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SHIFTED GENERALIZED MEHLER SEMIGROUPS ON WHITE NOISE FUNCTIONALS

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Abstract. We study shifted generalized Mehler semigroups on white noise functionals. We prove characterizations of invariant (white noise) distribution and the covariance property for shifted generalized Mehler semigroups. Also, we prove a Liouville type property of a shifted generalized Mehler semigroup or its infinitesimal generator.

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1. INTRODUCTION

The Gross Laplacian Δ_G and the number operator N have been introduced by Gross [14] and Piech [31], respectively, as infinite-dimensional Laplacians on certain functionals on abstract Wiener spaces. They have been studied as operators acting on white noise functionals in [22, 23, 24, 25]; they are continuous linear operators acting on the space of test white noise functionals [16, 29]. The Gross Laplacian and the number operator are characterized as quadratic white noise operators which are rotation-invariant (see [18, 27, 29]), i.e., invariant under infinite-dimensional rotations (see [39]). Invariance under subgroups of the (infinite-dimensional) rotation group corresponds to the covariance property studied in [1]. The infinite-dimensional Laplacians have been generalized to generalized Gross Laplacians [8] and conservation operators (as a generalization of the number operator) with general kernel distributions. The transformation groups generated by the Fourier–Gauss transform [9, 26] and the generalized Fourier–Gauss transform [8], associated with the infinite-dimensional Laplacians and their

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extensions, have been extensively studied in [22, 23, 16, 8, 9, 10, 11, 19, 30] and references therein.

The semigroups induced by the Fourier–Gauss transform and the generalized Fourier–Gauss transform correspond to Mehler semigroups and generalized Mehler semigroups, respectively, as integral representations. The generalized Mehler semigroups associated with Markov processes described by the stochastic differential equations of type

$$dX_t = BX_t dt + CdW_t$$

have been studied by many authors [1, 4, 13, 19, 35, 3]. Here B and C are certain operators on an infinite-dimensional space and W_t is an infinite-dimensional noise process. In [4, 13] (see also [3]), the generalized Mehler semigroups associated with Markov processes governed by different noise processes were studied. In [4] a characterization of the invariant measures for a generalized Mehler semigroup was proved. Also, in [34, 35], the Liouville theorem for a generalized Mehler semigroup was studied through the controllability of the system equation [33], and in [1], the covariance property of a generalized Mehler semigroup was studied.

In this paper, considering applications to

$$dX_t = BX_t dt + CdW_t + D\zeta dt,$$

we study so-called shifted generalized Mehler semigroups (see [35]) on white noise functionals. Based on the Gelfand triple $(E) \subset \Gamma(H) \subset (E)^*$, we construct a shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$ as a differentiable semigroup of continuous linear operators acting on (E) whose infinitesimal generator is the sum of a generalized Gross Laplacian, a conservation operator and an annihilation operator (see Section 4). The purpose of this paper is threefold:

(1) to characterize white noise distributions which are invariant for the adjoint semigroup $\{P_t^*\}_{t\geqslant 0}$ of the shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$,

(2) to characterize the covariance property of $\{P_t\}_{t\geqslant 0}$, under the orthogonal representation of a group, in terms of the invariance property of the kernels of the infinitesimal generator of $\{P_t\}_{t\geqslant 0}$,

(3) to prove a Liouville type property of the shifted generalized Mehler semigroup (or its infinitesimal generator).

This paper is organized as follows. In Section 2, we discuss the fundamental notions of white noise functionals. In Section 3, we review the basic results of white noise operator theory which are necessary for our study. In Section 4, we study shifted generalized Mehler semigroups on white noise functionals with their representation in terms of second quantization, generalized Gross Laplacian and annihilation operator, which provides an explicit form of the infinitesimal generator. In [5], an integral representation of the second quantization by means of a generalized Mehler formula was proved, and compactness, the Hilbert–Schmidt property and smoothing properties of the second quantization were studied. In Section 5, motivated by the concept of invariant measure [4] for a generalized Mehler semigroup, we prove a characterization of invariant white noise distributions for a shifted generalized Mehler semigroup. As an application, we prove a characterization of invariant Hida complex measures for shifted generalized Mehler semigroups. In Section 6, we first discuss invariant white noise operators under an orthogonal representation of a group G and investigate a necessary and sufficient condition for a shifted generalized Mehler semigroup (or its infinitesimal generator) to be G-covariant, i.e., covariant under the orthogonal representation of G. In Section 7, motivated by the Liouville property, studied in [34, 35], of generalized Mehler semigroups, we prove a Liouville type property for infinite-dimensional Laplacians and shifted generalized Mehler semigroups. In this case, the Liouville type property is defined as follows. For a semigroup $\{R_t\}_{t\geqslant 0} \subset \mathcal{L}((E),(E))$, we say that $\{R_t\}_{t\geqslant 0}$ has the *Liouville type property* if every harmonic function (not necessarily bounded) for $\{R_t\}_{t\geqslant 0}$ is constant.

2. WHITE NOISE FUNCTIONALS

Let H be a separable Hilbert space with Hilbertian norm $|\cdot|_0$ and let A be a densely defined positive selfadjoint operator on H. Suppose that there exists a complete orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers with $\lambda_1 > 1$ such that $Ae_n = \lambda_n e_n$ for all $n \in \mathbb{N}$, and

(A) A^{-1} is of Hilbert–Schmidt type, i.e. $\|A^{-1}\|_{\mathrm{HS}} = \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$.

For each $p \geqslant 0$, we define the Hilbert space $E_p = \{\xi \in H : |\xi|_p = |A^p\xi|_0 < \infty\}$. Let E_{-p} be the completion of H with respect to the norm $|\cdot|_{-p} = |A^{-p}\cdot|_0$. Then by defining the limit spaces

$$E := \operatorname*{proj\,lim}_{p\to\infty} E_p \quad \text{and} \quad E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

we obtain the Gelfand triple $E \subset H \cong H^* \subset E^*$. In this case, by assumption (**A**), E is a countably Hilbert nuclear space. Denote the canonical bilinear form on $E^* \times E$ by $\langle\cdot,\cdot\rangle$. Then the Bochner–Minlos theorem shows that there exists a standard Gaussian measure μ on $E_{\mathbb{R}}^*$ whose Fourier transform $\widehat{\mu}$ is given by

$$\widehat{\mu}(\xi) := \int_{E_{\mathbb{R}}^*} e^{i\langle x,\,\xi\rangle}\,\mu(dx) = \exp\left\{-\frac{1}{2}\langle\xi,\,\xi\rangle\right\}, \qquad \xi \in E,$$

where $E_{\mathbb{R}}$ is the real countably Hilbert space such that $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$. The (boson) *Fock space* $\Gamma(E_p)$ over E_p, $p \in \mathbb{R}$, is defined by

$$\Gamma(E_p) = \left\{\phi = (f_n)_{n=0}^{\infty} : f_n \in E_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2 < \infty\right\},$$

where $E_p^{\widehat{\otimes}n}$ $(n \geqslant 1)$ is the n-fold symmetric tensor product of E_p and $E_p^{\widehat{\otimes}0} = \mathbb{C}$. Then by taking the limit spaces, we obtain the Gelfand triple

$$(E) := \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p) \subset \Gamma(H) \subset \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}) \cong (E)^*.$$

We note that (E) becomes a countably Hilbert nuclear space equipped with the Hilbertian norms $\{\|\cdot\|_p\}_{p\geqslant 0}$.

The Wiener–Ito–Segal isomorphism states that $(L^2) = L^2(E_{\mathbb{R}}^*, \mu)$ is unitarily isomorphic to the Fock space $\Gamma(H)$. In fact, the unitary isomorphism between (L^2) and $\Gamma(H)$ is determined by the following correspondence:

$$\Gamma(H) \ni \phi_\xi := \left(\frac{1}{n!}\xi^{\otimes n}\right)_{n=0}^{\infty} \leftrightarrow \phi_\xi(x) := \exp\left\{\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right\} \in (L^2),$$

where ϕ_ξ is called the *exponential vector* associated with $\xi \in E$. Note that $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E).

The canonical bilinear form on $(E)^* \times (E)$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. The *S-transform* $S\Phi$ of $\Phi \in (E)^*$ is defined as a function $S\Phi : E \to \mathbb{C}$ by $S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle$, $\xi \in E$. In fact, for each $\Phi \in (E)^*$ and $F := S\Phi$, it is easy to see that the following conditions hold:

(S1) for all $\xi, \eta \in E$, $\mathbb{C} \ni z \mapsto F(\xi + z\eta) \in \mathbb{C}$ is an entire function,

(S2) there exist constants $K, c, p \geqslant 0$ such that $|F(\xi)| \leqslant K \exp c|\xi|_p^2$ for $\xi \in E$.

THEOREM 2.1 ([32, 17, 29, 26]). *A complex-valued function F on E is the S-transform of an element in $(E)^*$ if and only if F satisfies* (S1) *and* (S2).

3. WHITE NOISE OPERATORS

Let $\mathcal{X}, \mathcal{Y}$ be locally convex spaces. We denote the space of all continuous linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. An element in $\mathcal{L}((E), (E)^*)$ is called a *white noise operator*. For each $\Xi \in \mathcal{L}((E), (E)^*)$, its *symbol* $\widehat{\Xi} : E \times E \to \mathbb{C}$ is defined by $\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle$ for $\xi, \eta \in E$. Since the exponential vectors span a dense subspace of (E), each white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is uniquely determined by its symbol $\widehat{\Xi}$. Also, for $\Theta = \widehat{\Xi}$, it is easy to see that

(Θ**1**) for any $\xi_i, \eta_i \in E$ $(i = 1, 2)$, the function

$$\mathbb{C} \times \mathbb{C} \ni (z, w) \mapsto \Theta(\xi_1 + z\xi_2, \eta_1 + w\eta_2) \in \mathbb{C}$$

is entire,

(Θ**2**) there exist $K, c, p \geqslant 0$ such that

$$|\Theta(\xi, \eta)| \leqslant K \exp\{c(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi, \eta \in E.$$

Moreover, if $\Xi \in \mathcal{L}((E),(E))$, then $\Theta = \widehat{\Xi}$ satisfies the following condition:

($\Theta\mathbf{2}'$) for any $p \geqslant 0$ and $\epsilon > 0$, there exist constants $K, q \geqslant 0$ such that

$$|\Theta(\xi,\eta)| \leqslant K \exp\{\epsilon(|\xi|_{p+q}^2 + |\eta|_{-p}^2)\}, \quad \xi,\eta \in E.$$

THEOREM 3.1 ([28, 7]). *A complex-valued function Θ on $E \times E$ is the symbol of an operator $\Xi \in \mathcal{L}((E),(E)^*)$ if and only if Θ satisfies conditions* ($\Theta\mathbf{1}$) *and* ($\Theta\mathbf{2}$). *Moreover, Θ is the symbol of an operator $\Xi \in \mathcal{L}((E),(E))$ if and only if Θ satisfies conditions* ($\Theta\mathbf{1}$) *and* ($\Theta\mathbf{2}'$).

The adjoint of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ with respect to the canonical bilinear form $\langle\langle\cdot,\,\cdot\rangle\rangle$ is denoted by Ξ^*, that is, for all $\phi,\psi \in (E)$,

$$\langle\langle \Xi^*\phi,\,\psi\rangle\rangle = \langle\langle \phi,\,\Xi\psi\rangle\rangle = \langle\langle \Xi\psi,\,\phi\rangle\rangle.$$

EXAMPLE 3.1. (1) Let $\zeta \in E^*$ be given. A map $\Theta_1 : E \times E \to \mathbb{C}$ defined by $\Theta_1(\xi,\eta) = \langle\zeta,\,\xi\rangle e^{\langle\xi,\,\eta\rangle}$ for $\xi,\eta \in E$ satisfies conditions ($\Theta\mathbf{1}$) and ($\Theta\mathbf{2}'$), and so by Theorem 3.1 there exists a unique white noise operator in $\mathcal{L}((E),(E))$, denoted by $a(\zeta)$ and called the *annihilation operator*, such that $\widehat{a(\zeta)} = \Theta_1$. The adjoint of $a(\zeta)$ is denoted by $a^*(\zeta) := a(\zeta)^*$ and called the *creation operator*.

(2) For each $S \in \mathcal{L}(E,E^*)$, by applying Theorem 3.1, we see that there exists a unique white noise operator, denoted by $\Delta_G(S)$ and called the *generalized Gross Laplacian* (see [8]), in $\mathcal{L}((E),(E))$, such that

$$\widehat{\Delta_G(S)}(\xi,\eta) = \langle S\xi,\,\xi\rangle e^{\langle\xi,\,\eta\rangle}, \quad \xi,\eta \in E.$$

The generalized Gross Laplacian is uniquely determined by the action on exponential vectors: $\Delta_G(S)\phi_\xi = \langle S\xi,\,\xi\rangle\phi_\xi$ for $\xi \in E$. In particular, for the identity operator I, $\Delta_G := \Delta_G(I)$ is called the *Gross Laplacian*. For the adjoint operator of $\Delta_G(S)$, we write $\Delta_G^*(S) := \Delta_G(S)^*$.

(3) For $S \in \mathcal{L}(E,E^*)$ we have a unique white noise operator $\Lambda(S)$, called the *conservation operator*, in $\mathcal{L}((E),(E)^*)$ such that $\widehat{\Lambda(S)} = \langle S\xi,\,\eta\rangle e^{\langle\xi,\,\eta\rangle}$. Note that $\Lambda^*(S) := \Lambda(S)^* = \Lambda(S^*)$. If $S \in \mathcal{L}(E,E)$, then $\Lambda(S)$ belongs to $\mathcal{L}((E),(E))$. In particular, $N := \Lambda(I)$ is called the *number operator*.

(4) The *second quantization* $\Gamma(S)$ of $S \in \mathcal{L}(E,E^*)$ is defined by

$$\Gamma(S)\phi := (S^{\otimes n} f_n)_{n=0}^\infty, \quad \phi = (f_n)_{n=0}^\infty \in (E).$$

Then we have $\widehat{\Gamma(S)}(\xi,\eta) = e^{\langle S\xi,\,\eta\rangle}$ for $\xi,\eta \in E$, so that $\Gamma(S) \in \mathcal{L}((E),(E)^*)$ (see [29, 8]). Also, the second quantization $\Gamma(S)$ is uniquely determined by its action on exponential vectors: $\Gamma(S)\phi_\xi = \phi_{S\xi}$ for $\xi \in E$, and from the definition, we have $\Gamma(S)^* = \Gamma(S^*)$. If $S \in \mathcal{L}(E,E)$, then $\Gamma(S) \in \mathcal{L}((E),(E))$.

Furthermore, if $S \in \mathcal{L}(E, E)$ is an equicontinuous generator of a (semi)group $\{e^{tS}\}_{t\geqslant 0} \subset \mathcal{L}(E, E)$ (see [29] or (4.4)), then we have

$$\Lambda(S) = \frac{d}{d\epsilon}\Gamma(e^{\epsilon S})\Big|_{\epsilon=0}. \tag{3.1}$$

For each $K \in \mathcal{L}(E, E^*)$ and $S \in \mathcal{L}(E, E)$, the *generalized Fourier–Gauss* transform (see [8, 26]) is defined by $\mathcal{G}_{K,S} = \Gamma(S)e^{\Delta_G(K)} \in \mathcal{L}((E), (E))$. The adjoint operator of $\mathcal{G}_{K,S}$ is called the *generalized Fourier–Mehler transform* and denoted by $\mathcal{F}_{K,S}$. Then we have $\mathcal{F}_{K,S} = e^{\Delta^*_{\mathrm{G}}(K)}\Gamma(S^*) \in \mathcal{L}((E)^*, (E)^*)$.

4. SHIFTED GENERALIZED MEHLER SEMIGROUPS

We now study a shifted generalized Mehler semigroup acting on (E) (see [35]) and its generator in terms of white noise operators. Let $\{S_t\}_{t\geqslant 0}$ be a family of continuous linear operators acting on E and $\{T_t\}_{t\geqslant 0}$ be a family of continuous linear operators from E into E^*. Let $\{\eta_t\}_{t\geqslant 0} \subset E^*$. For each $t \geqslant 0$, put

$$P_t = \Gamma(S_t)e^{\Delta_G(T_t)}e^{a(\eta_t)}. \tag{4.1}$$

By applying the characterization theorem for the operator symbol, we can verify that P_t belongs to $\mathcal{L}((E), (E))$. To investigate the semigroup property we first observe the intertwining property of white noise operators.

PROPOSITION 4.1. *Let $S \in \mathcal{L}(E, E)$, $T \in \mathcal{L}(E, E^*)$ and $\eta \in E^*$ be given. Then*

$$e^{a(\eta)}\Gamma(S) = \Gamma(S)e^{a(S^*\eta)}, \qquad e^{\Delta_G(T)}\Gamma(S) = \Gamma(S)e^{\Delta_G(S^*TS)}.$$

Proof. For all $\xi, \zeta \in E$ one has

$$\langle\langle e^{a(\eta)}\Gamma(S)\phi_\xi,\, \phi_\zeta\rangle\rangle = e^{\langle S^*\eta,\,\xi\rangle + \langle S\xi,\,\zeta\rangle} = \langle\langle \Gamma(S)e^{a(S^*\eta)}\phi_\xi,\, \phi_\zeta\rangle\rangle,$$

so that the first equality holds. By the same argument we can easily see that the second equality holds (see also [19]). ∎

THEOREM 4.1. *Let $\{S_t\}_{t\geqslant 0}$ be a one-parameter semigroup in $\mathcal{L}(E, E)$. Suppose that $\{T_t\}_{t\geqslant 0} \subset \mathcal{L}(E, E^*)$ and $\{\eta_t\}_{t\geqslant 0} \subset E^*$. The family $\{P_t\}_{t\geqslant 0}$ of operators defined in* (4.1) *is a one-parameter semigroup in $\mathcal{L}((E), (E))$ if and only if for any $t, s \geqslant 0$,*

$$T_{t+s} = T_s + S^*_s T_t S_s, \qquad \eta_{t+s} = S^*_s \eta_t + \eta_s. \tag{4.2}$$

Proof. Apply Proposition 4.1. ∎

THEOREM 4.2. *Let $\{S_t\}_{t\geqslant 0}$ be a one-parameter semigroup in $\mathcal{L}(E,E)$. Suppose that $\{T_t\}_{t\geqslant 0}\subset\mathcal{L}(E,E^*)$, $\{\eta_t\}_{t\geqslant 0}\subset E^*$, and the maps $t\mapsto T_t$ and $t\mapsto\eta_t$ are differentiable at $t=0$. The family $\{P_t\}_{t\geqslant 0}$ of operators defined in* (4.1) *is a one-parameter semigroup in $\mathcal{L}((E),(E))$ if and only if for any $t\geqslant 0$,*

$$T_t=\int_0^t S_s^*TS_s\,ds,\qquad \eta_t=\int_0^t S_s^*\eta\,ds, \tag{4.3}$$

where $T=\frac{d}{dt}T_t\big|_{t=0}$ and $\eta=\frac{d}{dt}\eta_t\big|_{t=0}$.

Proof. Suppose that $\{P_t\}_{t\geqslant 0}$ is a one-parameter semigroup in $\mathcal{L}((E),(E))$. Then by Theorem 4.1, (4.2) holds. Therefore, by taking derivatives, we have $\frac{dT_t}{dt}=S_t^*TS_t^*$, $\frac{d\eta_t}{dt}=S_t^*\eta$, from which we have the explicit forms of T_t and η_t given in (4.3). The converse is straightforward. ∎

REMARK 4.1. In a Banach space setting, a family $\{T_t\}_{t\geqslant 0}$ of bounded linear operators satisfying the first equality of (4.2) with the explicit representation given in the first identity of (4.3) has been studied systematically in [37] (see also [36]) under the name of a Gaussian cylindrical Mehler semigroup.

DEFINITION 4.1. Let $\{S_t\}_{t\geqslant 0}$ be a one-parameter semigroup acting on E and $\{T_t\}_{t\geqslant 0}\subset\mathcal{L}(E,E^*)$. Let $\{\eta_t\}_{t\geqslant 0}\subset E^*$. If $\{S_t\}_{t\geqslant 0}$, $\{T_t\}_{t\geqslant 0}$ and $\{\eta_t\}_{t\geqslant 0}$ satisfy (4.3) for all $t,s\geqslant 0$, then the family $\{P_t\}_{t\geqslant 0}$ of operators defined in (4.1) is called a *shifted generalized Mehler semigroup*.

For an integral representation of a shifted generalized Mehler semigroup, we refer to [35] (see also (4.5) and (4.6)).

Let $\mathcal{X}$ be a locally convex Hausdorff space equipped with a family $\{\|\cdot\|_\alpha\}_{\alpha\in\Lambda}$ of seminorms. An operator $S\in\mathcal{L}(\mathcal{X},\mathcal{X})$ is called an *equicontinuous generator* if for any $r>0$, the family $\{(rS)^n/n!\}_{n=0}^\infty$ is equicontinuous, i.e. for any $\alpha\in\Lambda$, there exist $C\geqslant 0$ and $\beta\in\Lambda$ such that $\big\|\frac{(rS)^n}{n!}x\big\|_\alpha\leqslant C\|x\|_\beta$ for all $x\in\mathcal{X}$. For such a generator S the series

$$e^{rS}=\sum_{n=0}^{\infty}\frac{1}{n!}(rS)^n,\qquad t\in\mathbb{R}, \tag{4.4}$$

converges strongly in $\mathcal{X}$. Then $\{e^{zS}\}_{z\in\mathbb{C}}$ becomes a holomorphic one-parameter subgroup of $\mathrm{GL}(\mathcal{X})$ (see [29]).

THEOREM 4.3. *Let $\{S_t\}_{t\geqslant 0}$, $\{T_t\}_{t\geqslant 0}$ and $\{\eta_t\}_{t\geqslant 0}$ be as in Definition* 4.1. *Suppose that $\{S_t\}_{t\geqslant 0}$ a one-parameter semigroup with the equicontinuous generator $S\in\mathcal{L}(E,E)$ and the maps $t\mapsto T_t$ and $t\mapsto\eta_t$ are differentiable at $t=0$. Then the infinitesimal generator of the shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$ is given by $\Lambda(S)+\Delta_G(T)+a(\eta)$ where $T=\frac{d}{dt}T_t\big|_{t=0}$ and $\eta=\frac{d}{dt}\eta_t\big|_{t=0}$.*

Proof. We note that (4.2) implies that $S_0 = I$, $T_0 = 0$ and $\eta_0 = 0$. So for all $\xi, \zeta \in E$,

$$\frac{d}{dt}\widehat{P_t(\xi,\zeta)}\bigg|_{t=0} = (\langle S\xi, \zeta\rangle + \langle T\xi, \xi\rangle + \langle \eta, \xi\rangle)e^{\langle \xi, \zeta\rangle}$$
$$= \langle\langle(\Lambda(S) + \Delta_G(T) + a(\eta))\phi_\xi, \phi_\zeta\rangle\rangle,$$

which implies the assertion (see [29, 9]). ■

REMARK 4.2. Let $K_t \in \mathcal{L}(E_{\mathbb{R}}, E_{\mathbb{R}})$ be given for $t \geqslant 0$. Then the Bochner–Minlos theorem yields a unique Gaussian measure μ_t on $E_{\mathbb{R}}^*$ with covariance $K_t^* K_t$, that is,

$$\int_{E_{\mathbb{R}}^*} e^{i\langle y, \xi\rangle}\mu_t(dy) = e^{-\frac{1}{2}\langle K_t^* K_t\xi, \xi\rangle}, \quad \xi \in E.$$

Let $\{S_t^*\}_{t\geqslant 0} \subset \mathcal{L}(E_{\mathbb{R}}^*, E_{\mathbb{R}}^*)$ and $\{\eta_t\}_{t\geqslant 0} \subset E_{\mathbb{R}}^*$. For each $t \geqslant 0$, define

$$M_t\phi(x) := \int_{E_{\mathbb{R}}^*} \phi(S_t^* x + \eta_t + y)\mu_t(dy), \quad \phi \in (E),\ x \in E_{\mathbb{R}}^* \tag{4.5}$$

(see [35]). For the study of generalized Mehler semigroups, we refer to [4]. Applying an exponential vector ϕ_ξ to (4.5), we have

$$M_t\phi_\xi(x) = \Gamma(S_t)e^{\frac{1}{2}\Delta_G(S_t^* S_t + K_t^* K_t - I)}e^{a(\eta_t)}\phi_\xi(x),$$

so that

$$M_t = \Gamma(S_t)e^{\frac{1}{2}\Delta_G(S_t^* S_t + K_t^* K_t - I)}e^{a(\eta_t)}, \quad t \geqslant 0. \tag{4.6}$$

Thus if there exists a family $\{K_t\}_{t\geqslant 0} \subset \mathcal{L}(E_{\mathbb{R}}, E_{\mathbb{R}})$ such that

$$\int_0^t S_s^* T S_s ds = T_t = S_t^* S_t + K_t^* K_t - I, \quad t \geqslant 0,$$

for some $T \in \mathcal{L}(E, E^*)$, then we see that $\{M_t\}_{t\geqslant 0}$ coincides with $\{P_t\}_{t\geqslant 0}$, i.e.,

$$P_t = \Gamma(S_t)e^{\Delta_G(T_t)}e^{a(\eta_t)} = \int_{E_{\mathbb{R}}^*} \phi(S_t^* x + \eta_t + y)\, \mu_t(dy)$$
$$= \int_{E_{\mathbb{R}}^*} \phi(S_t^* x + \eta_t + K_t^* y)\, \mu(dy), \quad \phi \in (E),\ x \in E_{\mathbb{R}}^*,$$

which is an integral representation of P_t, and so $\{P_t\}_{t\geqslant 0}$ is the transition semigroup generated by the E^*-valued Markov process $\{J(t, x)\}_{t\geqslant 0}$ on $E_{\mathbb{R}}^*$ given as

$$E_{\mathbb{R}}^* \ni y \mapsto J(t, x)(y) = S_t^* x + K_t^* y + \eta_t \in E_{\mathbb{R}}^* \quad \text{with} \quad \eta_t = \int_0^t S_s^* \eta\, ds$$

(see Theorems 4.1 and 4.2), where $y \mapsto K_t^* y$ is considered as an $E_{\mathbb{R}}^*$-valued Gaussian process on $E_{\mathbb{R}}^*$ with covariance operator $K_t^* K_t$. Moreover, if the maps $t \mapsto K_t^* K_t$ and $t \mapsto \eta_t$ are differentiable at $t = 0$, the generator of $\{M_t\}_{t \geqslant 0}$ is given by $\Lambda(S) + \frac{1}{2}\Delta_G(S + S^* + V) + a(\eta)$, where S is the generator of $\{S_t\}_{t \geqslant 0}$, $V = \frac{d}{dt} K_t^* K_t\big|_{t=0}$ and $\eta = \frac{d}{dt}\eta_t\big|_{t=0}$.

5. INVARIANT WHITE NOISE DISTRIBUTIONS FOR THE SHIFTED GENERALIZED MEHLER SEMIGROUP

A complex measure ν on $E_{\mathbb{R}}^*$ is called a *Hida complex measure* if $(E) \subset L^1(\nu)$ and the linear functional $\varphi \mapsto \int_{E_{\mathbb{R}}^*} \varphi(x)\, d\nu(x)$ is continuous on (E) (see [26]). The Hida complex measure ν induces a white noise distribution $\Phi_\nu \in (E)^*$ such that $\langle\langle \Phi_\nu, \varphi \rangle\rangle = \int_{E_{\mathbb{R}}^*} \varphi(x)\, d\nu(x)$ for all $\varphi \in (E)$. A Hida complex measure is called a *Hida measure* if it is a measure. A white noise distribution Φ in $(E)^*$ is said to be *positive* if $\langle\langle \Phi, \varphi \rangle\rangle \geqslant 0$ for all nonnegative test functions $\varphi \in (E)$. A distribution $\Phi \in (E)^*$ is induced by a Hida measure if and only if Φ is positive. For details, we refer to [20, 38] (see also [26, Theorem 15.3]). A (complex) measure ν on $E_{\mathbb{R}}^*$ is said to be *invariant* for $\{P_t\}_{t \geqslant 0}$ if $(E) \subset L^1(\nu)$ and

$$\int_{E_{\mathbb{R}}^*} P_t \varphi(x)\, d\nu(x) = \int_{E_{\mathbb{R}}^*} \varphi(x)\, d\nu(x)$$

for all $t \geqslant 0$ and $\varphi \in (E)$ (see [4]). If $\Phi_\nu \in (E)^*$ is induced by a Hida measure ν which is invariant for $\{P_t\}_{t \geqslant 0}$, then

$$\langle\langle \Phi_\nu, \varphi \rangle\rangle = \int_{E_{\mathbb{R}}^*} \varphi(x)\, d\nu(x) = \int_{E_{\mathbb{R}}^*} P_t \varphi(x)\, d\nu(x) = \langle\langle P_t^* \Phi_\nu, \varphi \rangle\rangle$$

for all $\varphi \in (E)$. This observation motivates the following definition.

DEFINITION 5.1. A white noise functional $\Phi \in (E)^*$ is said to be *invariant* for $\{P_t^*\}_{t \geqslant 0}$ if $P_t^* \Phi = \Phi$ for all $t \geqslant 0$.

For any $\Phi, \Psi \in (E)^*$, by applying Theorem 2.1, we see that there exists a unique white noise distribution, denoted by $\Phi \diamond \Psi$ and called the *Wick product* of Φ and Ψ, in $(E)^*$ such that

$$S(\Phi \diamond \Psi) = S(\Phi) S(\Psi). \tag{5.1}$$

The following theorem provides a characterization of the existence of a vector in a topological space $\mathfrak{X}$ which is invariant for a one-parameter semigroup of sequentially continuous functions on $\mathfrak{X}$.

THEOREM 5.1. *Let $\{R_t\}_{t \geqslant 0}$ be a one-parameter semigroup of sequentially continuous functions R_t on a topological space $\mathfrak{X}$. Then there exists an element*

$x \in \mathfrak{X}$ such that x is invariant for $\{R_t\}_{t\geqslant 0}$, i.e. $R_t x = x$ for all $t \geqslant 0$, if and only if there exists an element $y \in \mathfrak{X}$ such that $\widetilde{y} := \lim_{t\to\infty} R_t y$ exists in $\mathfrak{X}$. In this case, $\widetilde{y}$ is invariant for $\{R_t\}_{t\geqslant 0}$.

Proof. Suppose that $x \in \mathfrak{X}$ is invariant for $\{R_t\}_{t\geqslant 0}$. Then it is obvious that $\lim_{t\to\infty} R_t x$ exists in $\mathfrak{X}$. Conversely, suppose that $\widetilde{y} := \lim_{s\to\infty} R_s y$ (for $y \in \mathfrak{X}$) exists in $\mathfrak{X}$. Then for any $t \geqslant 0$, by the sequential continuity of R_t, we have

$$R_t\widetilde{y} = R_t\Big(\lim_{s\to\infty} R_s y\Big) = \lim_{s\to\infty} R_{t+s} y = \widetilde{y},$$

which means that $\widetilde{y}$ is invariant for $\{R_t\}_{t\geqslant 0}$. ∎

COROLLARY 5.1. *Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup defined as in Definition* 4.1. *Then there exists $\Phi \in (E)^*$ such that Φ is invariant for $\{P_t^*\}_{t\geqslant 0}$ if and only if there exists $\Psi \in (E)^*$ such that $\widetilde{\Psi} := \lim_{t\to\infty} P_t^*\Psi$ exists in $(E)^*$. In this case, $\widetilde{\Psi}$ is invariant for $\{P_t^*\}_{t\geqslant 0}$.*

Proof. This is immediate from Theorem 5.1. ∎

THEOREM 5.2. *Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup defined as in Definition* 4.1. *Suppose that*

(**T**) *$T := \lim_{t\to\infty} T_t$ exists in $\mathcal{L}(E, E^*)$,*

(**η**) *$\eta := \lim_{t\to\infty} \eta_t$ exists in E^*.*

Then the following assertions are equivalent:

(i) *there exists $\Phi \in (E)^*$ such that Φ is invariant for $\{P_t^*\}_{t\geqslant 0}$*

(ii) *there exists $\Psi \in (E)^*$ such that $\overline{\Psi} := \lim_{t\to\infty} \Gamma(S_t^*)\Psi$ exists in $(E)^*$,*

(iii) *there exists $\Upsilon \in (E)^*$ such that Υ is invariant for $\{\Gamma(S_t^*)\}_{t\geqslant 0}$.*

In this case, the invariant vector Φ for $\{P_t^*\}_{t\geqslant 0}$ is explicitly given by

$$\Phi = \overline{\Psi} \diamond \Psi_T \diamond \phi_\eta,$$

where $\overline{\Psi} = \lim_{t\to\infty} \Gamma(S_t^*)\Psi$ for $\Psi \in (E)^*$ with $\lim_{t\to\infty} \Gamma(S_t^*)\Psi$ in $(E)^*$, $\Psi_T \in (E)^*$ (for $T \in \mathcal{L}(E, E^*)$) and $\phi_\eta \in (E)^*$ (for $\eta \in E^*$) are given by

$$\Psi_T = \left(\frac{\tau_T^{\widehat{\otimes} n}}{n!}\right), \qquad \phi_\eta = \left(\frac{\eta^{\widehat{\otimes} n}}{n!}\right), \tag{5.2}$$

where $\tau_T \in (E^*)^{\otimes 2}$ corresponds to $T \in \mathcal{L}(E, E^*)$ by the kernel theorem.

Proof. (i)⇒(ii) Suppose that Φ is invariant for $\{P_t^*\}_{t\geqslant 0}$. Then by Corollary 5.1, there exists $\Psi \in (E)^*$ such that $\lim_{t\to\infty} P_t^*\Psi$ exists in $(E)^*$. On the other hand, for any $\Phi \in (E)^*$, by applying (4.1), we have

$$S(P_t^*\Phi)(\xi) = \langle\langle P_t^*\Phi,\, \phi_\xi\rangle\rangle = e^{\langle T_t\xi,\,\xi\rangle+\langle\eta_t,\,\xi\rangle}\langle\langle\Gamma(S_t^*)\Phi,\, \phi_\xi\rangle\rangle,$$

which by the characterization of the S-transform (Theorem 2.1) implies that

$$P_t^*\Phi = \Gamma(S_t^*)\Phi \diamond \Psi_{T_t} \diamond \phi_{\eta_t}, \quad t \geqslant 0, \tag{5.3}$$

where $\Psi_T \in (E)^*$ (for $T \in \mathcal{L}(E, E^*)$) and $\phi_\eta \in (E)^*$ (for $\eta \in E^*$) are given as in (5.2). Also, from (5.3), we have

$$\Gamma(S_t^*)\Phi = P_t^*\Phi \diamond \Psi_{-T_t} \diamond \phi_{-\eta_t}, \quad t \geqslant 0. \tag{5.4}$$

On the other hand, it is known (see [26]) that for a family $\{\Upsilon_s\}_{s\geqslant 0} \subset (E)^*$, $\lim_{s\to\infty}\Upsilon_s$ exists in $(E)^*$ if and only if for each $\xi \in E$, $\lim_{s\to\infty} S(\Upsilon_s)(\xi)$ exists in $\mathbb{C}$ and there exist constants $K, C, p \geqslant 0$ such that $|S(\Upsilon_s)(\xi)| \leqslant C\exp\{K|\xi|_p^2\}$ for $\xi \in E$, $s \geqslant 0$. Hence we can see that the limits

$$\Psi_{-T} := \lim_{t\to\infty} \Psi_{-T_t}, \quad \phi_{-\eta} := \lim_{t\to\infty} \phi_{-\eta_t} \tag{5.5}$$

exist, and thus, by (5.4), since the Wick product is (separately) continuous and $\lim_{t\to\infty} P_t^*\Psi$ exists in $(E)^*$, $\lim_{t\to\infty}\Gamma(S_t^*)\Psi$ exists in $(E)^*$.

(ii)⇒(iii) Suppose there exists $\Psi \in (E)^*$ such that $\Upsilon := \overline{\Psi} := \lim_{t\to\infty}\Gamma(S_t^*)\Psi$ exists in $(E)^*$. Since $\{\Gamma(S_t^*)\}_{t\geqslant 0}$ is a one-parameter semigroup, by Theorem 5.1, Υ is invariant for $\{\Gamma(S_t^*)\}_{t\geqslant 0}$.

(iii)⇒(i) Suppose that there exists $\Upsilon \in (E)^*$ such that Υ is invariant for $\{\Gamma(S_t^*)\}_{t\geqslant 0}$. Then it is obvious that $\overline{\Upsilon} := \lim_{t\to\infty}\Gamma(S_t^*)\Upsilon = \Upsilon$ exists in $(E)^*$, and then from (5.3) and (5.5), we see that the limit

$$\Phi := \widetilde{\Upsilon} := \lim_{t\to\infty} P_t^*\Upsilon = \Big(\lim_{t\to\infty}\Gamma(S_t^*)\Upsilon\Big) \diamond \Psi_T \diamond \phi_\eta = \Upsilon \diamond \Psi_T \diamond \phi_\eta$$

exists in $(E)^*$. Hence by Corollary 5.1, $\Phi = \widetilde{\Upsilon}$ is invariant for $\{P_t^*\}_{t\geqslant 0}$. ∎

REMARK 5.1. Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup defined as in Definition 4.1. If conditions **(T)** and **($\boldsymbol{\eta}$)** of Theorem 5.2 hold for $\{T_t\}_{t\geqslant 0} \subset \mathcal{L}(E, E^*)$ and $\{\eta_t\}_{t\geqslant 0} \subset E^*$, then from (4.3), we have

$$T = \int_0^\infty S_s^* T S_s\, ds, \quad \eta = \int_0^\infty S_s^*\eta\, ds. \tag{5.6}$$

Therefore, as a sufficient condition for conditions **(T)** and **($\boldsymbol{\eta}$)** to hold we can consider some integrability conditions for the integrals in (5.6).

REMARK 5.2. Let $\Phi \in (E)^*$ and $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup defined as in Definition 4.1. Then by inspecting the proof of Theorem 5.2 (see (5.4)), it is easy to see that Φ is invariant for $\{P_t^*\}_{t\geqslant 0}$ if and only if

$$\Gamma(S_t^*)\Phi = \Phi \diamond \Psi_{-T_t} \diamond \phi_{-\eta_t}, \quad t \geqslant 0,$$

where $\Psi_T \in (E)^*$ and $\phi_\eta \in (E)^*$ are given in (5.2).

Let ν and ρ be Hida complex measures on $E_{\mathbb{R}}^*$ with corresponding white noise distributions Φ_ν and Φ_ρ. Then the convolution $\nu * \rho$ is also a Hida complex measure such that $\Phi_{\nu*\rho} = \Phi_\nu \diamond \Phi_\rho \diamond \Psi_{\frac{1}{2}I}$, where $\Psi_{\frac{1}{2}I}$ is given as in (5.2) (see [26]). Let $\mu_{2T,\eta}$ be the Gaussian measure with mean vector η and covariance operator $2T$, i.e. the Fourier transform of $\mu_{2T,\eta}$ is given by

$$\begin{aligned}\widetilde{\mu_{2T,\eta}}(\xi) &= \int_{E_{\mathbb{R}}^*} e^{i\langle x,\xi\rangle} d\mu_{2T,\eta}(x) = e^{i\langle \eta,\xi\rangle - \frac{1}{2}\langle 2T\xi,\xi\rangle} \\ &= \langle\langle \Phi_{\mu_{2T,\eta}}, \phi_{i\xi}\rangle\rangle e^{-\frac{1}{2}\langle \xi,\xi\rangle}, \quad \xi \in E,\end{aligned}$$

from which by applying (5.1) we see that $\Phi_{\mu_{2T,\eta}} \diamond \Psi_{\frac{1}{2}I} = \Psi_T \diamond \phi_\eta$. Therefore, for any Hida complex measure ν on $E_{\mathbb{R}}^*$, we have

$$\Phi_{\nu*\mu_{2T,\eta}} = \Phi_\nu \diamond \Phi_{\mu_{2T,\eta}} \diamond \Psi_{\frac{1}{2}I} = \Phi_\nu \diamond \Psi_T \diamond \phi_\eta.$$

Hence by applying Theorem 5.2, we can construct an invariant Hida (complex) measure for $\{P_t\}_{t\geqslant 0}$ as in the following corollary.

COROLLARY 5.2. *Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup defined as in Definition* 4.1. *Suppose that $\{T_t\}_{t\geqslant 0} \subset \mathcal{L}(E, E^*)$ and $\{\eta_t\}_{t\geqslant 0} \subset E^*$ satisfy conditions* (**T**) *and* ($\boldsymbol{\eta}$)*, respectively, such that $2T$ is a covariance operator, where $T := \lim_{t\to\infty} T_t$ exists in $\mathcal{L}(E, E^*)$. Then the following assertions are equivalent:*

(i) *an invariant Hida complex measure ν for $\{P_t\}_{t\geqslant 0}$ exists on $E_{\mathbb{R}}^*$,*

(ii) *there exists a Hida complex measure σ on $E_{\mathbb{R}}^*$ such that $\Phi_{\overline{\sigma}} := \overline{\Phi_\sigma} = \lim_{t\to\infty} \Gamma(S_t^*)\Phi_\sigma$ exists in $(E)^*$ and $\overline{\Phi_\sigma}$ corresponds to a Hida complex measure $\overline{\sigma}$ on $E_{\mathbb{R}}^*$, where $\Phi_\sigma \in (E)^*$ corresponds to the Hida complex measure σ,*

(iii) *an invariant Hida complex measure ρ for $\{\Gamma(S_t)\}_{t\geqslant 0}$ exists on $E_{\mathbb{R}}^*$.*

In this case, the invariant Hida complex measure ν for $\{P_t\}_{t\geqslant 0}$ is given by

$$\nu = \rho * \mu_{2T,\eta}. \tag{5.7}$$

Proof. (i)⇒(ii) Suppose that there exists a Hida complex measure ν on $E_{\mathbb{R}}^*$ such that ν is invariant for $\{P_t\}_{t\geqslant 0}$, which is equivalent to $\Phi_\nu \in (E)^*$ corresponding to ν being invariant for $\{P_t^*\}_{t\geqslant 0}$, which by conditions **(T)** and **($\boldsymbol{\eta}$)** implies that

$$\Phi_\nu = \lim_{t\to\infty} P_t^*\Phi_\nu = \Big(\lim_{t\to\infty} \Gamma(S_t^*)\Phi_\nu\Big) \diamond \Psi_T \diamond \phi_\eta.$$

Thus $\lim_{t\to\infty} \Gamma(S_t^*)\Phi_\nu$ exists and

$$\lim_{t\to\infty} \Gamma(S_t^*)\Phi_\nu = \Phi_\nu \diamond \Psi_{-T} \diamond \phi_{-\eta} = \Phi_{\nu*\mu_{-2T,-\eta}}.$$

(ii)⇒(iii) By assumption, it is obvious that $\overline{\sigma}$ is a Hida complex measure which is invariant for $\{\Gamma(S_t)\}_{t\geqslant 0}$.

(iii)⇒(i) Suppose that ρ is a Hida complex measure on $E_{\mathbb{R}}^*$ which is invariant for $\{\Gamma(S_t)\}_{t\geqslant 0}$. Then Φ_ρ corresponding to ρ is invariant for $\{\Gamma(S_t^*)\}_{t\geqslant 0}$, and so by Theorem 5.2, $\Phi_{\rho*\mu_{2T,\eta}} = \Phi_\rho \diamond \Psi_T \diamond \phi_\eta$ is invariant for $\{P_t^*\}_{t\geqslant 0}$, which is equivalent to $\rho * \mu_{2T,\eta}$ being invariant for $\{P_t\}_{t\geqslant 0}$. ■

REMARK 5.3. Corollary 5.2 can be considered as a generalization of [4, Theorem 4.4] (see also [12, Theorem 11.7]).

6. COVARIANCE PROPERTY OF A SHIFTED GENERALIZED MEHLER SEMIGROUP

In this section we study the covariance of a shifted generalized Mehler semigroup. We first recall the notion of infinite-dimensional rotation group (see [39, 18, 29]). Let $\mathcal{E} \subset \mathcal{H} \subset \mathcal{E}^*$ be a Gelfand triple and

$$O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}}) = \{g \in \mathrm{GL}(\mathcal{E}_{\mathbb{R}}) : |g\xi| = |\xi| \text{ for all } \xi \in \mathcal{E}_{\mathbb{R}}\},$$

where $\mathcal{E} = \mathcal{E}_{\mathbb{R}} + i\mathcal{E}_{\mathbb{R}}$, $\mathcal{H} = \mathcal{H}_{\mathbb{R}} + i\mathcal{E}_{\mathbb{R}}$ and $|\cdot|$ is the Hilbertian norm on the Hilbert space $\mathcal{H}$. An element $g \in O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}})$ is called a *rotation* and $O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}})$ is called the *infinite-dimensional rotation group* (see [29]). We note that each $g \in O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}})$ extends to an orthogonal operator, denoted by $\widetilde{g}$, on the Hilbert space $\mathcal{H}_{\mathbb{R}}$, and can also be extended to a continuous linear operator, denoted by the same symbol $\widetilde{g}$, on $\mathcal{E}_{\mathbb{R}}^*$. In fact, for each $g \in O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}})$ and its adjoint operator $g^* \in \mathcal{L}(\mathcal{E}_{\mathbb{R}}^*, \mathcal{E}_{\mathbb{R}}^*)$ with respect to the Hilbert space $\mathcal{H}_{\mathbb{R}}$, we have $\widetilde{g} = (g^{-1})^*$ (see [29, Proposition 5.4.3]). Similarly, we put

$$U(\mathcal{E};\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{E}) : |g\xi| = |\xi| \text{ for all } \xi \in \mathcal{E}\}.$$

By the canonical extension, $O(\mathcal{E}_{\mathbb{R}};\mathcal{H}_{\mathbb{R}})$ is considered as a subgroup of $U(\mathcal{E};\mathcal{H})$.

From now on, our study is based on the Gelfand triple $E \subset H \subset E^*$ constructed as in Section 2. Then the infinite-dimensional rotation group $O(E_{\mathbb{R}}; H_{\mathbb{R}})$ acts on the Gaussian space $E_{\mathbb{R}}^*$ by means of its adjoint $\langle g^*x,\, \xi\rangle = \langle x,\, g\xi\rangle$ for $x \in E_{\mathbb{R}}^*$ and $\xi \in E_{\mathbb{R}}$.

Throughout this section, we denote by G a group such that there is an orthogonal representation $U : G \ni g \mapsto U_g \in O(E_{\mathbb{R}}; H_{\mathbb{R}})$, and then we construct a unitary representation $\mathcal{U} : G \ni g \mapsto \mathcal{U}_g \in U((E); (L^2))$ of G by $\mathcal{U}_g \phi(x) = \phi(U_g^* x)$ for $\phi \in (L^2)$ and $x \in E_{\mathbb{R}}^*$. It is easy to see that $\mathcal{U}_g = \Gamma(U_g)$ for all $g \in G$. Then a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is said to be *covariant* under the action $\mathcal{U}$ of G, or simply *G-covariant* (or G-rotation-invariant), if

$$\mathcal{U}_g^* \Xi \mathcal{U}_g = \widetilde{\mathcal{U}_{g^{-1}}} \Xi \mathcal{U}_g = \Xi \quad \text{for all } g \in G$$

(see [1]), which is equivalent to $\Xi \mathcal{U}_g = \widetilde{\mathcal{U}_g} \Xi = \mathcal{U}_{g^{-1}}^* \Xi$ for all $g \in G$. If a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is G-covariant for the orthogonal group $G = O(E_{\mathbb{R}}; H_{\mathbb{R}})$, then Ξ is said to be *rotation-invariant* (see [27, 29]).

Every white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ admits the Fock expansion (see [29, Theorem 4.5.1])

$$\Xi \phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \phi, \quad \phi \in (E), \tag{6.1}$$

where the series converges in $(E)^*$. Here for all $l, m \geqslant 0$, $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, and $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$ is called the *integral kernel operator* with kernel distribution $\kappa_{l,m}$ and satisfies

$$\widehat{\Xi_{l,m}(\kappa_{l,m})}(\xi, \eta) = e^{\langle \xi, \eta \rangle} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle, \quad \xi, \eta \in E.$$

In (6.1), the family $\{\kappa_{l,m}\}_{l,m=0}^{\infty}$ of kernel distributions $\kappa_{l,m}$ is uniquely determined as $\kappa_{l,m} \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ (see [29, p. 83]).

For $F \in (E^{\otimes n})^*$, F is said to be *G-rotation-invariant* if $(U_g^{\otimes n}) F = F$ for all $g \in G$. For each $\kappa_{l,m} \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, by the kernel theorem, there exists a unique operator $K_{l,m} \in \mathcal{L}(E^{\otimes m}, E^{\otimes l})_{\mathrm{sym}}$ such that

$$\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle = \langle K_{l,m} \xi^{\otimes m}, \eta^{\otimes l} \rangle, \quad \xi, \eta \in E.$$

LEMMA 6.1. *Let $\kappa \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ correspond to $K \in \mathcal{L}(E^{\otimes m}, E^{\otimes l})_{\mathrm{sym}}$ by the kernel theorem. Then κ is G-rotation-invariant if and only if*

$$(U_g^*)^{\otimes l} K U_g^{\otimes m} = (\widetilde{U_{g^{-1}}})^{\otimes l} K U_g^{\otimes m} = K,$$

which is equivalent to $K U_g^{\otimes m} = (\widetilde{U_g})^{\otimes l} K$.

Proof. For any $\xi, \eta \in E$, by the kernel theorem, we have

$$\langle (U_g^*)^{\otimes(l+m)} \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle = \langle (U_g^*)^{\otimes l} K U_g^{\otimes m} \xi^{\otimes m}, \eta^{\otimes l} \rangle,$$

from which the proof of the assertion is straightforward. ∎

By modifying the proof of [29, Theorem 5.5.2] and applying Lemma 6.1, we can prove the following theorem.

THEOREM 6.1. *Let $\Xi \in \mathcal{L}((E),(E)^*)$ with Fock expansion*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}).$$

Then Ξ is G-covariant if and only if $\Xi_{l,m}(\kappa_{l,m})$ is G-covariant for all $l,m \geqslant 0$ if and only if $\kappa_{l,m}$ is G-rotation-invariant for all $l,m \geqslant 0$ if and only if

$$(U_g^*)^{\otimes l} K_{l,m} U_g^{\otimes m} = (\widetilde{U_{g^{-1}}})^{\otimes l} K_{l,m} U_g^{\otimes m} = K_{l,m}, \quad l,m \geqslant 0,$$

where $K_{l,m} \in \mathcal{L}(E^{\otimes m}, E^{\otimes l})_{\mathrm{sym}}$ corresponds to $\kappa_{l,m} \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$.

Every white noise operator $\Xi \in \mathcal{L}((E),(E))$ admits the Fock expansion given as in (6.1) with the family $\{\kappa_{l,m}\}_{l,m=0}^{\infty}$ of kernel distributions $\kappa_{l,m} \in (E^{\widehat{\otimes} l}) \otimes (E^{\otimes m})^*_{\mathrm{sym}}$ (see [29, Theorem 4.5.1]). In this case, the series in (6.1) converges in (E).

COROLLARY 6.1. *Let $\Xi \in \mathcal{L}((E),(E))$ with Fock expansion*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}).$$

Then Ξ is G-covariant if and only if $\Xi_{l,m}(\kappa_{l,m})$ is G-covariant for all $l,m \geqslant 0$ if and only if $\kappa_{l,m}$ is G-rotation-invariant for all $l,m \geqslant 0$ if and only if

$$K_{l,m} U_g^{\otimes m} = U_g^{\otimes l} K_{l,m}, \quad l,m \geqslant 0,$$

where $K_{l,m} \in \mathcal{L}(E^{\otimes m}, E^{\otimes l})_{\mathrm{sym}}$ corresponds to $\kappa_{l,m} \in (E^{\widehat{\otimes} l}) \otimes (E^{\otimes m})^*_{\mathrm{sym}}$.

Proof. This is immediate from Theorem 6.1. ∎

A one-parameter semigroup $\{R_t\}_{t\geqslant 0} \subset \mathcal{L}((E),(E))$ is said to be *covariant* under the action $\mathcal{U}$ of G, or simply *G-covariant*, if $R_t \mathcal{U}_g = \mathcal{U}_g R_t$ for any $t \geqslant 0$ and $g \in G$ (see [1]), which is equivalent to $\widetilde{\mathcal{U}_g} R_t^* = R_t^* \widetilde{\mathcal{U}_g}$ for any $t \geqslant 0$ and $g \in G$. If $\{R_t\}_{t\geqslant 0}$ is G-covariant for the orthogonal group $G = O(E_{\mathbb{R}}; H_{\mathbb{R}})$, then $\{R_t\}_{t\geqslant 0}$ is said to be *rotation-invariant* (see [29]).

THEOREM 6.2. *Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup as in Theorem* 4.3 *whose infinitesimal generator is given by $\mathcal{L} := \Lambda(S) + \Delta_G(T) + a(\eta)$. Then the following assertions are equivalent:*

(i) *the shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$ is G-covariant,*

(ii) *the infinitesimal generator $\mathcal{L} = \Lambda(S) + \Delta_G(T) + a(\eta)$ is G-covariant,*

(iii) *S commutes with U_g for all $g \in G$, and the operator $T \in \mathcal{L}(E, E^*)$ and the vector $\eta \in E^*$ satisfy $TU_g = \widetilde{U_g} T$ and $U_g^* \eta = \eta$.*

Proof. (i)⇒(ii) The proof is straightforward.

(ii)⇔(iii) By applying Corollary 6.1, we see that $\mathcal{L} = \Lambda(S) + \Delta_G(T) + a(\eta)$ is G-covariant if and only if $SU_g = U_g S$, $\tau_T \circ U_g^{\otimes 2} = \tau_T$ and $\eta \circ U_g = \eta$. Then for any $\xi, \zeta \in E$, we have

$$(\tau_T \circ U_g^{\otimes 2})(\zeta \otimes \xi) = \langle \tau_T, \, (U_g\zeta) \otimes (U_g\xi) \rangle = \langle U_g^* T U_g \xi, \, \zeta \rangle,$$

which implies that the equality $\tau_T \circ U_g^{\otimes 2} = \tau_T$ is equivalent to $U_g^* T U_g = T$, which is equivalent to $TU_g = \widetilde{U_g} T$. Also, for any $\xi \in E$, we have

$$(\eta \circ U_g)(\xi) = \eta(U_g \xi) = \langle \eta, \, U_g \xi \rangle = \langle U_g^* \eta, \, \xi \rangle,$$

from which we see that the equality $\eta \circ U_g = \eta$ is equivalent to $U_g^* \eta = \eta$.

(iii)⇒(i) For each $t \geqslant 0$, P_t is given as in (4.1) in which by Theorem 4.2, T_t and η_t are as in (4.3). On the other hand, for T_t and η_t given in (4.3), by applying (iii), for any $t \geqslant 0$ and $g \in G$, we can easily see that

$$U_g^* T_t U_g = T_t, \qquad U_g^* \eta_t = \eta_t, \qquad S_t U_g = U_g S_t.$$

Therefore, for any $\xi, \zeta \in E$, we obtain

$$\widehat{P_t \mathcal{U}_g}(\xi, \zeta) = \exp\{\langle T_t U_g \xi, \, U_g \xi \rangle + \langle \eta_t, \, U_g \xi \rangle + \langle S_t U_g \xi, \, \zeta \rangle\} = \widehat{\mathcal{U}_g P_t}(\xi, \zeta),$$

which implies that $P_t \mathcal{U}_g = \mathcal{U}_g P_t$, and hence $\{P_t\}_{t \geqslant 0}$ is G-covariant. ■

COROLLARY 6.2. *Let $\{P_t\}_{t\geqslant 0}$ be a shifted generalized Mehler semigroup as in Theorem* 4.3 *whose infinitesimal generator is given by $\mathcal{L} := \Lambda(S) + \Delta_G(T) + a(\eta)$. Then the shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$ is rotation-invariant, i.e. $\{P_t\}_{t\geqslant 0}$ is G-covariant with $G = O(E_{\mathbb{R}}; H_{\mathbb{R}})$ if and only if $\mathcal{L} := bN + a\Delta_G$ for some complex numbers a and b.*

Proof. By Theorem 6.2, $\{P_t\}_{t\geqslant 0}$ is rotation-invariant if and only if (iii) in Theorem 6.2 holds for $G = O(E_{\mathbb{R}}; H_{\mathbb{R}})$ if and only if $T = aI$ and $S = bI$ for some constants $a, b \in \mathbb{C}$ and $\eta = 0$ (see [27, Theorem 4.2] and [29, Theorem 5.5.4], and also [18, Theorems 5.1 and 5.2]). ■

7. LIOUVILLE TYPE PROPERTY FOR INFINITE-DIMENSIONAL LAPLACIAN

Let $\{P_t\}_{t\geqslant 0}$ be a differentiable one-parameter semigroup of continuous linear operators in $\mathcal{L}((E), (E))$ with infinitesimal generator L. A white noise functional $\phi \in (E)$ is said to be *harmonic* for $\{P_t\}_{t\geqslant 0}$ if ϕ is invariant for $\{P_t\}_{t\geqslant 0}$, i.e., $P_t\phi = \phi$ for $t \geqslant 0$. We note that $\phi \in (E)$ is harmonic for $\{P_t\}_{t\geqslant 0}$ if and only if $L\phi = 0$. Then we say that the operator (infinitesimal generator) L (or the differentiable one-parameter semigroup $\{P_t\}_{t\geqslant 0}$) has the *Liouville type property* if every harmonic white noise functional $\phi \in (E)$ for L (or $\{P_t\}_{t\geqslant 0}$) is constant.

THEOREM 7.1. *Let $S \in \mathcal{L}(E, E)$ be an equicontinuous generator of $\{e^{tS}\}_{t\geqslant 0}$. Suppose that*

(L) $\lim_{t\to\infty} e^{tS}\xi = 0$ *for any* $\xi \in E$.

Then the Ornstein–Uhlenbeck semigroup $\{\Gamma(e^{tS})\}_{t\geqslant 0} \subset \mathcal{L}((E),(E))$ has the Liouville type property.

Proof. Let $\phi = (f_n) \in (E)$ be a harmonic functional for $\{\Gamma(e^{tS})\}_{t\geqslant 0}$. For each $k \in \mathbb{N}$, put $\phi_k = (g_n)_{n=0}^{\infty} \in (E)$ with $g_n = 0$ for all $n \in \mathbb{N}$ with $n \neq k$ and $g_k = f_k$. Then for all $k \in \mathbb{N}$, ϕ_k is harmonic for $\{\Gamma(e^{tS})\}_{t\geqslant 0}$, and so for any $p, t \geqslant 0$, we obtain

$$\|\phi_k\|_p^2 = \|\Gamma(e^{tS})\phi_k\|_p^2 = k!|(e^{tS})^{\otimes k} f_k|_p^2. \tag{7.1}$$

On the other hand, from condition **(L)**, by applying the uniform boundedness principle, we see that $\{e^{tS}\}_{t\geqslant 0}$ is equicontinuous in $\mathcal{L}(E, E)$ and so for any $k \in \mathbb{N}$, $\{(e^{tS})^{\otimes k}\}_{t\geqslant 0}$ is equicontinuous in $\mathcal{L}(E^{\otimes k}, E^{\otimes k})$. From (7.1), by letting $t \to \infty$ and using **(L)** we see that

$$\|\phi_k\|_p^2 = k! \lim_{t\to\infty} |(e^{tS})^{\otimes k} f_k|_p^2 = 0,$$

i.e., $\phi_k = 0$ for all $k \in \mathbb{N}$. Therefore, $\phi = (f_0, 0, 0, \dots)$, i.e. $f_n = 0$ for all $n \in \mathbb{N}$ and hence ϕ is constant. ∎

REMARK 7.1. Condition (**L**) in Theorem 7.1 is somewhat comparable with the condition, given in terms of spectrums of the generator, in [35, Theorem 4.1].

PROPOSITION 7.1. *Let $L, M \in \mathcal{L}((E),(E))$ be given operators. Suppose that there exists an invertible operator $\mathcal{G} \in \mathcal{L}((E),(E))$ which preserves the constant functions and satisfies $M = \mathcal{G}^{-1}L\mathcal{G}$. Then L has the Liouville type property if and only if M has the Liouville type property.*

Proof. By assumption, we have $L\mathcal{G}\phi = \mathcal{G}M\phi$. Therefore, $L\mathcal{G}\phi = 0$ for $\phi \in (E)$ if and only if $M\phi = 0$, and hence $L\phi = 0$ implies that ϕ is constant if and only if $M\varphi = 0$ implies that φ is constant. ∎

PROPOSITION 7.2. *Let $V, S \in \mathcal{L}(E, E)$, $K \in \mathcal{L}(E, E^*)$ and $\zeta \in E^*$. Then*

(i) $e^{a(\zeta)}\Lambda(S) = (\Lambda(S) + a(S^*\zeta))e^{a(\zeta)}$,

(ii) $e^{\Delta_G(K)}\Lambda(S) = (\Lambda(S) + \Delta_G(S^*K + KS))e^{\Delta_G(K)}$.

Moreover, if we assume that V is invertible, then

(iii) $\Gamma(V)\Lambda(S) = \Lambda(VSV^{-1})\Gamma(V)$,

(iv) $\Gamma(V)\Delta_G(K) = \Delta_G((V^*)^{-1}KV^{-1})\Gamma(V)$.

Proof. (i) Note that $e^{a^*(\zeta)}\phi_\xi = \phi_{\zeta+\xi}$ for all $\xi \in E$ and so for any $\xi, \eta \in E$,

$$\langle\!\langle e^{a(\zeta)}\Lambda(S)\phi_\xi,\, \phi_\eta\rangle\!\rangle = \langle\!\langle \Lambda(S)\phi_\xi,\, \phi_{\zeta+\eta}\rangle\!\rangle = \langle S\xi,\, \zeta+\eta\rangle \langle\!\langle e^{a(\zeta)}\phi_\xi,\, \phi_\eta\rangle\!\rangle,$$

which implies (i).

(ii) We consider only the case of an equicontinuous generator S in $\mathcal{L}(E,E)$ of a (semi)group $\{e^{tS}\}_{t\geqslant 0} \subset \mathcal{L}(E,E)$. Then by (3.1), we obtain

$$\begin{aligned} e^{\Delta_G(K)}\Lambda(S) &= \frac{d}{d\epsilon} e^{\Delta_G(K)}\Gamma(e^{\epsilon S})\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\Gamma(e^{\epsilon S})e^{\Delta_G(e^{\epsilon S^*}Ke^{\epsilon S})}\Big|_{\epsilon=0} \\ &= (\Lambda(S) + \Delta_G(S^*K + KS))e^{\Delta_G(K)}. \end{aligned}$$

(iii) For $\xi, \eta \in E$, we obtain

$$\begin{aligned} \langle\!\langle \Gamma(V)\Lambda(S)\phi_\xi,\, \phi_\eta\rangle\!\rangle &= \langle\!\langle \Lambda(S)\phi_\xi,\, \phi_{V^*\eta}\rangle\!\rangle = \langle S\xi,\, V^*\eta\rangle e^{\langle \xi,\, V^*\eta\rangle} \\ &= \langle VS\xi,\, \eta\rangle \langle\!\langle \Gamma(V)\phi_\xi,\, \phi_\eta\rangle\!\rangle = \langle\!\langle \Lambda(VSV^{-1})\Gamma(V)\phi_\xi,\, \phi_\eta\rangle\!\rangle, \end{aligned}$$

which proves (iii).

(iv) For any $\xi \in E$, we obtain

$$\Gamma(V)\Delta_G(K)\phi_\xi = \langle K\xi,\, \xi\rangle\phi_{V\xi} = \Delta_G((V^*)^{-1}KV^{-1})\Gamma(V)\phi_\xi,$$

which implies (iv). ∎

For any given $V \in \mathcal{L}(E,E)$, $K \in \mathcal{L}(E,E^*)$ and $\zeta \in E^*$, put

$$\mathcal{G}_{K,V,\zeta} := \Gamma(V)e^{\Delta_G(K)}e^{a(\zeta)},$$

which is called the *shifted generalized Fourier–Gauss transform* (see [8, 11]). Then it is obvious that $\mathcal{G}_{0,I,0}$ is the identity operator. For $K, K' \in \mathcal{L}(E,E^*)$, $V, V' \in \mathcal{L}(E,E)$ and $\zeta, \zeta' \in E$, using the properties in Proposition 4.1, we obtain

$$\mathcal{G}_{K',V',\zeta'}\mathcal{G}_{K,V,\zeta} = \mathcal{G}_{K+V^*K'V,V'V,\zeta+V^*\zeta'},$$

from which we see that if V is invertible, then $\mathcal{G}_{K,V,\zeta}$ is invertible with inverse $\mathcal{G}_{K,V,\zeta}^{-1} = \mathcal{G}_{-(V^*)^{-1}KV^{-1},V^{-1},-(V^*)^{-1}\zeta}$.

PROPOSITION 7.3. *Let $S \in \mathcal{L}(E,E)$, $T \in \mathcal{L}(E,E^*)$ and $\eta \in E^*$ be given. Suppose that there exist an invertible operator $V \in \mathcal{L}(E,E)$, $K \in \mathcal{L}(E,E^*)$ and $\zeta \in E^*$ such that*

$$V^*TV = (V^{-1}SV)^*K + KV^{-1}SV, \qquad \eta = S^*(V^*)^{-1}\zeta. \tag{7.2}$$

Then

$$\mathcal{G}_{K,V,\zeta}\Lambda(V^{-1}SV)\mathcal{G}_{K,V,\zeta}^{-1} = \Lambda(S) + \Delta_G(T) + a(\eta). \tag{7.3}$$

Proof. For any $S' \in \mathcal{L}(E, E)$, by applying Proposition 7.2, we obtain

$$\begin{aligned} \mathcal{G}_{K,V,\zeta}\Lambda(S') &= \Gamma(V)e^{\Delta_G(K)}[\Lambda(S') + a(S'^*\zeta)]e^{a(\zeta)} \\ &= \Gamma(V)[\Lambda(S') + \Delta_G(S'^*K + KS') + a(S'^*\zeta)]e^{\Delta_G(K)}e^{a(\zeta)} \\ &= [\Lambda(VS'V^{-1}) + \Delta_G(\mathbf{V}) + a((V^*)^{-1}S'^*\zeta)]\mathcal{G}_{K,V,\zeta}, \end{aligned}$$

where $\mathbf{V} = (V^*)^{-1}(S'^*K + KS')V^{-1}$, from which by taking $S' = V^{-1}SV$ and using the conditions in (7.2), we have

$$\mathcal{G}_{K,V,\zeta}\Lambda(V^{-1}SV) = [\Lambda(S) + \Delta_G(T) + a(\eta)]\mathcal{G}_{K,V,\zeta}.$$

Since V is invertible and so $\mathcal{G}_{K,V,\zeta}$ is invertible, (7.3) holds. ∎

REMARK 7.2. The operator equation in (7.2) is a Sylvester–Rosenblum type equation (see [2] and references cited therein). If S is an equicontinuous generator of $\{e^{tS}\}_{t\geqslant 0}$ such that $\lim_{t\to\infty} e^{tS} = 0$ and the integral $\int_0^\infty V^*e^{tS^*}Te^{tS}V\,dt$ exists as an element of $\mathcal{L}(E, E^*)$, then a solution of the operator equation in (7.2) can be constructed by

$$K = -\int_0^\infty V^*e^{tS^*}Te^{tS}V\,dt.$$

THEOREM 7.2. *Let $S \in \mathcal{L}(E, E)$, $T \in \mathcal{L}(E, E^*)$ and $\eta \in E^*$. Suppose that there exist an invertible operator $V \in \mathcal{L}(E, E)$, $K \in \mathcal{L}(E, E^*)$ and $\zeta \in E^*$ such that* (7.2) *holds. If $S \in \mathcal{L}(E, E)$ is an equicontinuous generator of $\{e^{tS}\}_{t\geqslant 0}$ satisfying condition* (**L**) *in Theorem* 7.1*, then the infinitesimal generator $\Lambda(S) + \Delta_G(T) + a(\eta)$ of the shifted generalized Mehler semigroup $\{P_t\}_{t\geqslant 0}$ defined in Definition* 4.1 *has the Liouville type property.*

Proof. By Theorem 7.1, $\Lambda(S)$ has the Liouville type property and then by applying Theorem 7.1 again, we see that $\Lambda(V^{-1}SV)$ has the Liouville type property (see also Proposition 7.1). Therefore, by applying Propositions 7.3 and 7.1, we see that $\Lambda(S) + \Delta_G(T) + a(\eta)$ has the Liouville type property. ∎

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