PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 43, Fasc. 1 (2023), pp. 141–154
Published online 28.8.2023
doi:10.37190/0208-4147.00117

LARGE DEVIATIONS FOR BROWNIAN BRIDGES WITH PRESCRIBED TERMINAL DENSITIES

BY

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Abstract. We consider a family of Brownian bridges (depending on some small parameter) over a finite time interval whose initial position is deterministically fixed, and whose terminal position possesses a prescribed density. Large deviations for this family are studied with the help of the Girsanov transformation.

2020 Mathematics Subject Classification: Primary 60F10; Secondary 60J65, 35Q40.

Key words and phrases: large deviations, Brownian bridges, reciprocal processes.

1. INTRODUCTION

1.1. Reciprocal processes and their construction from Markov processes. A real-valued stochastic process $\{X_t, 0 \leqslant t \leqslant 1\}$ is said to be a *reciprocal process* if (see [10]) for any $0 \leqslant s < t < u \leqslant 1$ and bounded Borel-measurable function f,

$$E[f(X_t)|\mathcal{P}_s \vee \mathcal{F}_u] = E[f(X_t)|X_s,\, X_u],$$

where $\mathcal{P}_s = \sigma(X_v,\, 0 \leqslant v \leqslant s)$ is the forward filtration and $\mathcal{F}_u = \sigma(X_v, u \leqslant v \leqslant 1)$ is the backward filtration. In the literature, a reciprocal processes is also called a *Bernstein process* or a local (or two-sided) Markov process (cf. [4, 2]). Loosely speaking, the current state of a reciprocal process only depends on the nearest past and future. The properties of reciprocal processes have been summarized more recently in [14].

As noted in [10], any Markov process is a reciprocal process. Furthermore, given any probability measure μ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and any (real-valued) Markov process $\{Y_t, 0 \leqslant t \leqslant 1\}$ with Markov transition probability function $Q(s, x, t, E) = \int_E q(s, x; t, y)\, dy$ for some positive Markov transition density $q(s, x; t, y)$ and $0 \leqslant s < t \leqslant 1, x \in \mathbb{R}$ and Borel set E, the corresponding reciprocal process $\{X_t, 0 \leqslant t \leqslant 1\}$ can be constructed (see [10, Section 3]) such that (i) there is a reciprocal transition function $P(s,x;t,E;u,z) = \int_E p(s,x;t,y;u,z)\,dy$ with the reciprocal transition density $p(s,x;t,y;u,z)$ defined as

$$p(s,x;t,y;u,z) = \frac{q(s,x;t,y)\cdot q(t,y;u,z)}{q(s,x;u,z)}, \quad 0 \leqslant s < t < u \leqslant 1,$$

(ii) the joint endpoint distribution is $P(X_0 \in A, X_1 \in B) = \mu(A \times B)$ for $A, B \in \mathcal{B}(\mathbb{R})$, and (iii) $P(X_t \in A \,|\, X_s, X_u) = P(s, X_s; t, A; u, X_u)$ for any $A \in \mathcal{B}(\mathbb{R})$ and $0 \leqslant s < t < u \leqslant 1$.

1.2. Brownian bridges with prescribed terminal densities. For a small parameter $\hbar > 0$, let us consider a family of Markov processes $\{Y_t^\hbar = \sqrt{\hbar}\,W_t, 0 \leqslant t \leqslant 1\}$ with $\{W_t\}_{t\geqslant 0}$ being a standard Wiener process (starting from zero), the Markov transition densities $q^\hbar(s,x;t,y)$ of $\{Y_t^\hbar, 0 \leqslant t \leqslant 1\}$ are

$$q^\hbar(s,x;t,y) = (2\pi\hbar(t-s))^{-1/2} e^{-(y-x)^2/(2\hbar(t-s))}.$$

For brevity, we define

$$h_0(x,t,y) = (2\pi\hbar t)^{-1/2} e^{-(y-x)^2/(2\hbar t)};$$

then $q^\hbar(s,x;t,y) = h_0(x,t-s,y)$. With a given probability measure μ, the corresponding reciprocal processes $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ are called *Brownian bridges*.

In order that the Brownian bridges $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ become Markovian, suitable assumptions on μ should be imposed. According to [10, Theorem 3.1] and the remarks thereafter, if $\mu(A \times \mathbb{R}) = P(X_0^\hbar \in A) = \int_A p_0(x)\,dx$ and $\mu(\mathbb{R} \times B) = P(X_1^\hbar \in B) = \int_B p_1(x)\,dx$, then $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ is Markovian for each $\hbar > 0$ if there are non-negative functions $\eta^*(x)$ and $\eta(x)$ such that

$$\begin{cases} \eta^*(x) \int\limits_{\mathbb{R}} h_0(x,1,y)\eta(y)\,dy = p_0(x), \\ \eta(y) \int\limits_{\mathbb{R}} \eta^*(x) h_0(x,1,y)\,dx = p_1(y). \end{cases} \tag{1.1}$$

The existence and uniqueness of (non-negative) $\eta^*(x)$ and $\eta(x)$ have been discussed in [18, 2, 8, 3]. In the literature, (1.1) is called the *Schrödinger system*; see for example [13].

Throughout this paper, the family of Brownian bridges $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ constructed above is assumed to have an initial density $p_0(x) = \delta(x-a)$ being a Dirac delta function for some constant a, so that their initial position is deterministically fixed at a, which produces a Dirac measure $\mu(A \times \mathbb{R}) = P(X_0^\hbar \in A) = \delta_a(A)$ on a. For the terminal position, it is assumed that there exists a prescribed density $p_1(y) = p_1^\hbar(y)$ which may depend on $\hbar$ as well.

It is known from [4, Section 5.1, p. 156] that the Brownian bridges $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ thus constructed satisfy the following stochastic differential equation (SDE):

$$dX_t^\hbar = \sqrt{\hbar}\,dW_t + \hbar\nabla \ln \eta^\hbar(t, X_t^\hbar)\,dt, \quad X_0^\hbar = a, \tag{1.2}$$

and the distribution of $X_t^\hbar$ can be given as

$$P(X_t^\hbar \in A) = (\eta^\hbar(0,a))^{-1} \int_A \eta^{*\hbar}(t,x)\eta^\hbar(t,x)\,dx,$$

where $\eta^{*\hbar}$ and $\eta^\hbar$ are non-negative solutions of the adjoint partial differential equations, with $H = -\frac{\hbar^2}{2}\Delta$,

$$\begin{cases} -\hbar\frac{\partial\eta^{*\hbar}(t,x)}{\partial t} = H\eta^{*\hbar}(t,x), \\ \eta^{*\hbar}(0,x) = \eta^*(x), \end{cases} \quad \text{and} \quad \begin{cases} \hbar\frac{\partial\eta^{\hbar}(t,x)}{\partial t} = H\eta^{\hbar}(t,x), \\ \eta^{\hbar}(1,x) = \eta(x), \end{cases} \tag{1.3}$$

with non-negative $\eta^*(x)$ and $\eta(x)$ solving (1.1).

1.3. Main results. With suitable assumptions imposed on the prescribed terminal densities, the aim of this paper is to derive large deviation principles (LDPs) for the family of Brownian bridges $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ as $\hbar$ tends to zero. When the terminal positions are deterministically fixed (instead of densities), LDPs have been studied for various families of stochastic bridges including diffusion bridges [17, 9, 1], Bessel bridges [6], Lévy bridges [19] and Bernstein bridges [16]. However, when the terminal positions possess prescribed densities, LDPs for such stochastic bridges have not been investigated to the best of the authors' knowledge. We also note that in [7, Section 1 of II] LDPs of the empirical distribution of infinite-dimensional Brownian bridges with prescribed initial and terminal measures are considered. Furthermore, as noted in [16], when terminal positions are deterministically fixed (thus forming Dirac measures) the Girsanov transformation is usually not applicable over the entire time interval $[0,1]$ as singularities appear at $t = 1$. Therefore, it is natural to investigate whether one can derive LDPs for stochastic bridges with terminal Dirac measures using approximations involving smooth densities. This is actually our main motivation for studying LDPs for stochastic bridges with prescribed terminal densities, and more discussion on the validity of such approximations is included in Remark 1.1.

THEOREM 1.1. *Let $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ be the family of Brownian bridges defined in Section 1.2 with a deterministically fixed initial position $X_0^\hbar = a$ and a prescribed terminal density $p_1(y) = p_1^\hbar(y) > 0$. Assume that $p_1^\hbar(y)$ is continuously differentiable satisfying $\lim_{\hbar\to 0} \hbar \ln p_1^\hbar(y) = p(y)$ with a continuous function $p(y)$. For each $y_0 \in \mathbb{R}$ and small $\delta > 0$, suppose that there are functions p^- and p^+ such that $p^-(y_0,\hbar,\delta) \leqslant p_1^\hbar(y) \leqslant p^+(y_0,\hbar,\delta)$ for $|y - y_0| < \delta$,*

$$\lim_{\delta\to 0}\lim_{\hbar\to 0} \hbar \ln p^-(y_0,\hbar,\delta) = \lim_{\delta\to 0}\lim_{\hbar\to 0} \hbar \ln p^+(y_0,\hbar,\delta) = p(y_0) \tag{1.4}$$

and the limit as $\delta \to 0$ is uniform for bounded y_0. Furthermore, assume that $p_1^\hbar(y)$ satisfies the following condition: there are constants $\theta > 0$ and $N > 0$ such that *for all* $|y| > N$ *and small* $\hbar > 0$,

$$p_1^\hbar(y) \leqslant \alpha(\hbar) e^{-\theta y^2/\hbar} \quad \text{with}$$

(1.5)
$$\limsup_{\hbar\to 0} \hbar \ln \alpha(\hbar) < \infty \quad \text{and} \quad \limsup_{\hbar\to 0} \hbar \ln \max_{|y|\leqslant N} p_1^\hbar(y) < \infty.$$

Then the family $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ *satisfies an LDP as* $\hbar \to 0$ *with rate function* $S(\phi) = [\int_0^1 \phi'(t)^2\, dt - (\phi(1) - a)^2 - 2p(\phi(1))]/2$ *for absolutely continuous functions* ϕ *(otherwise it is* ∞*). That is, for any open set* $O \subseteq \mathbb{C}^a([0,1])$ *and closed set* $F \subseteq \mathbb{C}^a([0,1])$*, with* $\mathbb{C}^a([0,1])$ *denoting the space of continuous functions* $\phi(t)$ *on* $[0,1]$ *satisfying* $\phi(0) = a$,

$$\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F) \leqslant - \inf_{\phi\in F} S(\phi), \qquad \liminf_{\hbar\to 0} \hbar \ln P(X^\hbar \in O) \geqslant - \inf_{\phi\in O} S(\phi).$$

The proof of Theorem 1.1 is included in Section 2. The lower bound is relatively easier to handle, mainly based on the lower estimate p^- of the terminal densities. Condition (1.5) is only used for the proof of the upper bound.

The "continuously differentiable" assumption is imposed so that the drift term $\hbar\nabla \ln \eta^\hbar(t, X_t^\hbar)$ in the SDE (1.2) has no singularity for all $0 \leqslant t \leqslant 1$. This can be seen from the expression

$$\begin{aligned}\eta^\hbar(t,x) &= \int_{\mathbb{R}} h_0(x, 1-t, y)\eta(y)\, dy \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(1-t)\hbar}} e^{-(y-x)^2/(2\hbar(1-t))} \frac{p_1^\hbar(y)}{h_0(a,1,y)}\, dy \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hbar}} e^{-u^2/(2\hbar)} \frac{p_1^\hbar(x + u\sqrt{1-t})}{h_0(a,1,x+u\sqrt{1-t})}\, du.\end{aligned}$$

Condition (1.4) is imposed mainly to guarantee that the two limits in the proofs of the lower and upper bounds coincide, yielding the rate function. When (1.4) is not satisfied, it is still possible to derive large-deviation type results along the proof ideas of Theorem 1.1, but the two rate functions for lower and upper bounds do not coincide; see Example 1.3 below for a specific illustration of this phenomenon. Condition (1.5) is crucial in the proof of the upper bound. The control of decay $e^{-\theta y^2}$ for large y is used to offset the largeness effect from h_0, which is of size $e^{y^2/2}$, and the overall effect would be $e^{(1/2-\theta)y^2}$, which is controllable with $y \sim N(0,1)$.

1.4. LDP for Brownian bridges with deterministically fixed terminal positions. Before listing several explicit examples of Theorem 1.1, let us first recall the large deviations for Brownian bridges with deterministically fixed terminal positions. Note that such large deviations cannot be derived from Theorem 1.1 as there is no terminal density. The results are recalled for the purpose of comparing LDPs for Brownian bridges with and without terminal densities, as detailed in Remark 1.1.

Let $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ denote the family of Brownian bridges constructed in Section 1.2 (with a deterministically fixed initial position at a) when the terminal position is also deterministically fixed at b. As a by-product, it was shown in [16] that the family $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ satisfies an LDP as $\hbar \to 0$ with rate function $S^{a,b}(\phi) = (\int_0^1 \phi'(t)^2\,dt - (b-a)^2)/2$ for absolutely continuous ϕ. Here for completeness, we present a concise proof based on the contraction principle (see [5]). Let $\mathbb{C}^0([0,1])$ be the space of continuous functions $\phi(t)$ on $[0,1]$ with $\phi(0) = 0$, and $\mathbb{C}^{a,b}([0,1])$ the space of continuous functions $\phi(t)$ with $\phi(0) = a$ and $\phi(1) = b$. Now let $f : \mathbb{C}^0([0,1]) \to \mathbb{C}^{a,b}([0,1])$ be a continuous linear map defined as $f(x_\bullet)(t) = a(1-t) + bt + x(t) - tx(1), 0 \leqslant t \leqslant 1$. Then according to [12, Section 5.6], $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ and $\{f(\sqrt{\hbar}\,W_\bullet)(t), 0 \leqslant t \leqslant 1\}$ have the same distribution. It is well known that the family $\{\sqrt{\hbar}\,W_t, 0 \leqslant t \leqslant 1\}$ satisfies an LDP with rate function $S(\phi) = \frac{1}{2}\int_0^1 \phi'(t)^2\,dt$ for absolutely continuous ϕ, that is, for any open set $O \subseteq \mathbb{C}^0([0,1])$ and closed set $F \subseteq \mathbb{C}^0([0,1])$,

$$\begin{aligned}
&\limsup_{\hbar\to 0} \hbar \ln P(\sqrt{\hbar}\,W \in F) \leqslant -\inf_{\phi\in F} S(\phi),\\
&\liminf_{\hbar\to 0} \hbar \ln P(\sqrt{\hbar}\,W \in O) \geqslant -\inf_{\phi\in O} S(\phi).
\end{aligned} \tag{1.6}$$

Then the contraction principle implies that the family $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ satisfies an LDP with rate function $S^{a,b}(\phi) = \inf_{\varphi\in\mathbb{C}^0([0,1])}\{S(\varphi) : \phi = f(\varphi)\}$. Therefore,

$$\begin{aligned}
S^{a,b}(\phi) &= \inf_{\varphi\in\mathbb{C}^0([0,1]):\,\phi=f(\varphi)} \frac{1}{2}\int_0^1 \varphi'(t)^2\,dt\\
&= \inf_{\varphi\in\mathbb{C}^0([0,1])}\left[\frac{1}{2}\int_0^1 (\phi'(t)-(b-a))^2\,dt + \frac{1}{2}\varphi^2(1)\right]\\
&= \frac{1}{2}\int_0^1 (\phi'(t)-(b-a))^2\,dt.
\end{aligned}$$

1.5. Examples. In this section, we list several explicit examples of Theorem 1.1.

EXAMPLE 1.1. Let the terminal densities be

$$p_1(y) = p_1^\hbar(y) = \sum_{i=1}^k \alpha_i (2\pi\hbar)^{-1/2} e^{-(y-b_i)^2/(2\hbar)}$$

for constants α_i, b_i with $\sum_{i=1}^k \alpha_i = 1$. In this case condition (1.4) is satisfied with $p(y) = -\min_{1\leqslant i\leqslant k}(y-b_i)^2/2$ and

$$p^{\mp}(y_0,\hbar,\delta) = \sum_{i=1}^k \alpha_i (2\pi\hbar)^{-1/2} e^{-[(y_0-b_i)^2 \pm \varepsilon(\delta)]/(2\hbar)}$$

for some small $\varepsilon(\delta)$ depending on δ. Condition (1.5) is satisfied. Therefore the family $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ satisfies an LDP as $\hbar \to 0$ with rate function $S(\phi) = [\int_0^1 \phi'(t)^2\,dt - (\phi(1)-a)^2 + \min_{1\leqslant i\leqslant k}(\phi(1)-b_i)^2]/2$ for absolutely continuous functions ϕ (otherwise it is ∞).

REMARK 1.1 (to Example 1.1). If we consider a special case with $k = 1$, that is, $p_1(y) = p_1^\hbar(y) = (2\pi\hbar)^{-1/2}e^{-(y-b)^2/(2\hbar)}$, then the solution $\eta^\hbar(t,x)$ to the partial differential equation (1.3) can be found (for instance based on the moment generating functions of normal random variables) as

$$\eta^\hbar(t,x) = e^{(a^2-b^2)/(2\hbar)+x(b-a)/\hbar+(1-t)(b-a)^2/(2\hbar)},$$

which implies $\nabla \ln \eta^\hbar(t,x) = (b-a)/\hbar$. Therefore, the SDE (1.2) becomes

$$dX_t^\hbar = \sqrt{\hbar}\,dW_t + (b-a)\,dt, \quad X_0^\hbar = a,$$

whose solution is $X_t^\hbar = \sqrt{\hbar}\,W_t + (b-a)t + a$ for $0 \leqslant t \leqslant 1$. Note that in this case $p_1^\hbar(y) \to \delta(y-b)$ as $\hbar \to \infty$. Considering the classical family of Brownian bridges $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ discussed in Section 1.4 with initial position at a and terminal position at b both deterministically fixed, one can write that (in the sense of distribution) $X_t^{\hbar,a,b} = \sqrt{\hbar}\,W_t + (b-a)t + a - \sqrt{\hbar}\,tW_1$ for $0 \leqslant t \leqslant 1$ (see again [12, Section 5.6]). According to Example 1.1 and Section 1.4, LDPs in this case for the two families $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ and $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ admit the same rate function, and here we want to discuss how close the two families are. It is straightforward to see that, for any constant $\delta > 0$,

$$\begin{aligned} &P\Big(\max_{0\leqslant t\leqslant 1} |X_t^\hbar - X_t^{\hbar,a,b}\Big| > \delta) \\ &\quad = P\Big(\max_{0\leqslant t\leqslant 1} |\sqrt{\hbar}\,tW_1| > \delta\Big) = P(|W_1| > \delta/\sqrt{\hbar}) \sim \frac{2\sqrt{\hbar}}{\sqrt{2\pi}\,\delta}\cdot e^{-\delta^2/(2\hbar)}. \end{aligned}$$

Therefore, in terms of the rate $\hbar$ used in the LDPs, one has

$$\lim_{\hbar\to 0} h \ln P\Big(\max_{0\leqslant t\leqslant 1} |X_t^\hbar - X_t^{\hbar,a,b}| > \delta\Big) = -\delta^2/2.$$

As the above limit is not equal to $-\infty$, the two families $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ and $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ are not exponentially equivalent with rate $\hbar$ (see [5, Section 4.2.2] for the definition and discussion of exponential equivalence). This observation reveals that an LDP for $\{X_t^{\hbar,a,b}, 0 \leqslant t \leqslant 1\}$ cannot be straightforwardly and trivially deduced from an LDP for $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$, and vice versa.

REMARK 1.2 (to Example 1.1). As $\hbar \to 0$, the most probable paths of the family $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ are those ϕ such that $S(\phi) = 0$. More precisely, the most probable paths are $\phi_i(t) := a(1-t) + b_i t$, $1 \leqslant i \leqslant k$, which are lines connecting a and b_i. It is clear that $p_1^\hbar(y) \to \sum_{i=1}^k \alpha_i \delta(y - b_i)$ as $\hbar \to 0$, which implies that

$$P(X_1^\hbar \in A) = \int_A p_1^\hbar(y)\, dy \to \sum_{i=1}^k \alpha_i \delta_{b_i}(A).$$

This suggests that, as $\hbar \to 0$, the probability of the paths of $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ converging to the line $\phi_i(t)$ is α_i for each $1 \leqslant i \leqslant k$.

One can straightforwardly generalize Example 1.1 as follows.

EXAMPLE 1.2. Let the terminal densities be $p_1^\hbar(y) = \sum_{i=1}^k \alpha_i(\hbar) e^{f_i(y)/\hbar}$ for functions $\alpha_i(\hbar)$ and $f_i(y)$ such that for each $1 \leqslant i \leqslant k$, (a) the limit $\lim_{\hbar\to 0} \hbar \ln \alpha_i(\hbar)$ exists and is 0, (b) $f_i(y)$ is continuously differentiable, and there are constants $\theta > 0$ and $N > 0$ such that $f_i(y) \leqslant -\theta y^2$ for all $|y| > N$. Then the conditions in Theorem 1.1 are satisfied. More precisely, one can take $p^\mp(y_0, \hbar, \delta) = \sum_{i=1}^k \alpha_i e^{f_i(y) \mp \varepsilon(\delta)/\hbar}$ for some small $\varepsilon(\delta)$ depending on δ, and the limit is $p(y) = \max_{1\leqslant i\leqslant k} f_i(y)$. Therefore, the family $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ satisfies an LDP with rate function

$$S(\phi) = \Big[\int_0^1 \phi'(t)^2\, dt - (\phi(1) - a)^2 - 2 \max_{1\leqslant i\leqslant k} f_i(\phi(1))\Big]/2$$

for absolutely continuous functions ϕ (otherwise it is ∞). For instance, a particular example can be taken as $p_1^\hbar(y) = \alpha(\hbar) e^{-y^4/\hbar}$ for some normalizing factor $\alpha(\hbar)$.

The terminal densities $p_1^\hbar(y)$ considered in Examples 1.1 and 1.2 depend on the small parameter $\hbar$. Now we consider the case when the terminal density $p_1(y)$ is independent of $\hbar$; it turns out that the density does not then contribute to the rate functions.

EXAMPLE 1.3. Suppose that $p_1(y)$ is a function such that the drift term in SDE (1.2) has no singularity, $p_1(y) > 0$ for $y \in K$ and $p_1(y) = 0$ for $y \in K^c$ with K being a bounded set. Then it is clear that condition (1.5) is satisfied. However, condition (1.4) is not satisfied for $y_0 \in \partial K$ (in this case it can always happen that $p^-(y_0, \delta) = 0$ while $p^+(y_0, \delta) > 0$). Along the same ideas of the proof of Theorem 1.1 presented in Section 2, one can show that the family $\{X_t^\hbar, 0 \leqslant t \leqslant 1\}$ satisfies the following large-deviation type bounds: for any open set $O \subseteq \mathbb{C}^a([0,1])$ and closed set $F \subseteq \mathbb{C}^a([0,1])$,

$$\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F) \leqslant -\inf_{\phi\in F} S^u(\phi),$$
$$\liminf_{\hbar\to 0} \hbar \ln P(X^\hbar \in O) \geqslant -\inf_{\phi\in O} S_l(\phi),$$

where $S^u(\phi) = [\int_0^1 \phi'(t)^2\, dt - (\phi(1) - a)^2]/2$ for absolutely continuous functions ϕ with $\phi(1) \in \bar{K}$ (otherwise it is ∞), and $S^l(\phi) = [\int_0^1 \phi'(t)^2\, dt - (\phi(1) - a)^2]/2$

for absolutely continuous ϕ with $\phi(1) \in K^o$ (otherwise it is ∞), where $\bar{K}$ is the closure of K and K^o the interior of K. The difference between $S^u(\phi)$ and $S^l(\phi)$ appears on the boundary ∂K, because when $\phi(1) \in \partial K$ and $\phi(1)$ is very close to $a + \sqrt{\hbar}\, W_1$, it is not clear how the term $\eta^\hbar(1, a + \sqrt{\hbar}\, W_1)$ can be estimated (that is, condition (1.4) is not satisfied). Except for the non-trivial domain K, the rate functions $S^u(\phi)$ and $S^l(\phi)$ do not involve the terminal density $p_1(y)$ at all; in this sense the terminal density does not contribute to the rate functions.

2. LARGE DEVIATIONS FOR BROWNIAN BRIDGES

In this section, we prove the main result, Theorem 1.1. With prescribed terminal densities $p_1^\hbar(y)$, the solutions to (1.1) can be found as (keep in mind that in the sense of distributions, $g(x)\delta(x-a) = g(a)\delta(x-a)$ for smooth functions g)

$$\eta^*(x) = \delta(x-a), \qquad \eta(y) = p_1^\hbar(y)/h_0(a, 1, y).$$

The proof of Theorem 1.1 is based on the following Girsanov transformation in the space $\mathbb{C}^a([0,1])$, because of the SDE (1.2):

$$\mu_{X^\hbar}(A) = \int_{\{a+\sqrt{\hbar}\, W \in A\}} \frac{\eta^\hbar(1, a + \sqrt{\hbar}\, W_1)}{\eta^\hbar(0, a)}\, dP, \qquad A \subseteq \mathbb{C}^a([0,1]). \tag{2.1}$$

Note that from (1.3), $\eta^\hbar(t, x) = \int_{\mathbb{R}} h_0(x, 1-t, y) \cdot \eta(y)\, dy$, therefore $\eta^\hbar(0, a) = 1$.

2.1. Proof of the lower bound in Theorem 1.1. For any open $O \subseteq \mathbb{C}^a([0,1])$, let $\phi \in O$ be any point in this open set. Then for large enough n,

$$P(X^\hbar \in O) \geqslant P\Big(\max_{0\leqslant t\leqslant 1} |X_t^\hbar - \phi(t)| < 1/n\Big).$$

Now the Girsanov transformation (2.1) yields

$$P\Big(\max_{0\leqslant t\leqslant 1} |X_t^\hbar - \phi(t)| < 1/n\Big) = \int_{\{\max_{0\leqslant t\leqslant 1} |a+\sqrt{\hbar}\, W_t - \phi(t)| < 1/n\}} \eta^\hbar(1, a + \sqrt{\hbar}\, W_1)\, dP.$$

On the set $\{\max_{0\leqslant t\leqslant 1} |a+\sqrt{\hbar}\, W_t - \phi(t)| < 1/n\}$, from the conditions of Theorem 1.1 it follows that

$$p_1^\hbar(a + \sqrt{\hbar}\, W_1) \geqslant p^-(\phi(1), \hbar, 1/n). \tag{2.2}$$

Furthermore, the definition of h_0 gives

$$h_0(a, 1, a+\sqrt{\hbar}\, W_1) \leqslant (2\pi\hbar)^{-1/2} e^{-\min\{(\phi(1)-a+1/n)^2, (\phi(1)-a-1/n)^2\}/(2\hbar)}. \tag{2.3}$$

With the notation

$$\min\{(\phi(1)-a\mp 1/n)^2\} := \min\{(\phi(1)-a+1/n)^2, (\phi(1)-a-1/n)^2\},$$

it follows from (2.2) and (2.3) that

$$\eta^\hbar(1, a+\sqrt{\hbar}\,W_1) \geqslant \frac{p^-(\phi(1),\hbar,1/n)}{(2\pi\hbar)^{-1/2}e^{-\min\{(\phi(1)-a\mp 1/n)^2\}/(2\hbar)}}.$$

This implies that

$$\begin{aligned}
\liminf_{\hbar\to 0}\hbar\ln P(X^\hbar\in O) &\geqslant \liminf_{\hbar\to 0}\hbar\ln P\Big(\max_{0\leqslant t\leqslant 1}|X_t^\hbar-\phi(t)|<1/n\Big)\\
&= \liminf_{\hbar\to 0}\hbar\ln \int\limits_{\{\max_{0\leqslant t\leqslant 1}|a+\sqrt{\hbar}\,W_t-\phi(t)|<1/n\}} \eta^\hbar(1,a+\sqrt{\hbar}\,W_1)\,dP\\
&\geqslant \liminf_{\hbar\to 0}\hbar\ln\bigg[\frac{p^-(\phi(1),\hbar,1/n)}{(2\pi\hbar)^{-1/2}e^{-\min\{(\phi(1)-a\mp 1/n)^2\}/(2\hbar)}}\\
&\qquad\qquad\cdot P\Big(\max_{0\leqslant t\leqslant 1}|a+\sqrt{\hbar}\,W_t-\phi(t)|<1/n\Big)\bigg]\\
&\geqslant \liminf_{\hbar\to 0}\hbar\ln p^-(\phi(1),\hbar,1/n) - \min\{(\phi(1)-a\mp 1/n)^2\}/2\\
&\quad - \Big(2^{-1}\int_0^1\phi'(t)^2\,dt+\gamma(n)\Big)\\
&\to -S(\phi) \quad \text{as } n\to\infty,
\end{aligned}$$

where (1.4) has been used, together with the large deviation lower bound in (1.6) for the family $\{a+\sqrt{\hbar}\,W_t, 0\leqslant t\leqslant 1\}$ with $\gamma(n)\to 0$ as $n\to\infty$.

2.2. Proof of the upper bound in Theorem 1.1. Let us first assume that $F\subseteq \mathbb{C}^a([0,1])$ is a compact set with $\sup_{\phi\in F}\int_0^1\phi'(t)^2\,dt \leqslant C$ for some constant $C>0$. In this case, for any $\epsilon>0$, there is a finite ϵ-ball cover $F\subseteq\bigcup_{j=1}^{n(\epsilon)}\mathrm{Ball}_\epsilon(\phi_j)$ with $\phi_j\in F, 1\leqslant j\leqslant n(\epsilon)$. If $a+\sqrt{\hbar}\,W\in F$, then there must be some j such that $a+\sqrt{\hbar}\,W\in\mathrm{Ball}_\epsilon(\phi_j)$. When ϵ is small enough, the assumptions of Theorem 1.1 imply that

$$p_1^\hbar(a+\sqrt{\hbar}\,W_1)\leqslant p^+(\phi_j(1),\hbar,\epsilon). \tag{2.4}$$

What is more,

$$h_0(a,1,a+\sqrt{\hbar}\,W_1)\geqslant(2\pi\hbar)^{-1/2}e^{-\max\{(\phi_j(1)-a\mp\epsilon)^2\}/(2\hbar)}. \tag{2.5}$$

Then (2.4) and (2.5) imply that

$$\eta^\hbar(1,a+\sqrt{\hbar}\,W_1)\leqslant\frac{p^+(\phi_j(1),\hbar,\epsilon)}{(2\pi\hbar)^{-1/2}e^{-\max\{(\phi_j(1)-a\mp\epsilon)^2\}/(2\hbar)}}.$$

Then the Girsanov transformation (2.1) again implies that

$$\begin{aligned}
\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F) &= \limsup_{\hbar\to 0} \hbar \ln \int\limits_{\{a+\sqrt{\hbar}W\in F\}} \eta^\hbar(1, a+\sqrt{\hbar}\,W_1)\,dP \\
&\leqslant \limsup_{\hbar\to 0} \hbar \ln \int\limits_{\Omega} \sum_{i=j}^{n(\epsilon)} 1_{\{a+\sqrt{\hbar}W\in \mathrm{Ball}_\epsilon(\phi_j)\}} \eta^\hbar(1, a+\sqrt{\hbar}\,W_1)\,dP \\
&= \max_{j=1}^{n(\epsilon)} \limsup_{\hbar\to 0} \hbar \ln \int\limits_{\{a+\sqrt{\hbar}W\in \mathrm{Ball}_\epsilon(\phi_j)\}} \eta^\hbar(1, a+\sqrt{\hbar}\,W_1)\,dP \\
&\leqslant \max_{j=1}^{n(\epsilon)} \Big(\limsup_{\hbar\to 0} \hbar \ln p^+(\phi_j(1), \hbar, \epsilon) - \max\{(\phi_j(1)-a\mp\epsilon)^2\}/2 \\
&\qquad\qquad - \inf_{\phi\in\overline{\mathrm{Ball}_\epsilon(\phi_j)}} \int\limits_0^1 \phi'(t)^2\,dt/2 \Big),
\end{aligned}$$

where the large deviation upper bound in (1.6) for the family $\{a+\sqrt{\hbar}\,W_t, 0\leqslant t\leqslant 1\}$ has been used. As the functional $\int_0^1 \phi'(t)^2\,dt$ is lower semicontinuous, it then follows that

$$\inf_{\phi\in\overline{\mathrm{Ball}_\epsilon(\phi_j)}} \int\limits_0^1 \phi'(t)^2\,dt/2 \geqslant \int\limits_0^1 \phi_j'(t)^2\,dt/2 - \varepsilon_j,$$

for small $\varepsilon_j = \varepsilon_j(\epsilon) \to 0$ as $\epsilon \to 0$, uniformly in j. As $\sup_{\phi\in F} \int_0^1 \phi'(t)^2\,dt \leqslant C$ (implying that $\sup_{\phi\in F}\|\phi\| \leqslant C'$ for some constant C'), and the limit $\lim_{\epsilon\to 0}\limsup_{\hbar\to 0} \hbar\ln p^+(\phi_j(1),\hbar,\epsilon)$ is uniform for bounded $\phi_j(1)$, there are small $\gamma_j(\epsilon)\to 0$ uniformly in $1\leqslant j\leqslant n(\epsilon)$ as $\epsilon\to 0$ such that

$$\begin{aligned}
\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F) &\leqslant \max_{j=1}^{n(\epsilon)} (-S(\phi_j) + \gamma_j(\epsilon)) \\
&\leqslant \max_{j=1}^{n(\epsilon)} \Big(-\inf_{\phi\in F} S(\phi) + \gamma_j(\epsilon) \Big) \to -\inf_{\phi\in F} S(\phi).
\end{aligned} \tag{2.6}$$

Now for a general closed set $F \subseteq \mathbb{C}^a([0,1])$, let us consider the compact set $\Phi(s) := \{\phi\in\mathbb{C}^a([0,1]) : \int_0^1 \phi'(t)^2\,dt/2 \leqslant s\}$ for each $s>0$. The set F can be split into $F\cap\Phi(s)$ and $F\cap\Phi^c(s)$. Therefore,

$$\begin{aligned}
&\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F) \\
&\quad = \limsup_{\hbar\to 0} \hbar \ln [P(X^\hbar \in F\cap\Phi(s)) + P(X^\hbar\in F\cap\Phi^c(s))] \\
&\quad = \max\Big\{ \limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F\cap\Phi(s)),\ \limsup_{\hbar\to 0} \hbar\ln P(X^\hbar\in F\cap\Phi^c(s)) \Big\} \\
&\quad \leqslant \max\Big\{ -\inf_{\phi\in F\cap\Phi(s)} S(\phi),\ -s/p + c \Big\} \leqslant \max\Big\{ -\inf_{\phi\in F} S(\phi),\ -s/p+c \Big\} \\
&\quad \to -\inf_{\phi\in F} S(\phi) \qquad \text{as } s\to\infty,
\end{aligned}$$

where the first term $-\inf_{\phi\in F\cap\Phi(s)} S(\phi)$ comes from (2.6) and the second term $-s/p+c$ is from the following Lemma 2.1. This completes the proof of the upper bound.

LEMMA 2.1. *Under the assumptions of Theorem* 1.1*, there are constants* $p>1$ *and* $c>0$ *such that for each* $s>0$*,*

$$\limsup_{\hbar\to 0} \hbar \ln P(X^\hbar \in F\cap \Phi^c(s)) \leqslant -s/p + c.$$

Proof. Let us first apply the Girsanov transformation (2.1) and rewrite

$$P(X^\hbar\in F\cap\Phi^c(s)) = \int_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s)\}} \eta^\hbar(1, a+\sqrt{\hbar}\,W_1)\,dP;$$

recall that $\eta^\hbar(1,y)=\eta(y)=p_1^\hbar(y)/h_0(a,1,y)$. We need the following fact: for any constants $\alpha<1/2$ and β, and a standard normal random variable $Z\sim N(0,1)$,

$$E(e^{\alpha Z^2+\beta Z}) = \frac{1}{\sqrt{1-2\alpha}}\cdot e^{\beta^2/(2(1-2\alpha))}. \tag{2.7}$$

The Girsanov transformation again implies that

$$\begin{aligned} P(X^\hbar\in F\cap\Phi^c(s)) &= \int_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s)\}} \eta^\hbar(1, a+\sqrt{\hbar}\,W_1)\,dP \\ &= E\big(1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|>N\}}\eta^\hbar(1,a+\sqrt{\hbar}\,W_1) \\ &\quad + 1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|\leqslant N\}}\eta^\hbar(1,a+\sqrt{\hbar}\,W_1)\big). \end{aligned} \tag{2.8}$$

The second term on the right hand side in (2.8) is bounded above as

$$\begin{aligned} &E\big(1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|\leqslant N\}}\eta^\hbar(1,a+\sqrt{\hbar}\,W_1)\big) \\ &\quad\leqslant \sqrt{2\pi\hbar}\cdot \max_{|y|\leqslant N} p_1^\hbar(y)\cdot e^{(N+|a|)^2/(2\hbar)}\cdot P\big(a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s)\big). \end{aligned} \tag{2.9}$$

For the first term, condition (1.5) implies that

$$\begin{aligned} &E\big(1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|>N\}}\eta^\hbar(1,a+\sqrt{\hbar}\,W_1)\big) \\ &\quad\leqslant \sqrt{2\pi\hbar}\cdot\alpha(\hbar)\cdot \\ &\qquad\cdot E\big(1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|>N\}}e^{(-\theta(a+\sqrt{\hbar}\,W_1)^2+\hbar W_1^2/2)/\hbar}\big) \\ &\quad= \sqrt{2\pi\hbar}\cdot\alpha(\hbar)\cdot \\ &\qquad\cdot E\big(1_{\{a+\sqrt{\hbar}\,W\in F\cap\Phi^c(s), |a+\sqrt{\hbar}\,W_1|>N\}}e^{(1/2-\theta)W_1^2-2\theta a W_1/\sqrt{\hbar}-\theta a^2/\hbar}\big) \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\leqslant \sqrt{2\pi\hbar}\cdot\alpha(\hbar)\cdot e^{-\theta a^2/\hbar}\cdot\big[P(a+\sqrt{\hbar}W\in F\cap\Phi^c(s))\big]^{1/p}\cdot\\
&\quad\cdot\big[E(e^{(1/2-\theta)qW_1^2-2\theta aW_1q/\sqrt{\hbar}})\big]^{1/q}\\
&=\sqrt{2\pi\hbar}\cdot\alpha(\hbar)\cdot e^{-\theta a^2/\hbar}\cdot\big[P(a+\sqrt{\hbar}W\in F\cap\Phi^c(s))\big]^{1/p}\\
&\quad\cdot\left[\frac{1}{\sqrt{1-2(1/2-\theta)q}}e^{(2\theta aq)^2/(2\hbar(1-2(1/2-\theta)q))}\right]^{1/q},
\end{aligned}
$$

for which we have applied (2.7) and Hölder's inequality with $q>1$ such that $(1/2-\theta)q<1/2$. By inserting (2.9) and (2.10) back to (2.8), it follows that

$$
\begin{aligned}
&\limsup_{\hbar\to0}\hbar\ln P(X^\hbar\in F\cap\Phi^c(s))\\
&=\limsup_{\hbar\to0}\hbar\ln\int_{\{a+\sqrt{\hbar}W\in F\cap\Phi^c(s)\}}\eta^\hbar(1,a+\sqrt{\hbar}W_1)\,dP\\
&\leqslant\max\Big\{\limsup_{\hbar\to0}\hbar\ln\max_{|y|\leqslant N}p_1^\hbar(y)+(N+|a|)^2/2\\
&\qquad+\limsup_{\hbar\to0}\hbar\ln P(a+\sqrt{\hbar}W\in F\cap\Phi^c(s)),\\
&\qquad\qquad\limsup_{\hbar\to0}\hbar\ln\alpha(\hbar)+(2\theta aq)^2/(2q(1-2(1/2-\theta)q))-\theta a^2\\
&\qquad\qquad+\frac{1}{p}\limsup_{\hbar\to0}\hbar\ln P(a+\sqrt{\hbar}W\in F\cap\Phi^c(s))\Big\}\\
&\leqslant\max\{-s+c_1,-s/p+c_2\}\leqslant -s/p+c,
\end{aligned}
$$

where

$$
\begin{aligned}
c_1&:=\limsup_{\hbar\to0}\hbar\ln\max_{|y|\leqslant N}p_1^\hbar(y)+(N+|a|)^2/2,\\
c_2&:=\limsup_{\hbar\to0}\hbar\ln\alpha(\hbar)+(2\theta aq)^2/(2q(1-2(1/2-\theta)q))-\theta a^2,
\end{aligned}
$$

and $c=\max\{c_1,c_2\}$. ∎

3. REMARKS ON GENERALIZATIONS AND FUTURE WORK

Two types of generalizations of our results seem to be natural: (a) more general stochastic bridges with prescribed terminal densities (such as diffusion bridges and Bernstein bridges); and (b) stochastic bridges with initial and terminal densities both prescribed. The main challenge is to get enough properties, besides the existence and uniqueness, of the solutions $\eta^*(x)$ and $\eta(x)$ to the Schrödinger system (1.1). In particular, one needs estimates of these solutions in order to derive the result in Lemma 2.1 which is essential in the proof of the upper bound. It is also noted here that in [10, 11] Jamison studied the Schrödinger problem (see [13] for a recent survey) and its solution based on reciprocal transitions, while the Schrödinger problem itself has connections with optimal transport (see [13, 15]). It would be interesting to explore the role of large deviations of reciprocal processes within the framework of the Schrödinger problem and optimal transport. These will be the subject of our future work.

Acknowledgements. The authors thank for the support from The Swedish International Development Cooperation Agency (SIDA) through the bilateral programme *UEM-Sweden Research Cooperation Programme* with the subprogramme *Capacity Building in Mathematics, Statistics and Its Applications*. The authors are grateful to the Editor and the two anonymous reviewers for valuable comments and suggestions which led to a better presentation of the paper.

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Received 22.6.2022;
accepted 18.7.2023