psi_b^{-1}(0)))$ (which implies $\Psi_m^{-1}(\alpha) < \Psi_b^{-1}(0)$) the identity

$$\Phi_\alpha(x) := \mathbb{P}_x[e^{-\alpha\sigma_0}; \sigma_0 < \mathsf{I}] = \int e^{-xz}\,\nu(\mathrm{d}z), \quad x \in [0,\infty),$$

holds for no measure $\nu$ on $\mathcal{B}_{[0,\infty)}$. Suppose otherwise. Then for any starting value $x \in (0,\infty)$, taking $\mu = \delta_x$ in Theorem 2.1, the process $(\Phi_\alpha(Y_t)e^{-\alpha(t\wedge\tau_0)}\mathbb{1}_{\{t\wedge\tau_0<\zeta\}})_{t\in[0,\infty)}$ would be a bounded martingale (essentially by the Markov property; we refer to [20, Proposition 3.1] for the detailed computation) and so by Theorem 2.1(v) and Remark 2.2 we would have $\mathcal{A}\Phi_\alpha = \alpha\Phi_\alpha$ on $(0,\infty)$, i.e.

$$\int (x\Psi_b(z) + \Psi_m(z))e^{-xz}\,\nu(\mathrm{d}z) = \alpha \int e^{-xz}\,\nu(\mathrm{d}z), \quad x \in (0,\infty).$$

Rearranging and applying Tonelli–Fubini gives

$$\int e^{-xz}\Psi_b(z)\,\nu(\mathrm{d}z) = \int_0^\infty e^{-xz} \int_{[0,z]} (\alpha - \Psi_m(y))\,\nu(\mathrm{d}y)\,\mathrm{d}z, \quad x \in (0,\infty);$$

therefore

$$\Psi_b(z)\,\nu(\mathrm{d}z) = \mathrm{d}z \int_{[0,z]} (\alpha - \Psi_m(y))\,\nu(\mathrm{d}y), \qquad z \in [0,\infty).$$

Comparing the sign of the measure on the l.h.s. with the sign of the measure on the r.h.s. we see that $\nu$ must be carried by $[\Psi_m^{-1}(\alpha),\infty)$. In that case, since $\nu \neq 0$, we infer that the r.h.s. has unbounded support and is a negative measure unless it is carried by $\Psi_m^{-1}(\{\alpha\}) = \{\Psi_m^{-1}(\alpha)\}$ (because $\alpha > 0$) in which case the r.h.s. is the zero measure. The former case is in contradiction with the fact that the l.h.s. is nonnegative on $[\Psi_b^{-1}(0),\infty)$. The latter case requires that $\nu$ be also carried by $\Psi_b^{-1}(\{0\})$, therefore $\Psi_b(\Psi_m^{-1}(\alpha)) = 0$, which contradicts $\Psi_b^{-1}(0) > \Psi_m^{-1}(\alpha)$.

This means that we were not "completely dumb" by failing, for $\alpha$ belonging to $(0, \Psi_m(\Psi_b^{-1}(0)))$, to recognize a different would-be (nonnegative) "density kernel" in (3.2); the completely monotone [19, Definition 1.3] character of $(0,\infty) \ni x \mapsto \mathbb{P}_x[e^{-\alpha\sigma_0}; \sigma_0 < \mathsf{l}]$, being true for $\alpha \in [\Psi_m(\Psi_b^{-1}(0)),\infty)$, in fact does not extend to $\alpha \in (0, \Psi_m(\Psi_b^{-1}(0)))$ (by Bernstein's theorem [19, Theorem 1.4]).

We turn to explosions.

THEOREM 3.2 (Explosion times of CBMs). *Let* $\alpha, \bar\alpha \in [0,\infty)$. *Suppose* $\Psi_m(\Psi_b^{-1}(\bar\alpha)) < \alpha$. *If* $\bar\alpha = 0$, *assume further that* $\int_{0+} |\Psi_b|^{-1} < \infty$ (*explosivity condition for the associated CSBP with branching mechanism* $\Psi_b$ [12, *Theorem* 12.3], *in particular then* $\Psi_b^{-1}(0) > 0$). *For* $x \in [0,\infty)$ *put*

$$\begin{aligned} (3.6)\qquad \Psi_{\alpha,\bar\alpha}(x) &:= 1 - \alpha Z_{\alpha,\bar\alpha}(x) \\ &:= 1 - \alpha \int_0^{\Psi_b^{-1}(\bar\alpha)} \frac{\mathrm{d}z}{\bar\alpha - \Psi_b(z)} \exp\left(-xz - \int_0^z \frac{\alpha - \Psi_m(u)}{\bar\alpha - \Psi_b(u)}\,\mathrm{d}u\right). \end{aligned}$$

*Then for* $a \leqslant x$ *from* $[0,\infty)$*:*

(i) *when* $\int_{0+} |\Psi_b|^{-1} < \infty$ *(taking* $\bar\alpha = 0$*),*

$$(3.7)\qquad \mathbb{P}_x[e^{-\alpha\mathsf{l}}; \mathsf{l} < \sigma_a] = \Psi_{\alpha,0}(x) - \frac{\Phi_{\alpha,0}(x)}{\Phi_{\alpha,0}(a)}\Psi_{\alpha,0}(a);$$

(ii) *for* $\bar\alpha > 0$*,*

$$(3.8)\qquad \mathbb{P}_x\Big[\int_0^{\sigma_a\wedge\mathsf{l}} e^{-\alpha s - \bar\alpha\int_0^s Y_u\,\mathrm{d}u}\,\mathrm{d}s\Big] = Z_{\alpha,\bar\alpha}(x) - \frac{\Phi_{\alpha,\bar\alpha}(x)}{\Phi_{\alpha,\bar\alpha}(a)} Z_{\alpha,\bar\alpha}(a).$$

Just as in Theorem 3.1, the case $\bar\alpha = 0$ is of main interest here. However, (3.8) has the following interpretation. The quantity

$$\alpha\mathbb{P}_x\Big[\int_0^{\sigma_a\wedge\mathsf{l}} e^{-\alpha s - \bar\alpha\int_0^s Y_u\,\mathrm{d}u}\,\mathrm{d}s\Big] = \int_0^\infty \alpha e^{-\alpha s}\mathbb{P}_x\Big[e^{-\bar\alpha\int_0^s Y_u\,\mathrm{d}u}; s < \sigma_a \wedge \mathsf{l}\Big]\,\mathrm{d}s$$

is the probability that a CBM starting from $x$, and killed independently at rate $\alpha$, has neither reached $a$, nor exploded, nor has its running lifetime-to-date process $\int_0^{\cdot} Y_u\,\mathrm{d}u$ exceeded an independent exponential random variable of rate $\bar{\alpha}$, before it was killed. In the Lamperti transform it corresponds to, ceteris paribus, $X$ being killed at rate $\alpha$, $L$ being killed at rate $\bar{\alpha}$, and then asking for the probability that starting from $x$, the process $Y$ is killed by $X$ before it has had a chance to be killed by $L$, to reach $a$, or to explode.

*Proof of Theorem 3.2.* $\Psi_{\alpha,\bar{\alpha}} : [0,\infty) \to \mathbb{R}$ is well-defined, finite, strictly increasing, continuous with limit 1 at infinity, which follows easily from the assumptions made. These properties in turn are mirrored in those of $Z_{\alpha,\bar{\alpha}}$. Next, we check that

$$\mathcal{A}\Psi_{\alpha,\bar{\alpha}}(x) = \alpha\Psi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Psi_{\alpha,\bar{\alpha}}(x) - \bar{\alpha}x, \quad x \in (0,\infty),$$

which is again just a straightforward computation:

$$\begin{aligned}
\mathcal{A}\Phi_{\alpha,\bar{\alpha}}(x) &= \mathcal{L}^{\Psi_m}\Psi_{\alpha,\bar{\alpha}}(x) + x\mathcal{L}^{\Psi_b}\Psi_{\alpha,\bar{\alpha}}(x) \\
&= -\alpha \int_0^{\Psi_b^{-1}(\bar{\alpha})} \frac{\mathrm{d}z}{\bar{\alpha} - \Psi_b(z)} \exp\left(-xz - \int_0^z \frac{\alpha - \Psi_m(u)}{\bar{\alpha} - \Psi_b(u)}\,\mathrm{d}u\right) \\
&\qquad\qquad \times(\Psi_m(z) + x\Psi_b(z)) \\
&= \alpha\Psi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Psi_{\alpha,\bar{\alpha}}(x) - \alpha - \bar{\alpha}x \\
&\quad + \alpha \int_0^{\Psi_b^{-1}(\bar{\alpha})} \mathrm{d}z \frac{\alpha - \Psi_m(z)}{\bar{\alpha} - \Psi_b(z)} \exp\left(-xz - \int_0^z \frac{\alpha - \Psi_m(u)}{\bar{\alpha} - \Psi_b(u)}\,\mathrm{d}u\right) \\
&\quad + \alpha x \int_0^{\Psi_b^{-1}(\bar{\alpha})} \mathrm{d}z \exp\left(-xz - \int_0^z \frac{\alpha - \Psi_m(u)}{\bar{\alpha} - \Psi_b(u)}\,\mathrm{d}u\right) \\
&= \alpha\Psi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Psi_{\alpha,\bar{\alpha}}(x) - \alpha - \bar{\alpha}x \\
&\quad - \alpha\left[\exp\left(-xz - \int_0^z \frac{\alpha - \Psi_m(u)}{\bar{\alpha} - \Psi_b(u)}\,\mathrm{d}u\right)\Bigg|_{z=0}^{\Psi_b^{-1}(\bar{\alpha})}\right] \\
&= \alpha\Psi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}x\Psi_{\alpha,\bar{\alpha}}(x) - \bar{\alpha}x.
\end{aligned}$$

It follows from Theorem 2.1(v) & Remark 2.2 that for any $a \in (0,\infty)$, the process

$$\begin{aligned}
M_t &:= \Psi_{\alpha,\bar{\alpha}}(Y(t))e^{-\alpha t - \bar{\alpha}\int_0^t Y_s\,\mathrm{d}s} - \Psi_{\alpha,\bar{\alpha}}(x) + \bar{\alpha}\int_0^t e^{-\alpha s - \bar{\alpha}\int_0^s Y_u\,\mathrm{d}u} Y_s\,\mathrm{d}s \\
&= -\alpha\left(Z_{\alpha,\bar{\alpha}}(Y(t))e^{-\alpha t - \bar{\alpha}\int_0^t Y_s\,\mathrm{d}s} - Z_{\alpha,\bar{\alpha}}(x) + \int_0^t e^{-\alpha s - \bar{\alpha}\int_0^s Y_u\,\mathrm{d}u}\,\mathrm{d}s\right),
\end{aligned}$$

given for $t \in [0,\zeta)$, stopped at $\tau_a$, is a local martingale on $[0,\zeta)$ under $\mathbb{P}^{\delta_x}$, for $x \in [a,\infty)$.

Because martingales have a constant expectation (the first form of $M$ is the most convenient, exploiting $\lim_\infty \Psi_{\alpha,0} = 1$), and from (3.3), setting $\bar{\alpha} = 0$ we get the $\mathbb{P}_x$-Laplace transform for the explosion time $\mathsf{l}$ on $\{\mathsf{l} < \sigma_a\}$, $x \in [a, \infty)$, $a \in (0, \infty)$. Letting $a \downarrow 0$ gives (3.7) also for $a = 0$.

For $\bar{\alpha} > 0$, we get (3.8) similarly (but now the second form of $M$ appears to be more handy, using $\lim_\infty Z_{\alpha,\bar{\alpha}} = 0$). ∎

COROLLARY 3.4. *The CSBP with branching mechanism* $\Psi_b$ *is explosive* (*an equivalent integral condition is* $\int_{0+} |\Psi_b|^{-1} < \infty$ [12, *Theorem* 12.3]) *iff the CBM process* $Y$ *is explosive* (*i.e.* $\mathbb{P}^\mu(\zeta < \infty) > 0$ *for some, equivalently all, initial distributions* $\mu$ *that are not concentrated at* 0).

*Proof.* We already know that if for some initial distribution $\mu$ (not concentrated at 0), $\mathbb{P}^\mu(\zeta < \infty) > 0$, then the CSBP with branching mechanism $\Psi_b$ is explosive (Corollary 2.1). Now suppose the latter, i.e. $\int_{0+} |\Psi_b|^{-1} < \infty$. Taking $a = 0$ in (3.7) (with an arbitrary $\alpha \in (\Psi_m(\Psi_b^{-1}(\bar{\alpha})) \vee 0, \infty)$) we get $\mathbb{P}_x(\mathsf{l} < \infty) > 0$ for $x \in (0, \infty)$ (just because $\Psi_{\alpha,0}$ is strictly increasing, while $\Phi_{\alpha,0}$ is strictly decreasing). ∎

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Matija Vidmar
Faculty of Mathematics and Physics
Department of Mathematics
University of Ljubljana
Jadranska ulica 21
1000 Ljubljana, Slovenia
*E-mail*: matija.vidmar@fmf.uni-lj.si

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