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GENERALIZATIONS OF THE FOURTH MOMENT THEOREM

BY

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Abstract. Azmoodeh et al. established a criterion regarding convergence of the *second* and *other* even moments of random variables in a Wiener chaos with fixed order guaranteeing the central convergence of the random variables. This was a major step in studies of the fourth moment theorem. In this paper, we provide further generalizations of the fourth moment theorem by building on their ideas. More precisely, further criteria implying central convergence are provided: (i) the convergence of the *fourth* and *any other* even moment, (ii) the convergence of the *sixth* and *some other* even moments.

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1. INTRODUCTION

The fourth moment theorem (Nualart–Peccati criterion), discovered by Nualart and Peccati [9], provides a concise criterion for central convergence of random variables $\{Z_n\}_{n=1}^{\infty}$ belonging to a Wiener chaos of fixed order. More precisely, Nualart and Peccati showed that if $\boldsymbol{E}[Z_n^2] \to 1$ and $\boldsymbol{E}[Z_n^4] \to 3$ as $n \to \infty$, then $\{Z_n\}_{n=1}^{\infty}$ converges in law to a standard Gaussian random variable N. Subsequently, many researchers began studying generalizations and applications of the theorem. For example, Peccati and Tudor [11] extended it to the multidimensional case, and Nualart and Ortiz-Latorre [8] provided another proof for the theorem in terms of Malliavin calculus. Nourdin and Peccati [5] established Berry–Esséen bounds in the Breuer–Major central limit theorem by combining Malliavin calculus and Stein's method.

An extension by Ledoux [3] was a major step in the ongoing study of the fourth moment theorem. He provided another proof for the fourth moment theorem in the framework of diffusive Markov generators inspired by a proof based on Malliavin

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calculus. More sophisticated and generalized results were given by Azmoodeh, Campese, and Poly [1]. These papers were devoted to answering the following question stated in [2] by Azmoodeh, Malicet, Mijoule, and Poly:

What moment conditions ensure central convergence?

This paper is also devoted to answering this question.

In order to be more precise, we introduce some notation. Let $X = \{X(h)\}_{h\in\mathfrak{H}}$ be an isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$. For every $p \in \mathbb{N} \cup \{0\}$, we write $\mathcal{H}_p$ to denote the pth Wiener chaos of X. For precise definitions, see [7, 6]. Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathcal{H}_p$ for some integer $p \geqslant 2$. We denote by $\mathcal{I}$ a finite subset of even numbers.

The question above may be reduced to equivalence of (CL) and (CM) for a finite subset $\mathcal{I}$ of even numbers, where

(CL) $Z_n \to N$ in law as $n \to \infty$.

(CM) $\boldsymbol{E}[Z_n^{2i}] \to \boldsymbol{E}[N^{2i}]$ as $n \to \infty$ for all $2i \in \mathcal{I}$.

Of course, the fourth moment theorem involves equivalence of (CL) and (CM) for $\mathcal{I} = \{2, 4\}$, and after it was shown, some researchers wondered whether the equivalence held for *any* set of two distinct even numbers. The authors of [2] showed the equivalence of (CL) and (CM) for $\mathcal{I} = \{2, 2k\}$ with $2k \geqslant 4$. One of the ingredients in their proofs was a formulation of central convergence in terms of polynomials (this will be stated in Lemma 2.1).

In this paper, we build on their formulation to suggest directions for generalization of the fourth moment theorem. Although we cannot provide a full answer to the above question, we provide interesting examples of central convergence based on a lemma in [2]. Our main theorem is as follows:

THEOREM 1.1. *Let $\mathcal{I}$ be any of the following:*

(1) $\mathcal{I} = \{2, 2k\}$, *where $2k \geqslant 4$ is an arbitrary even integer.*

(2) $\mathcal{I} = \{4, 2k\}$, *where $2k \geqslant 6$ is an arbitrary even integer.*

(3) $\mathcal{I} = \{6, 8\}, \{6, 10\}$.

(4) $\mathcal{I} = \{6, 12, 14, 2k\}$, *where $2k \geqslant 16$ is an arbitrary even integer.*

(5) $\mathcal{I} = \{6, 12, 18, 30, 32, 2k\}$, *where $2k \geqslant 34$ is an arbitrary even integer.*

Then (CL) *and* (CM) *for $\mathcal{I}$ are equivalent.*

For the readers' convenience, this theorem contains preceding results: assertion (1), a part of assertion (2) ($\mathcal{I} = \{4, 6\}$, $\{4, 8\}$, $\{4, 10\}$) and assertion (3) have already been demonstrated in [2, Theorem 1.2 and Section 5]. The first contribution of this paper, in (2), is that convergence of the *fourth* and *any* even moment implies central convergence. We also show that the method of [2] is only effective in cases (1), (2), and (3) (Proposition 3.1). Assertions (4) and (5) are entirely new and are our third contribution.

Some remarks on our results are in order.

• The case $\mathcal{I} = \{6, 12\}$ cannot be treated with the method in [2] due to Proposition 3.1; it is the first truely nontrivial case after cases (1), (2), and (3). Hence the second smallest number in assertions (4) and (5) should be greater than or equal to 12. If we replace 12 by 10, we obtain the equivalence due to assertion (3).

• At this stage, we have no counterexample for $\mathcal{I} = \{6, 12\}$.

• Assertions (4) and (5) are nontrivial and their proofs are interesting from the viewpoint of the properties of the polynomials that appear in the proof. For more discussion on our main theorem, see Section 4.

The remainder of this paper is organized as follows. Section 2 reviews the principal part of [2]. Section 3 is devoted to proving our main theorem. In Section 4, we discuss our main theorem. Section 5 investigates asymptotic characteristics of the hypergeometric function.

Throughout this paper, we use the following notation. Let N be a standard Gaussian random variable with the density function $w(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Set $\mu_i = \boldsymbol{E}[N^{2i}] = (2i-1)!!$ for $i \in \mathbb{N} \cup \{0\}$ with the convention $(-1)!! = 0$. We consider the following functions:

- The Hermite polynomials: $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$ for $n \in \mathbb{N} \cup \{0\}$.
- The Gamma function: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$ for $a > 0$.
- The Beta function: $B(a, b) = \int_0^1 (1-u)^{a-1} u^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ for $a, b > 0$.
- The hypergeometric function:

$$F(a, b, c; z) = \frac{1}{B(a, c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-uz)^{-b}\, du$$

  for $0 < a < c$ and $|z| < 1$.

We define $\{\kappa_i(m)\}_{m \geqslant i \geqslant 2}$ and $\{\xi_i(m)\}_{m,i \geqslant 2}$ by

$$\begin{aligned} \kappa_i(m) &= B\left(i-1, 1/2\right) F(i-1, -(m-i), i-1/2, 1/2) \\ &= \int_0^1 u^{i-2}(1-u)^{-1/2}(1-u/2)^{m-i}\, du, \end{aligned} \tag{1.1}$$

and

$$\xi_i(m) = \begin{cases} \frac{(m-1)!}{(m-i)!} \kappa_i(m), & 2 \leqslant i \leqslant m, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

2. REVIEW OF AZMOODEH ET AL. [2]

In this section, we summarize the most important part of [2] and extend it. For every $i \geqslant 2$, we define even polynomials W_i and ψ_i of degree $2i$ by

$$W_i(x) = (2i-1)\Phi[H_i H_{i-2}](x), \qquad \psi_i(x) = \boldsymbol{E}[W_i(xN)],$$

where Φ is defined as

$$\Phi[Q](x) = x \int_0^x Q(t)\,dt - Q(x).$$

Note that W_i is monic. Let T be a monic even polynomial of degree $2k \geqslant 4$ of the form

$$T(x) = \sum_{i=2}^{k} \alpha_i W_i(x) \tag{2.1}$$

for some $\alpha_2, \ldots, \alpha_{k-1} \in \mathbb{R}$ and $\alpha_k = 1$. The next lemma is a major result of [2].

LEMMA 2.1 ([2, Lemma 4.2]). *Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathcal{H}_p$ for some integer $p \geqslant 2$, and let T be a monic even polynomial of degree $2k \geqslant 4$ of the form* (2.1) *with positive α_2, nonnegative $\alpha_3, \ldots, \alpha_{k-1}$, and $\alpha_k = 1$. Then $Z_n \to N$ in law as $n \to \infty$ if and only if $\boldsymbol{E}[T(Z_n)] \to 0$ as $n \to \infty$.*

Lemma 2.1 tells us that we can obtain central convergence of $\{Z_n\}_{n=1}^{\infty}$ by finding a suitable polynomial T. In general, a monic even polynomial T of degree $2k \geqslant 4$ is defined as

$$T(x) = \sum_{i=1}^{k} a_i x^{2i} + a_0 \tag{2.2}$$

for some $a_0, \ldots, a_{k-1}$ and $a_k = 1$. To use Lemma 2.1, we seek to determine what conditions on $a_0, \ldots, a_k$ imply that T is of the form (2.1) with some $\alpha_2, \ldots, \alpha_k$, and we provide a formula for calculating $\alpha_2, \ldots, \alpha_k$ from $a_0, \ldots, a_k$. We know that

$$\boldsymbol{E}[T(N)] = \lim_{n\to\infty} \boldsymbol{E}[T(Z_n)] = 0$$

if $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{H}_p$ satisfies $\boldsymbol{E}[Z_n^{2i}] \to \mu_i$ as $n \to \infty$. This is equivalent to $\phi(1) = 0$, where $\phi(x) = \boldsymbol{E}[T(xN)]$.

PROPOSITION 2.1. *Let T be an even polynomial of degree $2k \geqslant 4$ and set $\phi(x) = \boldsymbol{E}[T(xN)]$. The following are equivalent:*

(1) $\phi(1) = \phi'(1) = 0$*; in other words,*

$$\sum_{i=1}^{k} a_i \mu_i + a_0 = 0, \qquad \sum_{i=1}^{k} a_i 2i \mu_i = 0. \tag{2.3}$$

(2) *There exist constants $\alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that* (2.1) *holds.*

Proof. In this proof, we use $\psi_i(1) = \psi_i'(1) = 0$ for $i \geqslant 2$ (see [2, Lemma 4.1]).

We first show that (1) implies (2). Since W_i is an even polynomial of degree $2i$, there exists a unique expansion of the form

$$T(x) = \sum_{i=2}^{k} \alpha_i W_i(x) + \beta x^2 + \gamma.$$

We see that $\beta = \gamma = 0$ as follows. We have

$$\phi(x) = \sum_{i=2}^{k} \alpha_i \boldsymbol{E}[W_i(xN)] + \beta \boldsymbol{E}[(xN)^2] + \gamma = \sum_{i=2}^{k} \alpha_i \psi_i(x) + \beta x^2 + \gamma.$$

Since $\phi(1) = \phi'(1) = 0$ and $\psi_i(1) = \psi_i'(1) = 0$ for $i \geqslant 2$, it follows that $\beta + \gamma = 0$ and $2\beta = 0$ so $\beta = \gamma = 0$. Hence, (2) holds.

Next, we show that (2) implies (1). The assumption implies that $\phi(x) = \sum_{i=2}^{k} \alpha_i \psi_i(x)$. This expression and the identity $\psi_i(1) = \psi_i'(1) = 0$ yield (1). ∎

Hereafter, we assume $\phi(1) = \phi'(1) = 0$. Then, as a result of Proposition 2.1, $a_0, \ldots, a_k$ in (2.2) and $\alpha_2, \ldots, \alpha_k$ in (2.1) are related. We will find an explicit formula for $\alpha_2, \ldots, \alpha_k$ in terms of $a_0, \ldots, a_k$. More precisely, setting $c_i = (2i-1)i!(i-2)!$ for $i \geqslant 2$, we demonstrate the next proposition, an analogue of [2, Proposition 4.1] proved in a similar manner.

PROPOSITION 2.2. *For every* $2 \leqslant i \leqslant k$,

$$\alpha_i c_i = \frac{1}{2^{i-1}} \sum_{m=i}^{k} \frac{m!\kappa_i(m)}{(m-i)!} a_m \mu_m.$$

Here, $\{\kappa_i(m)\}_{m \geqslant i \geqslant 2}$ *are defined by* (1.1).

The next corollary follows immediately from Proposition 2.2. It will be used in Section 3 and will play an important role in the proof of the main theorem.

COROLLARY 2.1. *Let* $1 \leqslant l < k$ *and assume that* $a_m = 0$ *for all* $1 \leqslant m \leqslant l-1$. *Then, for every* $2 \leqslant i \leqslant k$,

$$\alpha_i c_i = \frac{1}{2^{i-1}} \sum_{m=l+1}^{k} \{\xi_i(m) - \xi_i(l)\} m \mu_m a_m.$$

Here, $\{\xi_i(m)\}_{i,m \geqslant 2}$ *are defined by* (1.2).

Proof. From Proposition 2.2, for all $2 \leqslant i \leqslant k$, it follows that

$$\alpha_i c_i 2^{i-1} = \sum_{m=i}^{k} \xi_i(m) m \mu_m a_m = \sum_{m=2}^{k} \xi_i(m) m \mu_m a_m = \sum_{m=l}^{k} \xi_i(m) m \mu_m a_m.$$

In the above, we used $\xi_i(m) = 0$ for $2 \leqslant m \leqslant i-1$ and $a_m = 0$ for $1 \leqslant m \leqslant l-1$. Since $\phi'(1) = 0$ (see (2.3)) and $a_m = 0$ for $1 \leqslant m \leqslant l-1$ imply

$$0 = \sum_{m=1}^{k} m\mu_m a_m = \sum_{m=l}^{k} m\mu_m a_m,$$

we have

$$\xi_i(l) l\mu_l a_l = - \sum_{m=l+1}^{k} \xi_i(l) m\mu_m a_m.$$

Substituting this equality into $\alpha_i c_i 2^{i-1}$ yields the assertion. ∎

For the readers' convenience, we provide a proof of Proposition 2.2. For details, see [2, Appendix A]. We introduce even polynomials Q and R of degree $2(k-1) \geqslant 2$ as

$$Q(x) = \sum_{i=2}^{k} \alpha_i (2i-1) H_i(x) H_{i-2}(x), \qquad R(x) = \sum_{i=1}^{k} a_i \mu_i \sum_{r=0}^{i-1} \frac{x^{2r}}{\mu_r}.$$

Then $\Phi[Q] = T = \Phi[R]$ from direct computation, and $Q = R$ as a consequence of [2, Lemma A.2].

LEMMA 2.2. *For all* $1 \leqslant n \leqslant k-1$,

$$\int_{-\infty}^{\infty} Q(x) H_{2n}(x) w(x)\, dx = \frac{(2n)!}{(n-1)!(n+1)!} \sum_{m=n+1}^{k} \frac{\alpha_m c_m}{(m-(n+1))!},$$
$$\int_{-\infty}^{\infty} R(x) H_{2n}(x) w(x)\, dx = 2^n \sum_{m=n+1}^{k} a_m \mu_m \sum_{r=n}^{m-1} \frac{r!}{(r-n)!}.$$

Proof. We refer to [2, Lemma A.1]. The product formula and the orthogonality of Hermite polynomials imply that

$$\int_{-\infty}^{\infty} H_i(x) H_{i-2}(x) H_{2n}(x) w(x)\, dx = \frac{(2n)!}{(n+1)!(n-1)!} \frac{i!(i-2)!}{(i-(n+1))!} 1_{n+1 \leqslant i}.$$

Hence, the first equality holds. The second assertion follows from

$$\frac{1}{\mu_r} \int_{-\infty}^{\infty} x^{2r} H_{2n}(x) w(x)\, dx = \frac{1}{\mu_r} \frac{(2r)!}{2^{r-n}(i-n)!} 1_{n \leqslant r} = \frac{2^n r!}{(r-n)!} 1_{n \leqslant r}. \quad ∎$$

Proof of Proposition 2.2. Set

$$f(x) = \sum_{i=2}^{k} \frac{\alpha_i c_i}{(i-1)!} x^{i-1}, \qquad g(x) = \sum_{i=1}^{k} a_i \mu_i \sum_{r=0}^{i-1} x^r.$$

Since $f^{(n)}(0) = \alpha_{n+1}c_{n+1}$ for every $1 \leqslant n \leqslant k-1$, we look for other expressions of $f^{(n)}(0)$. First, we show that

$$f(1-2x) - f(1) = \int_0^1 (1-u)^{-1/2} u^{-1} \frac{d}{du}\{ug(1-ux)\}\, du \tag{2.4}$$

and next we consider the nth derivatives of both sides at $x = 1/2$. We obtain the assertion as a consequence.

For every $n \in \mathbb{N}$,

$$f^{(n)}(x) = \sum_{i=n+1}^{k} \frac{\alpha_i c_i}{(i-(n+1))!} x^{i-(n+1)},$$
$$g^{(n)}(x) = \sum_{i=n+1}^{k} a_i \mu_i \sum_{r=n}^{i-1} \frac{r!}{(r-n)!} x^{r-n}.$$

Combining Lemma 2.2 with the above yields

$$\frac{(2n)!}{(n-1)!(n+1)!} f^{(n)}(1) = 2^n g^{(n)}(1).$$

Since $\frac{(2n)!}{(n-1)!(n+1)!} = \frac{2^{2n}}{n+1} \frac{1}{B(1/2,n)}$ as a consequence of [10, (5.4.6), (5.5.5) and (5.12.1)],

$$f^{(n)}(1) = \frac{n+1}{2^n} B(1/2, n) g^{(n)}(1).$$

By the above,

$$\begin{aligned} f(1-2x) - f(1) &= \sum_{n=1}^{k-1} \frac{f^{(n)}(1)}{n!}(-2x)^n \\ &= \sum_{n=1}^{k-1} \frac{1}{n!}\left(\frac{n+1}{2^n} \int_0^1 (1-u)^{-1/2} u^{n-1}\, du \right) g^{(n)}(1)(-2)^n x^n \\ &= \int_0^1 (1-u)^{-1/2} \left(\sum_{n=1}^{k-1} \frac{g^{(n)}(1)}{n!}(n+1)u^{n-1}\,(-1)^n x^n \right) du. \end{aligned}$$

Here, noting that $g(1) = \sum_{i=1}^{k} a_i \mu_i i = 0$ and applying the Taylor formula to $g(1-ux)$ yields

$$\frac{d}{du}\{ug(1-ux)\} = \frac{d}{du}\left\{ u \sum_{n=1}^{k-1} \frac{g^{(n)}(1)}{n!}(-ux)^n \right\} = \sum_{n=1}^{k-1} \frac{g^{(n)}(1)}{n!}(n+1)(-ux)^n.$$

The two equalities imply (2.4).

Next, we consider the nth derivative of (2.4) at $x = 1/2$. Substituting

$$\frac{d}{du}\{ug(1-ux)\} = \frac{d}{du}\sum_{m=1}^{k} \alpha_m \mu_m u \frac{1-(1-ux)^m}{1-(1-ux)}$$
$$= \sum_{m=1}^{k} a_m \mu_m \cdot m(1-ux)^{m-1}$$

into (2.4) yields

$$f(1-2x) - f(1) = \sum_{m=1}^{k} a_m \mu_m \cdot m \int_0^1 (1-u)^{-1/2} u^{-1} (1-ux)^{m-1}\, du.$$

Furthermore, for every $n \leqslant m-1$,

$$\frac{d^n}{dx^n}(1-u)^{-1/2}u^{-1}(1-ux)^{m-1}$$
$$= \frac{(m-1)!}{(m-1-n)!}(1-u)^{-1/2}u^{-1}(-u)^n(1-ux)^{m-1-n}$$

and

$$\sup_{x\in(1/4,3/4)} |\text{RHS above}| \leqslant \frac{(m-1)!}{(m-1-n)!}(1-u)^{-1/2}u^{n-1}\left(1-\frac{u}{4}\right)^{m-1-n}.$$

Hence, by Lebesgue's differentiation theorem,

$$(-2)^n f^{(n)}(0) = (-1)^n \sum_{m=n+1}^{k} a_m \mu_m \cdot \frac{m(m-1)!}{(m-1-n)!}\kappa_{n+1}(m),$$

where $\kappa_{n+1}(m)$ is the constant defined by (1.1). This is the conclusion of the proposition. ■

3. EXPRESSION OF T AND PROOF OF MAIN THEOREMS

In this section, we consider the positivity of $\{\alpha_i\}_{2\leqslant i\leqslant k}$ in several cases and prove our main theorem. Set $k \geqslant 2$, and write $\tilde{\alpha}_i = \tilde{\alpha}_i(k) = \frac{\alpha_i c_i 2^{i-1}}{k\mu_k}$. From Corollary 2.1, we have

$$\tilde{\alpha}_i = \tilde{\alpha}_i(k) = \sum_{m=l+1}^{k} \{\xi_i(m) - \xi_i(l)\}\frac{m\mu_m}{k\mu_k} a_m. \tag{3.1}$$

3.1. $T(x) = x^{2k} + a_l x^{2l} + a_0$ for $2k > 2l \geqslant 2$. Then the function $\phi(x) = \boldsymbol{E}[T(xN)]$ satisfies $\phi(1) = \phi'(1) = 0$ if and only if $a_l = -\frac{k\mu_k}{l\mu_l}$ and $a_0 = (k/l-1)\mu_k$.

PROPOSITION 3.1. *The polynomial $T(x) = x^{2k} + a_l x^{2l} + a_0$ is expressed as in* (2.1) *with positive coefficients $\alpha_2, \ldots, \alpha_k$ if and only if $k > l = 1$ or $k > l = 2$ or $(k, l) = (4, 3), (5, 3)$.*

Proof. Substituting $a_{l+1} = \cdots = a_{k-1} = 0$ and $a_k = 1$ into (3.1) we have $\tilde{\alpha}_i(k) = \xi_i(k) - \xi_i(l)$ for every $2 \leqslant i \leqslant k$. Recall $\xi_i(l) \neq 0$ only for $2 \leqslant i \leqslant l$ due to (1.2).

We can obtain the "only if" part of the assertion by focusing on α_2. If $l = 3$, then

$$\tilde{\alpha}_2(k) = \xi_2(k) - \xi_2(3) \leqslant \xi_2(6) - \xi_2(3) = \frac{166}{63} - \frac{8}{3} < 0$$

for all $k \geqslant 6$ (see Propositions 5.2(3) and 5.1). If $l \geqslant 4$, then $\tilde{\alpha}_2(k) = \xi_2(k) - \xi_2(l)$ < 0 for all $k > l$ (see Proposition 5.2(3)). This yields the "only if" part.

Now, we show the "if" part. If $l = 1$, then $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for all $2 \leqslant i \leqslant k$. If $l = 2$, then

$$\tilde{\alpha}_i(k) = \begin{cases} \xi_i(k), & 3 \leqslant i \leqslant k, \\ \xi_2(k) - \xi_2(2), & i = 2. \end{cases}$$

Hence we have $\tilde{\alpha}_i(k) > 0$ for $i = 2$ (resp. $3 \leqslant i \leqslant k$) due to Proposition 5.2(1) (resp. $\xi_i(k) > 0$). If $l = 3$, then $\tilde{\alpha}_i(k) = \xi_i(k) - \xi_i(3) > 0$ for $k = 4, 5$ for the same reason as in the case $l = 2$. This completes the proof. ∎

3.2. $T(x) = x^{2k} + ax^{14} + bx^{12} + a_3x^6 + a_0$ for $k \geqslant 8$. Here, a_3 and a_0 are chosen to ensure that $\phi(1) = \phi'(1) = 0$. Then, from Corollary 2.1,

$$\tilde{\alpha}_i(k) = \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(7) - \xi_i(3)\}\frac{7\mu_7}{k\mu_k}a + \{\xi_i(6) - \xi_i(3)\}\frac{6\mu_6}{k\mu_k}b.$$

In what follows, we consider the case $\alpha_7 = 0$ and $\alpha_6 = 0$ and show that $\alpha_k, \ldots,$ $\alpha_8, \alpha_5, \alpha_4, \alpha_3, \alpha_2$ are positive. In this case, we have

$$\frac{7\mu_7}{k\mu_k}a = -\frac{\xi_7(k)}{\xi_7(7)}, \qquad \frac{6\mu_6}{k\mu_k}b = \frac{\xi_7(k)}{\xi_6(7)}\frac{\xi_6(7)}{\xi_6(6)} - \frac{\xi_6(k)}{\xi_6(6)}.$$

Hence,

$$\begin{aligned} \tilde{\alpha}_i(k) &= \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(7) - \xi_i(3)\}\left(-\frac{\xi_7(k)}{\xi_7(7)}\right) \\ &\quad + \{\xi_i(6) - \xi_i(3)\}\left(\frac{\xi_7(k)}{\xi_7(7)}\frac{\xi_6(7)}{\xi_6(6)} - \frac{\xi_6(k)}{\xi_6(6)}\right) \\ &= \xi_i(k) + \left[-\frac{\xi_i(7) - \xi_i(3)}{\xi_7(7)} + \frac{\xi_6(7)(\xi_i(6) - \xi_i(3))}{\xi_7(7)\xi_6(6)}\right]\xi_7(k) \\ &\quad + \left[-\frac{\xi_i(6) - \xi_i(3)}{\xi_6(6)}\right]\xi_6(k) - \xi_i(3). \end{aligned}$$

Since $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for $8 \leqslant i \leqslant k$, we consider $\tilde{\alpha}_i(k)$ for $i = 2, 3, 4, 5$.

LEMMA 3.1. *Let $k \geqslant 8$. Then $\tilde{\alpha}_i(k) > 0$ for any $i = 2, 3, 4, 5$.*

Proof. For $8 \leqslant k < 3000$, the assertion follows by direct computation using Mathematica. For the source code used, see Listing 1. Next we show the assertion for $k \geqslant 3000$. As a consequence of Proposition 5.3, $\{\xi_i(k)\}_{k=2}^{\infty}$ converges to $2^{i-1}(i-2)!$ as $k \to \infty$, and we estimate the error of this convergence. Setting $r_i(k) = \xi_i(k) - 2^{i-1}(i-2)!$ yields

$$\tilde{\alpha}_i(k) = \begin{cases} \xi_2(k) + \frac{1}{3072}\xi_6(k) + \frac{1}{15360}\xi_7(k) - \frac{8}{3}, & i = 2, \\ \xi_3(k) - \frac{29}{768}\xi_6(k) + \frac{121}{7680}\xi_7(k) - \frac{8}{3}, & i = 3, \\ \xi_4(k) - \frac{7}{32}\xi_6(k) + \frac{1}{12}\xi_7(k), & i = 4, \\ \xi_5(k) - \frac{5}{8}\xi_6(k) + \frac{29}{160}\xi_7(k), & i = 5, \end{cases}$$
$$= \begin{cases} r_2(k) + \frac{1}{3072}r_6(k) + \frac{1}{15360}r_7(k) + \frac{1}{12}, & i = 2, \\ r_3(k) - \frac{29}{768}r_6(k) + \frac{121}{7680}r_7(k) + \frac{280}{3}, & i = 3, \\ r_4(k) - \frac{7}{32}r_6(k) + \frac{1}{12}r_7(k) + 488, & i = 4, \\ r_5(k) - \frac{5}{8}r_6(k) + \frac{29}{160}r_7(k) + 1008, & i = 5, \end{cases}$$

which implies

$$\tilde{\alpha}_i(k) \geqslant \begin{cases} -(|r_2(k)| + \frac{1}{3072}|r_6(k)| + \frac{1}{15360}|r_7(k)|) + \frac{1}{12}, & i = 2, \\ -(|r_3(k)| + \frac{29}{768}|r_6(k)| + \frac{121}{7680}|r_7(k)|) + \frac{280}{3}, & i = 3, \\ -(|r_4(k)| + \frac{7}{32}|r_6(k)| + \frac{1}{12}|r_7(k)|) + 488, & i = 4, \\ -(|r_5(k)| + \frac{5}{8}|r_6(k)| + \frac{29}{160}|r_7(k)|) + 1008, & i = 5, \end{cases}$$
$$> 0.$$

The last inequality follows from Proposition 5.3. ∎

Listing 1. Proof of Lemma 3.1

```
kappa[i_,m_]:=Beta[i-1,1/2]*Hypergeometric2F1[i-1,-(m-i),
    i-1/2,1/2];
xi[i_,m_]:=(m-1)!/(m-i)!*kappa[i,m]/;m>=i;
xi[i_,m_]:=0/;m<i;
tildeA[i_,k_]:=xi[i,k]+(-(xi[i,7]-xi[i,3])/xi[7,7]+
     xi[6,7]*(xi[i,6]-xi[i,3])/(xi[7,7]*xi[6,6]))*xi[7,k]+
     (-(xi[i,6]-xi[i,3])/xi[6,6])*xi[6,k]-xi[i,3]

(*Are tildeA[i,k]>0 for i=2,3,4,5 and k<3001?*)
Table[Map[tildeA[#,k]&,{2,3,4,5}],{k,8,3000}];
AllTrue[Flatten[%],Positive]

(*Are tildeA[i,k]>0 for i=2,3,4,5 and k>3000?*)
Map[tildeA[#,k]&,{2,3,4,5}]/.Array[xi[#,k]->2^(#-1)*(#-2)!+
     r[#,k]&,7,2]//Expand;
CoefficientArrays[%,Map[r[#,k]&,Range[2,16]]]//Normal;
Map[Abs,%];
%[[1]]+%[[2]].Map[-r[#,k]&,Range[2,16]];
%/.MapThread[#1->2^#2&,{Map[r[#,k]&,Range
    [2,16]],{-18,-9,-5,-2,2,6,10,14,18,23,28,32,37,42,47}}];
AllTrue[Flatten[%],Positive]
```

3.3. $T(x) = x^{2k} + ax^{32} + bx^{30} + cx^{18} + dx^{12} + a_3x^6 + a_0$ for $k \geqslant 17$. Here, a_3 and a_0 are chosen to ensure that $\phi(1) = \phi'(1) = 0$. Then, from Corollary 2.1,

$$\tilde{\alpha}_i(k) = \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(16) - \xi_i(3)\}\frac{16\mu_{16}}{k\mu_k}a + \{\xi_i(15) - \xi_i(3)\}\frac{15\mu_{15}}{k\mu_k}b + \{\xi_i(9) - \xi_i(3)\}\frac{9\mu_9}{k\mu_k}c + \{\xi_i(6) - \xi_i(3)\}\frac{6\mu_6}{k\mu_k}d.$$

Here, we choose a, b, c, d to ensure that $\alpha_i = 0$ for all $i \in \{6, 7, 12, 13\}$. It follows from this expression that $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for any $17 \leqslant i \leqslant k$, and we can demonstrate the next lemma in the same manner as Lemma 3.1.

LEMMA 3.2. *Let $k \geqslant 17$. For every $i \in \{2, \dots, 16\} \setminus \{6, 7, 12, 13\}$, we have $\tilde{\alpha}_i(k) > 0$.*

3.4. Proof of the main theorem. Proposition 3.1 implies that $T(x) = x^{2k} - \frac{k\mu_k}{l\mu_l}x^{2l} + (\frac{k}{l} - 1)\mu_k$ can be written as in (2.1) with positive $\alpha_2, \dots, \alpha_k$ for $k > l = 1$ or $k > l = 2$ or $(k, l) = (4, 3), (5, 3)$. Combining this fact with Lemma 2.1 yields assertions (1)–(3).

In the same manner, combining Lemmas 2.1, 3.1 and 3.2 yields (4) and (5). ∎

4. DISCUSSION OF THE MAIN THEOREM

After [9, 2] and the present paper, the next conjecture is still open:

CONJECTURE 4.1. *Let $\mathcal{I} = \{2l, 2k\}$ for $6 \leqslant 2l < 2k$. Then* (CL) *and* (CM) *for $\mathcal{I}$ are equivalent.*

As stated in Section 1, we cannot show Conjecture 4.1 by the method of [2] (see Proposition 3.1). In this section, we discuss this in more detail. Write $\mathcal{I} = \{2l_1, \dots, 2l_M, 2k\}$ with $2 \leqslant 2l_1 < \cdots < 2l_M < 2k$.

Since our proof of (CM)⇒(CL) relies on Lemma 2.1, T should be expressed as in (2.1) and $\phi(1) = \phi'(1) = 0$ should be satisfied (see Proposition 2.1). Note that the conditions $\phi(1) = \phi'(1) = 0$ give a system of two linear equations (2.3) with k unknowns $a_0, a_1, \dots, a_{k-1}$ ($a_k = 1$ because T is monic). Since we should obtain $\boldsymbol{E}[T(Z_n)] \to 0$ from convergence of moments in $\mathcal{I}$, we should set $a_i = 0$ for $i \in \{1, \dots, k-1\} \setminus \{l_1, \dots, l_M\}$. Hence we have two linear equations with $M+1$ unknowns $a_0, a_{l_1}, \dots, a_{l_M}$. If $M = 1$ (that is, $\mathcal{I} = \{2l_1, 2k\}$ with $2 \leqslant 2l_1 < 2k$), the solution (a_0, a_{l_1}) to the two linear equations is unique. If $M \geqslant 2$, the solution $(a_0, a_{l_1}, \dots, a_{l_M})$ is not unique.

After finding $a_0, a_1, \dots, a_{k-1}$, we can calculate $\alpha_2, \dots, \alpha_{k-1}$ ($\alpha_k = 1$ since T and W_k are monic) from $a_0, a_1, \dots, a_{k-1}$ due to Proposition 2.2. If $M = 1$, we see that $a_0, a_1, \dots, a_{k-1}$ are unique and so are $\alpha_2, \dots, \alpha_{k-1}$. Furthermore, for some cases (e.g. $\mathcal{I} = \{6, 12\}$), we have $\alpha_2 < 0$ and we cannot show the equivalence of (CM) and (CL). If $M \geqslant 2$, we see that $a_0, a_1, \dots, a_{k-1}$ are not unique and hence

$\alpha_2, \dots, \alpha_{k-1}$ are not unique. Therefore we may be able to choose $a_0, a_1, \dots, a_{k-1}$ so that $\alpha_2, \dots, \alpha_{k-1}$ are nonnegative.

From the observation above, we can find $\mathcal{I}$ in assertions (4) and (5) of Theorem 1.1 so that $\alpha_2, \dots, \alpha_{k-1}$ are nonnegative. This procedure needs numerical calculation (see Listing 2 in Mathematica). Other than $\mathcal{I}$ in assertions (4) and (5), we consider the next examples:

• The largest number of $\mathcal{I}$ in assertions (4) ($\mathcal{I} = \{6, 12, 14, 2k\}$) and (5) ($\mathcal{I} = \{6, 12, 18, 30, 32, 2k\}$) of Theorem 1.1 is arbitrary, but the theorem does not hold in general. For example, all of $\alpha_2, \dots, \alpha_k$ are nonnegative (resp. at least one of $\alpha_2, \dots, \alpha_k$ is negative) for $\mathcal{I} = \{6,\ 12,\ 16,\ 2k\}$ with $18 \leqslant 2k \leqslant 40$ (resp. $42 \leqslant 2k \leqslant 100$).

• The smallest number of $\mathcal{I}$ may be arbitrary. For example, $\alpha_2, \dots, \alpha_k$ are nonnegative for $\mathcal{I} = \{8,\ 12,\ 14,\ 18,\ 26,\ 32,\ 34,\ 36,\ 38,\ 1000\}$, $\mathcal{I} = \{8,\ 12,\ 14,\ 18,\ 28,\ 30,\ 34,\ 36,\ 38,\ 1000\}$ and $\mathcal{I} = \{10,\ 14,\ 16,\ 18,\ 24,\ 28,\ 30,\ 32,\ 34,\ 36,\ 38,\ 1000\}$. However, we cannot find a rule guaranteeing that $\alpha_2, \dots, \alpha_{k-1}$ are nonnegative. These examples suggest the following conjecture, which is a relaxed version of Conjecture 4.1.

CONJECTURE 4.2. *Let $2l_1 \geqslant 8$ be an arbitrary even integer, and choose $M-1$ suitable even integers $2l_2, \dots, 2l_M$ with $2l_1 < \cdots < 2l_M$, where $M \geqslant 1$. Let $2k > 2l_M$ be an arbitrary even integer. Set*

$$\mathcal{I} = \{2l_1, \dots, 2l_M, 2k\}.$$

Then (CL) *and* (CM) *for $\mathcal{I}$ are equivalent.*

Of course the cases $2l_1 = 2, 4, 6$ are obtained in Theorem 1.1 and this conjecture might be shown by the method of [2].

Listing 2. How to find examples

```
he[k_,x_]:=he[k,x]=2^(-k/2)HermiteH[k,x/Sqrt[2]];
(* Define w *)
w[l_,x_]:=w[l,x]=Module[{coeffList},coeffList=CoefficientList[he[l
    /2,t]*he[l/2-2,t],t];
(2*l/2-1)*({0,0}~Join~(coeffList*Map[1/#&,Range[Length[coeffList
    ]]])-(coeffList~Join~{0,0})).Map[x^#&,Range[0,l]]]//Expand;
(* Set list={l,...,k}. Consider an identity with respect to x so
    that a_0+a_lx^l+...+a_kx^k = b_4 w[4,x]+...+ b_k w[k,x] *)
equalities[list_]:=equalities[list]=Map[#==0&,CoefficientList[
    Plus@@Map[Subscript[a, #]*x^#&,{0}~Join~list]-Plus@@Map[
    Subscript[b, #]*w[#,x]&,Range[4,Last[list],2]],x^2]];
(* Find example a_k,...,a_1,b_1,...,b_k so that a_k=1, b_k=1, b_k
    ,...,b_4 are nonnegative *)
example[list_]:=FindInstance[Join[{Subscript[a, Last[list]]==1,
    Subscript[b, Last[list]]==1},Map[Subscript[b, #]>=0&,Range[4,
    Last[list],2]],equalities[list]],Map[Subscript[a, #]&,{0}~Join
    ~list]~Join~Map[Subscript[b, #]&,Range[4,Last[list],2]]]
list = {6, 12, 16, 100};
equalities[list]
example[list]
```

5. APPENDIX

In this section, we study properties of $\xi_i(k)$ defined by (1.2). First, we obtain the next proposition by direct calculation.

PROPOSITION 5.1. *The first few exact values of* $\{\xi_i(m)\}_{m\geqslant i\geqslant 2}$ *are*

$$\xi_2(2)=2,\qquad \xi_2(3)=\frac{8}{3}=2.66\ldots,\qquad \xi_2(4)=\frac{14}{5}=2.8,$$
$$\xi_2(5)=\frac{96}{35}=2.74\ldots,\qquad \xi_2(6)=\frac{166}{63}=2.63\ldots,\qquad \xi_2(7)=\frac{584}{231}=2.52\ldots,$$
$$\xi_3(3)=\frac{8}{3}=2.66\ldots,\qquad \xi_3(4)=\frac{24}{5}=4.8,\qquad \xi_3(5)=\frac{208}{35}=5.94\ldots.$$

In addition, we can obtain more information regarding $\xi_i(k)$ by studying the hypergeometric function:

PROPOSITION 5.2. *Let* $i=2$. *Then:*

(1) $\xi_2(2)=2$ *and* $2<\xi_2(k)$ *for* $k\geqslant 3$.

(2) $\xi_2(2)<\xi_2(3)<\xi_2(4)$.

(3) $\xi_2(4)>\xi_2(5)>\xi_2(6)>\cdots$.

PROPOSITION 5.3. *For every* $i\geqslant 2$, $\{\xi_i(k)\}_{k=2}^{\infty}$ *converges to* $2^{i-1}(i-1)!$ *as* $k\to\infty$. *In addition, for all* $2\leqslant i\leqslant 16$ *and* $k\geqslant 3000$, $r_i(k)=\xi_i(k)-2^{i-1}(i-2)!$ *satisfies*

$$|r_i(k)|\leqslant 2^{p_i}.$$

The values of p_i *are listed in Table* 1.

TABLE 1. Definition of p_i

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
p_i	-18	-9	-5	-2	2	6	10	14	18	23	28	32	37	42	47

5.1. Proof of Proposition 5.2. First,

$$(1-u)^{-1/2}\leqslant(1-u/2)^{-2}\quad\text{for } 0\leqslant u\leqslant 1/2, \tag{5.1}$$

$$(1-u)^{-1/2}\geqslant(1-u/2)^{-1}\quad\text{for } 0\leqslant u\leqslant 1. \tag{5.2}$$

Then, assertions (1) and (2) for $k=2,3$ follow from Proposition 5.1. For $k\geqslant 4$, it follows from (5.2) that

$$\xi_2(k)\geqslant(k-1)\int_0^1\left(1-\frac{u}{2}\right)^{-1}\left(1-\frac{u}{2}\right)^{k-2}du=2+\frac{2}{k-2}\left(1-\frac{k-1}{2^{k-2}}\right).$$

Since the last term is positive for $k\geqslant 4$, we have $\xi_2(k)>2$ for $k\geqslant 4$.

Now, we demonstrate (3). Since $\xi_2(4) > \xi_2(5) > \xi_2(6) > \xi_2(7)$ from Proposition 5.1, we will show that $\xi_2(k) > \xi_2(k+1)$ for $k \geqslant 7$. If we set $\delta_k = \xi_2(k+1) - \xi_2(k)$, then

$$\delta_k = \int_0^1 (1-u)^{-1/2}\left(1-\frac{u}{2}\right)^{k-2}\left(1-\frac{k}{2}u\right)du.$$

Noting that $1-\frac{k}{2}u \gtrless 0$ for $u \lessgtr \frac{2}{k}$ and using estimates (5.1) and (5.2) yields

$$\begin{aligned}\delta_k &\leqslant \int_0^{2/k}\left(1-\frac{u}{2}\right)^{k-4}\left(1-\frac{k}{2}u\right)du + \int_{2/k}^1\left(1-\frac{u}{2}\right)^{k-3}\left(1-\frac{k}{2}u\right)du\\ &= 2\biggl[-\frac{2}{(k-3)(k-2)}\\ &\qquad + \frac{3k^2}{(k-3)(k-2)(k-1)^2}\left(1-\frac{1}{k}\right)^k + \frac{2(k^2-2k+2)}{(k-2)(k-1)}\left(\frac{1}{2}\right)^k\biggr].\end{aligned}$$

Here, the facts that (i) $k \mapsto (1-\frac{1}{k})^k$ is increasing and converges to $1/e$ $(< 7/19)$ as $k \to \infty$, (ii) $k \mapsto \frac{k^2}{(k-1)^2}$ is decreasing, and (iii) $k \mapsto \frac{2(k^2-2k+2)}{(k-2)(k-1)}$ is decreasing, imply that for $k \geqslant 7$, δ_k is no greater than

$$\begin{aligned}&2\left[-\frac{2}{(k-3)(k-2)} + \frac{3\cdot 7^2}{(k-3)(k-2)(7-1)^2}\frac{7}{19} + \frac{2(7^2-2\cdot 7+2)}{(7-2)(7-1)}\left(\frac{1}{2}\right)^k\right]\\ &\qquad = 2\left[-\frac{113}{228}\,\frac{1}{(k-3)(k-2)} + \frac{37}{15}\left(\frac{1}{2}\right)^k\right]\\ &\qquad = -\frac{2\cdot 37}{15(k-3)(k-2)2^k}\left[\frac{15}{37}\,\frac{113}{228}2^k - (k-3)(k-2)\right].\end{aligned}$$

Since the last term is negative if $k \geqslant 7$, the assertion is demonstrated. ∎

5.2. Proof of Proposition 5.3. Now, we examine the hypergeometric function $F(a,b,c;z)$.

LEMMA 5.1 (Watson's lemma, [4, Proposition 2.1]). *Let $\phi\colon (0,1) \to \mathbb{R}$ be integrable. Assume that there exist constants $\sigma > 0$ and $0 < \rho < 1$ and a smooth function ψ on $[0,\rho]$ such that $\phi(s) = \psi(s)s^{\sigma-1}$. Then*

$$\begin{aligned}&\left|\int_0^1 \phi(s)e^{-\lambda s}\,ds - \frac{\psi(0)\Gamma(\sigma)}{\lambda^\sigma}\right|\\ &\qquad \leqslant \frac{2|\psi(0)|}{\rho\lambda e^{\rho\lambda}} + \frac{(\max_{0\leqslant s\leqslant\rho}|\psi'(s)|)\Gamma(\sigma+1)}{\lambda^{\sigma+1}} + \frac{1}{e^{\rho\lambda}}\int_\rho^1 |\phi(s)|\,ds\end{aligned}$$

for any $\lambda \geqslant 2\sigma/\rho$.

Proof. Follow the proof in [4], using the monotonicity of $t \mapsto e^{-\frac{t}{2}}t^{\sigma-1}$ on $[2\sigma, \infty)$ in estimating an incomplete Gamma function. ■

LEMMA 5.2. *Let* $a \geqslant 1$, $0 < c - a < 1$, *and* $0 < z < 1$. *Then*

$$\left| B(a, c-a)F(a, -b, c; z) - \frac{\Gamma(a)}{z^a (b+1)^a} \right| \leqslant M_{a,c;z}(-(b+1)\log(1-z))$$

whenever $-(b+1)\log(1-z) > 2a/\rho$. *Here,* $0 < \rho < 1$ *is an arbitrary constant and*

$$M_{a,c;z}(\lambda) = \frac{1}{(1-z)^{a+1}} \left(\frac{2}{\rho\lambda e^{\rho\lambda}} + \frac{(1-\rho)^{c-a-2}\Gamma(a+2)}{\lambda^{a+1}} + \frac{B(a, c-a)}{e^{\rho\lambda}} \right).$$

Proof. We expand

$$B(a, c-a)F(a, -b, c; z) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^b \, du$$

with respect to $b+1$ making use of Lemma 5.1.

Set $v = \frac{\log(1-zu)}{\log(1-z)}$, $\xi = \frac{-\log(1-z)}{z}$, $\eta = \frac{-(1-z)\log(1-z)}{z}$, and $h(w) = \frac{e^w - 1}{w}$. Then

$$\frac{u}{v} = \xi h(v\log(1-z)), \quad \frac{1-u}{1-v} = \eta h((v-1)\log(1-z)), \quad 1 - zu = e^{v\log(1-z)},$$

so that

$$\begin{aligned} u^{a-1}(1-u)^{c-a-1}(1-zu)^b &= \left(\frac{u}{v}\right)^{a-1} \left(\frac{1-u}{1-v}\right)^{c-a-1} v^{a-1}(1-v)^{c-a-1}(1-zu)^b \\ &= \xi^{-1}\phi(v)e^{vb\log(1-z)} = \xi^{-1}\psi(v)v^{a-1}e^{vb\log(1-z)}, \end{aligned}$$

where

$$\phi(v) = \psi(v)v^{a-1}, \quad \psi(v) = Kg(v)(1-v)^{c-a-1},$$
$$K = \xi^a \eta^{c-a-1}, \quad g(v) = h(v\log(1-z))^{a-1} h((v-1)\log(1-z))^{c-a-1}.$$

Combining this with $\frac{du}{dv} = \xi e^{v\log(1-z)}$ and writing $\lambda = -(b+1)\log(1-z)$ yields

$$B(a, c-a)F(a, -b, c; z) = \int_0^1 \phi(v)e^{-\lambda v} \, dv.$$

In what follows, we expand the integral above with respect to λ making use of Lemma 5.1. First, we list the properties of h:

- h is strictly increasing and positive;
- $h(\log(1-z)) = \xi^{-1}$, $h(0) = 1$ and $h(-\log(1-z)) = \eta^{-1}$;

- h'/h is strictly increasing and $0 < (h'/h)(w) < 1$ for $w \in \mathbb{R}$;
- $(h'/h)(0) = 1/2$ and $(h'/h)(-\log(1-z)) = 1/z + 1/\log(1-z)$;
- $0 < (h'/h)'(w) \leqslant |(h'/h)'(0)| = 1/12$ for $w \in \mathbb{R}$.

Now, $\psi(0) = Kg(0) = Kh(0)^{a-1}h(-\log(1-z))^{c-a-1} = \xi^a$. From this and $0 < \xi \leqslant (1-z)^{-1}$, it follows that

$$\frac{\psi(0)\Gamma(a)}{\lambda^a} = \frac{\Gamma(a)}{z^a(b+1)^a}, \qquad |\psi(0)| \leqslant (1-z)^{-a}.$$

Next, we estimate $\max_{0\leqslant v\leqslant \rho} |\psi'(v)|$. Note that $g'(v) = g(v)f(v)$, where

$$f(v) = \left\{(a-1)\frac{h'(v\log(1-z))}{h(v\log(1-z))}\right.$$
$$\left. + (c-a-1)\frac{h'((v-1)\log(1-z))}{h((v-1)\log(1-z))}\right\}\log(1-z),$$

implying that

$$\psi'(v) = Kg(v)(1-v)^{c-a-2}\{f(v)(1-v) - (c-a-1)\}.$$

It follows from $a - 1 \geqslant 0$, $-1 < c - a - 1 < 0$, and the properties of h that

$$\max_{0\leqslant v\leqslant 1} |g(v)| \leqslant h(0)^{a-1}h(0)^{c-a-1} = 1,$$
$$\max_{0\leqslant v\leqslant 1} |f(v)| \leqslant \{|a-1| + |c-a-1|\}|\log(1-z)| \leqslant a|\log(1-z)|.$$

Hence, using $K = \xi^{c-1}(1-z)^{c-a-1} \leqslant (1-z)^{-a}$ $(c \geqslant 1)$ and $|\log(1-z)| \vee 1 \leqslant (1-z)^{-1}$ for $0 < z < 1$ yields

$$\max_{0\leqslant v\leqslant \rho} |\psi'(v)| \leqslant K(1-\rho)^{c-a-2}\{a|\log(1-z)| + 1\}$$
$$\leqslant (1-\rho)^{c-a-2}(a+1)(1-z)^{-(a+1)}.$$

and finally,

$$\int_\rho^1 |\phi(v)|\, dv \leqslant K \max_{0\leqslant s\leqslant 1} |g(s)| \int_0^1 v^{a-1}(1-v)^{c-a-1}\, dv \leqslant (1-z)^{-a}B(a, c-a).$$

Hence, the remainder is bounded by

$$\frac{1}{(1-z)^a}\left(\frac{2}{\rho\lambda e^{\rho\lambda}} + \frac{(1-\rho)^{c-a-2}(a+1)}{1-z}\,\frac{\Gamma(a+1)}{\lambda^{a+1}} + \frac{1}{e^{\rho\lambda}}B(a, c-a)\right).$$

This bound and $1 \leqslant (1-z)^{-1}$ complete the proof. ■

Proof of Proposition 5.3. Recalling (1.1) and applying Lemma 5.2 with $a = i-1$, $b = k-i$ and $c = i-1/2$ yields $r_i(k) = \xi_i(k) - 2^{i-1}(i-2)! = r_{1,i}(k) + r_{2,i}(k)$, where

$$r_{1,i}(k) = \frac{(k-1)!}{(k-i)!}\left\{B(i-1,1/2)F(i-1,-(k-i),i-1/2;1/2) - \frac{2^{i-1}(i-2)!}{(k-i+1)^{i-1}}\right\},$$
$$r_{2,i}(k) = \frac{(k-1)!}{(k-i)!}\,\frac{2^{i-1}(i-2)!}{(k-i+1)^{i-1}} - 2^{i-1}(i-2)!.$$

Write $c_i(k) = \frac{(k-1)!}{(k-i)!(k-i+1)^{i-1}}$. Then $c_i(k) = \prod_{\alpha=1}^{i-1}\bigl(1 + \frac{\alpha-1}{k-i+1}\bigr)$ is monotonically convergent to 1 as $k \to \infty$.

Setting $\lambda = (k-i+1)\log 2$ yields

$$|r_{1,i}(k)| \leqslant \frac{(k-1)!}{(k-i)!}M_{i-1,i-1/2;1/2}(\lambda) = \frac{c_i(k)}{(\log 2)^{i-1}} \cdot \lambda^{i-1}M_{i-1,i-1/2;1/2}(\lambda)$$

for all $\lambda \geqslant 2(i-1)/\rho$. Since, for every $n \geqslant 0$, the functions $\lambda \mapsto \lambda^n e^{-\rho\lambda}$ and $\lambda \mapsto \lambda^{-n}$ are decreasing on $[n/\rho,\infty)$, the function $[0,\infty) \ni \lambda \mapsto \lambda^{i-1}M_{i-1,i-1/2;1/2}(\lambda)$ is decreasing on $[(i-1)/\rho,\infty)$ and converges to 0 as $\lambda \to \infty$. In addition,

$$0 \leqslant r_{2,i}(k) = 2^{i-1}(i-2)!\{c_i(k) - 1\}.$$

From the above, it follows that $\xi_i(k) \to 2^{i-1}(i-2)!$ as $k \to \infty$.

Choose $k_0 \in \mathbb{N}$ and $i \in \mathbb{N}$ so that $(k_0 - i + 1)\log 2 \geqslant 2(i-1)/\rho$, in other words, $\frac{(k_0+1)\rho\log 2+2}{\rho\log 2+2} \geqslant i$. Then, for all $k \geqslant k_0$,

$$|r_i(k)| \leqslant \frac{c_i(k_0)}{(\log 2)^{i-1}} \cdot \lambda^{i-1}M_{i-1,i-1/2;1/2}(\lambda)|_{\lambda=(k_0-i+1)\log 2} + 2^{i-1}(i-2)!\{c_i(k_0) - 1\}.$$

Since $\frac{(k_0+1)\rho\log 2+2}{2+\rho\log 2} \geqslant 64$ for $k_0 = 3000$ and $\rho = 2^{-4}$, we can choose $i = 2,\dots,16$ and obtain the estimate of $|r_i(k)|$ for $i = 2,\dots,16$. ∎

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