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LIMIT THEOREMS FOR A HIGHER ORDER TIME DEPENDENT MARKOV CHAIN MODEL

BY

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Abstract. The paper establishes a strong law of large numbers and a central limit theorem for a sequence of dependent Bernoulli random variables modeled as a higher order Markov chain. The model under consideration is motivated by problems in quality control where acceptability of an item depends on the past k acceptability scores. Moreover, the model introduces dependence that may evolve over time and thus advances the theory for models with time invariant dependence. We establish explicit assumptions that incorporate this dynamic dependence and show how it enters into the limits describing long-term behavior of the system.

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1. INTRODUCTION

A great many useful stochastic models have been developed using the Markov assumption. Such models have applications in diverse fields including quality control, reliability theory, neural networks, etc.; see e.g. [14], with more references to follow. In the present paper, we study a new higher order Markov chain model and establish limit theorems for it.

Let $\{X_n, n \geqslant 1\}$ be a sequence of Bernoulli random variables defined as follows:

$$\begin{aligned} &P(X_1 = 1) = p, \\ &P(X_i = 1 \mid \mathscr{F}_{i-1}) = (1-\omega_i)p + \frac{\omega_i}{i-1} S_{i-1} \quad \text{for } 2 \leqslant i \leqslant k+1, \\ &P(X_i = 1 \mid \mathscr{F}_{i-1}) = (1-\omega_i)p + \frac{\omega_i}{k} S_{i,k} \quad \text{for } k+2 \leqslant i \leqslant n, \end{aligned} \tag{1.1}$$

where $0 < \omega_i < 1$, $0 < p < 1$, $\mathscr{F}_i$ denotes the σ-field generated by random variables $X_1, \dots, X_i$ and

$$(1.2) \qquad S_n = \sum_{i=1}^{n} X_i, \quad S_{i,k} = S_{i-1} - S_{i-k-1} = \sum_{j=i-k}^{i-1} X_j, \quad i \geqslant k+1.$$

An important feature of the above model is that the dependence on the previous average changes at time k. For the initial period of time, up to time $k+1$, all previous observations are used. Once k or more observations are available, only the most recent k observations are used. This corresponds to a typical scenario in quality control processes: we need to compare the current outcome to the previous k outcomes. If k outcomes are not available yet, we use those that are available; see e.g. [2]. In forecasting, we often observe data over a long period. Only the most recent k observations should be used for forecasting; see e.g. [5, Chapter 10].

With a suitable interpretation of the average, model (1.1) can be written as $P(X_i = 1 \mid \mathscr{F}_{i-1}) = p + \omega_i(\bar{X} - p)$. The size of the ω_i quantifies the dependence of the current observation on the previous observations. We refer to them as the *dependence parameters*. If the ω_i form a monotone sequence, then the conditional probabilities either increase or decrease whenever $\bar{X} > p$. In this case, our model preserves the monotone dependence in the sense of [11]. Time-varying dependence has been used in many profound applications such as reliability of systems and neural networks. In regression, time-varying models play a vital role. The simplest model $y_i = x_i^T\theta_i + \epsilon_i$ with time-dependent coefficients θ_i is often more useful than the standard model with a fixed θ; see e.g. [13] and references therein. [22] presents a statistical method for estimating time-dependent trial to trial variation in spike trains in animal brains.

First order Markov chain models have been successfully used to model complex systems. However, in such models, the future state of a system depends only on its current state and not on any previous observations. Higher order Markov chain models do not suffer from this restriction and are of importance in various applications such as quality control, neural networks, speech recognition, and conformity of production. For example, in a production process, over a long period, the conformity of an item will depend on many previous observations. As noted above, only a fixed number of previous observations are used. For more details and applications, we refer to [16, 18] and references therein.

Asymptotic behavior of sums of dependent random variables has been a focus of probability theory for many decades. Strong laws of large numbers, laws of the iterated logarithm, central limit theorems, and strong invariance principles for dependent random variables have been studied by many authors. Many classical results are reviewed in [6]. A few recent papers are fairly closely related to our research. [9] considered a conditional probability model and established a weak law of large numbers and a central limit theorem. [21] proposed a conditional model where Bernoulli random variables depend on several previous observations. [12] established strong limit theorems for a conditional model. [19] gave new limit theorems for dependent and non-identically-distributed Bernoulli random variables.

We now explain how model (1.1) extends previously studied models. If the memory parameter k is absent, model (1.1) reduces to the previous all-sum dependent model considered in [12], where the strong law of large numbers, the central limit theorem, and the law of the iterated logarithm were established. The model of [12] extends a simpler model of [9] in which $\omega_i = \omega$ for all $i \geqslant 2$. If $k = 1$ and $\omega_i = \omega$, then, for all $2 \leqslant i \leqslant n$, model (1.1) reduces to

$$P(X_i = 1 \,|\, \mathscr{F}_{i-1}) = (1 - \omega)p + \omega X_{i-1}.$$

The above model is a well known Markov chain dependent model; see [4, 1]. In [20] limit theorems are derived for sums of random variables when $\omega_i = \omega$ in model (1.1).

In this paper, we consider model (1.1) with the novel feature of varying ω_i. We find conditions on the sequence $\{\omega_i\}$ that guarantee the strong law of large numbers and the central limit theorem, and show how these results look like in the case of nonconstant ω_i. In Section 2, we state the main results. The proofs are given in Section 3.

2. MAIN RESULTS

We use the following notation: E denotes expectation, $\mathbb{V}$ variance and Cov covariance. We denote by I the indicator function, by $\mathscr{F}_n$ the σ-field generated by random variables $\{X_1, \ldots, X_n\}$. We use the usual initialisms: CLT, WLLN and SLLN.

We derive our asymptotic results under the following assumption.

ASSUMPTION 2.1.

(i) *The limits of the following averages exist for all* $m = 1, \ldots, k-1$*:*

$$\frac{1}{n}\sum_{i=1}^{n} \omega_i, \qquad \frac{1}{n}\sum_{i=1}^{n} \omega_i^2, \qquad \frac{1}{n}\sum_{i=k+1}^{n} \omega_i^2 \prod_{j=0}^{m-1}\left(1 + \frac{2\omega_{i-j-1}}{k-j-1}\right).$$

(ii) *For all* $i \geqslant 1$, $\delta < \omega_i < 1 - \delta$ *for some* δ *satisfying*

$$0 \leqslant 1 - 2\sqrt{p(1-p)} < \delta < 1/2.$$

The formulation of Assumption 2.1 is a significant contribution of this work. Condition (i) is trivially satisfied if $\omega_i = \omega$ does not depend on i, the case considered in previous work. It is also satisfied if $\omega_i \to \omega_\infty$ as $i \to \infty$ for some $\omega_\infty \in (0, 1)$, in which case the three limits are ω_∞, ω_∞^2, $\omega_\infty^2 \prod_{j=0}^{m-1}\left(1 + \frac{2\omega_\infty}{k-j-1}\right)$. Condition (ii), in which the lower bound is informative, also extends an analogous condition formulated in the case of constant ω. Together, they say that the ω_i must be separated from 0 and 1, and limits of specific averages must exist. As we will see, these limits show up in the asymptotic variance of the X_i.

We begin with a SLLN that is proven in Section 3.2.

THEOREM 2.1. *Under model* (1.1) *and Assumption* 2.1*, the SLLN holds, that is,*

$$S_n/n \xrightarrow{\text{a.s.}} p.$$

The remaining results are proven in Section 3.3. Lemma 2.1 establishes the existence and positivity of the asymptotic variance that appears in Theorem 2.2.

LEMMA 2.1. *If Assumption* 2.1 *holds, then the limit*

$$V_\infty := \lim_{n\to\infty} \frac{1}{n} \sum_{i=k+1}^{n} \omega_i^2 \mathbb{V}(S_{i,k})$$

exists and

$$\sigma_k^2 := p(1-p) - \frac{V_\infty}{k^2} > 0.$$

The next result gives two forms of the CLT. The first one may provide a more accurate approximation in finite samples.

THEOREM 2.2. *Recall* σ_k^2 *defined in Lemma* 2.1*. Under model* (1.1) *and Assumption* 2.1*,*

$$\left(1 - \frac{1}{n}\sum_{i=1}^{n}\omega_i\right)\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N(0, \sigma_k^2), \tag{2.1}$$

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{\sigma_k^2}{(1-\omega_\infty)^2}\right), \tag{2.2}$$

where $\omega_\infty = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\omega_i$.

In the following corollary, we consider special cases of model (1.1).

COROLLARY 2.1. *If* $\omega_i = \omega$*, then*

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{\sigma_k^2}{(1-\omega)^2}\right),$$

where

$$\sigma_1^2 = p(1-p)(1-\omega^2), \tag{2.3}$$

$$\sigma_k^2 = p(1-p) - \frac{p(1-p)\omega^2}{k^2}\,\frac{k - B(k, 2\omega)^{-1}}{1-2\omega} \quad \textit{for } k > 1, \tag{2.4}$$

where $B(\cdot,\cdot)$ *is the beta function.*

A natural question is whether model (1.1), under our assumptions, has short-range or long-range dependence. Maybe it can have either one, depending on the sequence $\{\omega_i\}$? Lemma 2.1 is not precise enough to answer this question. Long-range dependence (LRD), also referred to as long memory, is typically defined for stationary time series; see e.g. [3]. However, a more general definition was proposed and studied by [10, 9] where the LRD was defined as follows. Suppose Y_i are mean zero and set

$$U_n = \frac{(\sum_{i=1}^n Y_i)^2}{\sum_{i=1}^n Y_i^2}. \tag{2.5}$$

If $U_n \xrightarrow{\text{P}} \infty$, we say that the model *exhibits LRD*, otherwise it is a *short-range dependent model*. Corollary 2.2 shows that the model we consider is short-range dependent.

COROLLARY 2.2. *Under model* (1.1) *and Assumption* 2.1*, the sequence*

$$U_n = \frac{(S_n - np)^2}{\sum_{i=1}^n (X_i - p)^2} \tag{2.6}$$

is bounded in probability.

3. PROOFS OF THE RESULTS OF SECTION 2

3.1. Preliminaries. For ease of reference, we state here a few known results, Recall that $\mathscr{F}_n$ is the σ-field generated by the random variables $X_1, \ldots, X_n$.

DEFINITION 3.1. Suppose $\{X_n\}$ is a sequence of random variables and set $Y_n = X_n - E[X_n \mid \mathscr{F}_{n-1}]$. The sequence $\{Y_n\}$ is called a *martingale difference sequence* if $E|Y_n| < \infty$ and $EY_n = 0$ for each n.

The next two results can be found in [8, Theorem 2.17].

LEMMA 3.1. *Let $\{Y_n, \mathscr{F}_n, n \geqslant 1\}$ be a martingale difference sequence. If $\sum_{i=1}^{\infty} E[Y_i^2 \mid \mathscr{F}_{i-1}] < \infty$ almost surely, then $\sum_{i=1}^{n} Y_i$ converges almost surely.*

LEMMA 3.2. *Let $\{Y_n, \mathscr{F}_n, n \geqslant 1\}$ be a bounded martingale difference sequence, i.e., $|Y_n| \leqslant M$ a.s. for a constant M. Assume that there exist positive constants σ_n such that $\sigma_n \to \infty$ as $n \to \infty$, and*

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{n} E[Y_i^2 \mid \mathscr{F}_{i-1}] \xrightarrow{\text{P}} 1, \tag{3.1}$$

and the conditional Lindeberg condition holds, i.e., for every $\epsilon > 0$,

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{n} E[Y_i^2 I(|Y_i| > \epsilon\sigma_n) \mid \mathscr{F}_{i-1}] \xrightarrow{\text{P}} 0.$$

Then

$$\frac{1}{\sigma_n} \sum_{i=1}^{n} Y_i \xrightarrow{\text{d}} N(0,1).$$

The next result is the well-known Kronecker lemma that can be found e.g. in [17].

LEMMA 3.3. *If* $\sum_{n=1}^{\infty} x_n$ *converges to a finite limit and* $\{b_n\}$ *is a sequence that is nondecreasing and diverging to infinity, then* $b_n^{-1} \sum_{k=1}^{n} b_k x_k \to 0$.

Next, we state a version of the law of large numbers for bounded random variables. A proof of this result can be found, e.g., in [15].

LEMMA 3.4. *Suppose that* $\{Y_i, i \geqslant 1\}$ *is a sequence of random variables that are absolutely bounded by* 1 *and set* $\bar{Y}_n := n^{-1} \sum_{i=1}^{n} Y_i$. *If*

$$\sum_{n=1}^{\infty} n^{-1} E[\bar{Y}_n^2] < \infty,$$

then $\bar{Y}_n \xrightarrow{\text{a.s.}} 0$.

3.2. Proof of Theorem 2.1. We begin with a lemma that lists useful properties of certain autocorrelations in model (1.1). They are used in the proofs, but are also of independent interest because they show asymptotic decorrelation. To lighten the notation, denote

$$E_i[\cdot] := E[\cdot \mid \mathcal{F}_i].$$

LEMMA 3.5. *Under model* (1.1) *and Assumption* 2.1 *there exists a constant* $c > 0$*, only depending on* δ*, such that*

$$|\mathrm{Cov}(X_{m+j-1}, X_j)| \leqslant c(1-\delta)^{m/k}, \quad j \geqslant k+1,\, m \geqslant k+1. \tag{3.2}$$

Proof. The proof uses induction on m. Recall that by Assumption 2.1, $\delta < \omega_i < 1-\delta$ for some $0 < \delta < 1/2$.

Using (1.1) and induction, it is easy to check that

$$EX_i = p, \quad i \geqslant 1. \tag{3.3}$$

Now, for any $m \geqslant 2$ we observe that

$$\mathrm{Cov}(X_{m+j-1}, X_j) = E[X_j E_{m+j-2} X_{m+j-1}] - EX_{m+j-1} EX_j.$$

According to (3.3), we have $EX_{m+j-1} EX_j = p^2$. Furthermore, using model (1.1), (3.3) and the condition $j \geqslant k+1$, we get

$$\begin{aligned} &E[X_j E_{m+j-2} X_{m+j-1}] \\ &\quad = (1-\omega_{m+j-1})p^2 + \frac{\omega_{m+j-1}}{k}\big[E[X_j X_{m+j-1-k}] + \cdots + E[X_j X_{m+j-2}]\big]. \end{aligned}$$

Combining these two insights shows that

$$
\begin{aligned}
(3.4)\qquad & |\mathrm{Cov}(X_{m+j-1}, X_j)| \\
&= \left| \frac{\omega_{m+j-1}}{k} \big[E[X_j X_{m+j-1-k}] + \cdots + E[X_j X_{m+j-2}] \big] - \omega_{m+j-1} p^2 \right| \\
&= \left| \frac{\omega_{m+j-1}}{k} [\mathrm{Cov}(X_j, X_{m+j-1-k}) + \cdots + \mathrm{Cov}(X_j, X_{m+j-2})] \right| \\
&\leqslant \frac{1-\delta}{k} [|\mathrm{Cov}(X_j, X_{m+j-1-k})| + \cdots + |\mathrm{Cov}(X_j, X_{m+j-2})|] \\
&\leqslant (1-\delta) \max\{|\mathrm{Cov}(X_j, X_{m+j-1-k})|, \ldots, |\mathrm{Cov}(X_j, X_{m+j-2})|\}.
\end{aligned}
$$

Now, we can apply the same argument to any of the covariances on the right and (since the minimum index of X decreases by k every time), we can apply it another $\lfloor (m-1)/k \rfloor$ times, yielding

$$|\mathrm{Cov}(X_{m+j-1}, X_j)| \leqslant (1-\delta)^{1+\lfloor (m-1)/k \rfloor} \leqslant c(1-\delta)^{m/k}$$

for a constant $c \leqslant (1-\delta)^{-2}$. ■

Proof of Theorem 2.1. Consider the random variables $Y_i = X_i - p$ and their average

$$\bar{Y}_n = \frac{S_n - np}{n}.$$

It is enough to show that

$$\bar{Y}_n \to 0 \quad \text{a.s.} \tag{3.5}$$

To apply Lemma 3.4, observe that

$$E(\bar{Y}_n)^2 = \frac{\mathbb{V}(S_n)}{n^2}.$$

It is easy to show that

$$|\mathbb{V}(S_n)| \leqslant np(1-p) + 2 \sum_{l=1}^{n-1} \sum_{i=1}^{n-l} |\mathrm{Cov}(X_i, X_{i+l})|.$$

By Lemma 3.5, $\sum_{i=1}^{n-l} |\mathrm{Cov}(X_i, X_{i+l})| \leqslant A$ for some absolute constant A, so

$$|\mathbb{V}(S_n)| \leqslant np(1-p) + 2nA.$$

Thus

$$\sum_{n=1}^{\infty} n^{-1} E(\bar{Y}_n)^2 = \sum_{n=1}^{\infty} \frac{1}{n^3} \mathbb{V}(S_n) < \infty.$$

Lemma 3.4 then yields (3.5). ■

3.3. Proof of Theorem 2.2

Proof of Lemma 2.1. We first show that for model (1.1), the variance of $S_{i,k}$ is given by

$$\mathbb{V}(S_{i,k}) = p(1-p)\left\{1 + \sum_{m=1}^{k-1} \prod_{j=0}^{m-1} \left(1 + \frac{2\omega_{i-j-1}}{k-j-1}\right)\right\}, \quad i \geqslant k+1. \tag{3.6}$$

Using model (1.1), it is easy to verify that $\mathrm{Cov}(X_i, S_{i,k}) = \frac{\omega_i}{k}\mathbb{V}(S_{i,k})$. Hence

$$\mathbb{V}(S_{i+1,k+1}) = \mathbb{V}(X_i + S_{i,k}) = p(1-p) + \left(1 + \frac{2\omega_i}{k}\right)\mathbb{V}(S_{i,k}).$$

Recursively, we obtain

$$\begin{aligned}\mathbb{V}&(S_{i+1,k+1})\\ &= p(1-p)\left\{1 + \left(1 + \frac{2\omega_i}{k}\right)\right\} + \left(1 + \frac{2\omega_i}{k}\right)\left(1 + \frac{2\omega_{i-1}}{k-1}\right)\mathbb{V}(S_{i-1,k-1}).\end{aligned}$$

This leads to

$$\begin{aligned}\mathbb{V}(S_{i+1,k+1}) = p(1-p)&\left\{1 + \sum_{m=1}^{k-1} \prod_{j=0}^{m-1} \left(1 + \frac{2\omega_{i-j}}{k-j}\right)\right\}\\ &+ \left\{\prod_{j=0}^{k-1} \left(1 + \frac{2\omega_{i-j}}{k-j}\right)\right\}\mathbb{V}(S_{i-k+1,1}).\end{aligned}$$

We know that $\mathbb{V}(S_{i-k+1,1}) = \mathbb{V}(X_{i-k}) = p(1-p)$. This gives

$$\mathbb{V}(S_{i+1,k+1}) = p(1-p)\left\{1 + \sum_{m=1}^{k} \prod_{j=0}^{m-1} \left(1 + \frac{2\omega_{i-j}}{k-j}\right)\right\},$$

completing the proof of (3.6).

By Assumption 2.1 and (3.6), the limit V_∞ in Lemma 2.1 exists. For $k = 1$, it is easy to show that $\sigma_1^2 > 0$. Since $\delta < \omega_i < 1 - \delta$, it follows that

$$\frac{1}{k^2 n} \sum_{i=k+1}^{n} \omega_i^2 \mathbb{V}(S_{i,k}) \leqslant \frac{1}{4n} \sum_{i=k+1}^{n} \omega_i^2 < \frac{(1-\delta)^2}{4},$$

and Assumption 2.1 yields

$$\frac{1}{k^2 n} \sum_{i=k+1}^{n} \omega_i^2 \mathbb{V}(S_{i,k}) < \frac{(1-\delta)^2}{4} < p(1-p),$$

showing that $\sigma_k^2 > 0$. ■

Lemmas 3.6–3.9 are needed to prove the CLT. The first one gives the SLLN for weighted sums of random variables.

LEMMA 3.6. *Under model* (1.1), *we define* $\Phi_n = \sum_{i=1}^{n} a_i X_i$, *where* $a_i \in (0,1)$. *Then*

$$\left| \frac{\Phi_n}{n} - \frac{p}{n} \sum_{i=1}^{n} a_i \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Proof. Set

$$T_n = \frac{\Phi_n - E(\Phi_n)}{n}, \quad \text{where} \quad \Phi_n = \sum_{i=1}^{n} a_i X_i.$$

This gives $E(T_n) = 0$ and

$$E(T_n)^2 = \frac{\mathbb{V}(\Phi_n)}{n^2}.$$

To apply Lemma 3.4, it is enough to show that $\mathbb{V}(\Phi_n) = O(n)$. We know that

$$|\mathbb{V}(\Phi_n)| \leqslant p(1-p) \sum_{i=1}^{n} a_i^2 + A_2 \sum_{l=1}^{n-1} \sum_{m=1}^{n-l} |\text{Cov}(X_m, X_{m+l})|,$$

where $|a_l a_m| \leqslant A_2$. Using Lemma 3.5, we get

$$|\mathbb{V}(\Phi_n)| \leqslant p(1-p) \sum_{i=1}^{n} a_i^2 + A_3 \sum_{l=1}^{n-1} (n-l) r^{l/k}.$$

where A_3 is constant and $r = 1 - \delta$. This gives $\mathbb{V}(\Phi_n) = O(n)$ and

$$\sum_{n=1}^{\infty} n^{-1} E(T_n)^2 = \sum_{n=1}^{\infty} \frac{1}{n^3} \mathbb{V}(\Phi_n) < \infty,$$

completing the proof. ■

Now we establish suitable SLLNs for weighted sums of the $S_{i,k}$. Lemmas 3.7 and 3.8 follow from Lemma 3.6 and model formulation.

LEMMA 3.7. *Under model* (1.1) *and Assumption* 2.1,

$$\left| \frac{1}{n} \sum_{i=k+1}^{n} \omega_i S_{i,k} - \frac{kp}{n} \sum_{i=k+1}^{n} \omega_i \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We know that

$$\frac{1}{n} \sum_{i=k+1}^{n} \omega_i S_{i,k} = \frac{1}{n} \left\{ \sum_{i=k+1}^{n} \omega_i X_{i-1} + \sum_{i=k+1}^{n} \omega_i X_{i-2} + \cdots + \sum_{i=k+1}^{n} \omega_i X_{i-k} \right\}.$$

Using Lemma 3.6, we get

$$\frac{1}{n} \sum_{i=k+1}^{n} \omega_i S_{i,k} - \frac{kp}{n} \sum_{i=k+1}^{n} \omega_i \to 0$$

with probability 1. This completes the proof. ■

LEMMA 3.8. *Under model* (1.1) *and Assumption* 2.1*,*

$$\left|\frac{1}{n}\sum_{i=k+1}^{n}\omega_i(1-\omega_i)S_{i,k}-\frac{kp}{n}\sum_{i=k+1}^{n}\omega_i(1-\omega_i)\right|\xrightarrow{\text{a.s.}}0.$$

Proof. Use an argument analogous to the one used to prove Lemma 3.7. ∎

The next result gives the WLLN for a suitably defined sequence of correlated random variables.

LEMMA 3.9. *Set* $Z_i = \omega_i X_i S_{i,k}$ *for* $i \geqslant k+1$. *Under model* (1.1) *and Assumption* 2.1*,*

$$\left|\frac{1}{n}\sum_{i=k+1}^{n}Z_i-\frac{1}{n}\sum_{i=k+1}^{n}\mu_i\right|\xrightarrow{\text{P}}0,$$

where $\mu_i = kp^2\omega_i + \frac{\omega_i^2}{k}\mathbb{V}(S_{i,k})$.

Proof. Since $S_{i,k}$ is $\mathscr{F}_{i-1}$-measurable, we can express the expectation of Z_i as

$$E[Z_i]=E[E(Z_i\,|\,\mathscr{F}_{i-1})]=E[\omega_i S_{i,k}E(X_i\,|\,\mathscr{F}_{i-1})].$$

Now, model (1.1) gives

$$\begin{aligned}E(Z_i)&=E\left[\omega_i S_{i,k}\left\{(1-\omega_i)p+\frac{\omega_i}{k}S_{i,k}\right\}\right]\\&=\omega_i(1-\omega_i)pE(S_{i,k})+\frac{\omega_i^2}{k}E(S_{i,k})^2\\&=kp^2\omega_i+\frac{\omega_i^2}{k}\mathbb{V}(S_{i,k})=\mu_i.\end{aligned}$$

As a consequence, the difference $\frac{1}{n}\sum_{i=k+1}^{n}Z_i-\frac{1}{n}\sum_{i=k+1}^{n}\mu_i$ is centered, and we can use Chebyshev's inequality to show convergence to 0. More precisely,

$$P\left(\left|\frac{1}{n}\sum_{i=k+1}^{n}Z_i-\frac{1}{n}\sum_{i=k+1}^{n}\mu_i\right|\geqslant\epsilon\right)\leqslant\frac{\mathbb{V}(\sum_{i=k+1}^{n}Z_i)}{n^2\epsilon^2}. \tag{3.8}$$

So, to obtain the lemma it suffices to show that

$$\mathbb{V}\Big[\sum_{i=k+1}^{n}Z_i\Big]=O(n). \tag{3.9}$$

To establish (3.9), we prove the stronger result

$$S_Z(n):=\sum_{i,j=k+1}^{n}|\text{Cov}(Z_i,Z_j)|=O(n). \tag{3.10}$$

The proof of (3.10) begins by grouping the terms on the left according to the distance of i, j (larger distances translate into weaker dependence). More precisely, we rewrite the left-hand side of (3.10) as

$$S_Z(n) = \sum_{b=1}^{n/k} \sum_{(b-1)k \leqslant |i-j| < bk} |\mathrm{Cov}(Z_i, Z_j)|. \tag{3.11}$$

Here, without loss of generality, we assume that n is divisible by k (adapting the argument for the case of nondivisibility is easy and therefore omitted). In the following, we establish that for a pair (i, j) with $(b-1)k \leqslant |i-j| < bk$ we have

$$|\mathrm{Cov}(Z_i, Z_j)| \leqslant Kb(1-\delta)^b \tag{3.12}$$

for some universal constant K and $\delta \in (0, 1/2)$ defined in Assumption 2.1. If (3.12) is true, it follows directly that (3.11) is of order $O(n)$ (due to summability of the sequence $(b(1-\delta)^b)_{b\in\mathbb{N}}$ over b), implying (3.10), thereby (3.9) and thus the lemma according to (3.8).

In the remainder of this proof, we demonstrate (3.12) for $b \geqslant 2$ (the case $b = 1$ is easier and therefore omitted). Let us henceforth consider a fixed but arbitrary pair (i, j) which satisfies for some $b \geqslant 2$ the inequality $(b-1)k \leqslant |i-j| < bk$ and without loss of generality suppose $i < j$. We can bound the absolute value of the covariance between Z_i, Z_j as follows:

$$\begin{aligned} |\mathrm{Cov}(Z_i, Z_j)| &= \big|E\big[(Z_i - E\{Z_i\})w_j X_j S_{j,k}\big]\big| \\ &= \Big|\sum_{l=1}^{k} w_j E\big[(Z_i - E\{Z_i\})X_j X_{j-l}\big]\Big| \\ &\leqslant k(1-\delta) \max_{l=1,\dots,k} \big|E\big[(Z_i - E\{Z_i\})X_j X_{j-l}\big]\big|. \end{aligned} \tag{3.13}$$

Here, in the first equality, we have used the definition of $Z_j = \omega_j X_j S_{j,k}$, together with the fact that the covariance of a random variable with a constant is 0. In the second equality, we have plugged in the definition of $S_{j,k}$ (see (1.2)). In the final step, we have used the triangle inequality and bounded the sum by the number of terms times the maximum term. Moreover, we have used the fact that $w_j \leqslant 1-\delta$ for all j (see Assumption 2.1(ii)). Let us now suppose that the maximum over l is assumed for some $l^{(1)}$. Conditioning on $\mathscr{F}_{j-1}$ then yields

$$\begin{aligned} &k(1-\delta)\big|E\big[E\big[(Z_i - E\{Z_i\})X_j X_{j-l^{(1)}} \,\big|\, \mathscr{F}_{j-1}\big]\big]\big| \\ &\quad = k(1-\delta)\big|E\big[(Z_i - E\{Z_i\})E(X_j \,|\, \mathscr{F}_{j-1})X_{j-l^{(1)}}\big]\big| \\ &\quad = k(1-\delta)\frac{w_j}{k}\left|E\left[(Z_i - E\{Z_i\})\left(\frac{\omega_j}{k}S_{j,k} + (1-\omega_j)p\right)X_{j-l^{(1)}}\right]\right| \\ &\quad \leqslant k(1-\delta)\frac{w_j}{k}\big|E\big[(Z_i - E\{Z_i\})S_{j,k}X_{j-l^{(1)}}\big]\big| \\ &\qquad + k(1-\delta)(1-w_j)\big|E\big[(Z_i - E\{Z_i\})X_{j-l^{(1)}}\big]\big| =: R_1 + R_2. \end{aligned} \tag{3.14}$$

Here, R_1 and R_2 on the right are defined in the obvious way. In the first equality of (3.14), we have exploited that all variables except X_j are $\mathscr{F}_{j-1}$-measurable. In the second equality, we have used that by model (1.1) (and since $j > k$)

$$E[X_j = 1 \mid \mathscr{F}_{j-1}] = \frac{\omega_j}{k} S_{j,k} + (1 - \omega_j)p\,.$$

We now treat the two terms R_1, R_2 separately. Recalling that $w_j < 1 - \delta$ yields

$$R_1 \leqslant (1-\delta)^2 \big| E\big[(Z_i - E\{Z_i\}) S_{j,k} X_{j-l^{(1)}}\big]\big|. \tag{3.15}$$

For R_2, we employ the (subsequently derived) upper bound (3.21) with parameters $b' = b - 1$ and $j' = j - l^{(1)}$, together with the fact that $1 - w_j < 1 - \delta$, which yields

$$R_2 \leqslant [k(1-\delta)(1-w_j)][k(1-\delta)^{b-2}] \leqslant k^2(1-\delta)^b. \tag{3.16}$$

Since

$$|\mathrm{Cov}(Z_i, Z_j)| \leqslant R_1 + R_2, \tag{3.17}$$

(3.15) and (3.16) suffice to establish (3.12) for the case $b = 2$. Notice that the constant K in (3.12) can be chosen as k^2, since it follows by simple calculations (bounding the expectation of a random variable by its maximum absolute value) that for any $i, j, l^{(1)}$,

$$\big| E\big[(Z_i - E\{Z_i\}) S_{j,k} X_{j-l^{(1)}}\big]\big| \leqslant k^2.$$

In particular, (3.15) implies $R_1 \leqslant k^2(1-\delta)^b$. So, our previous considerations imply

$$|\mathrm{Cov}(Z_i, Z_j)| \leqslant R_1 + R_2 \leqslant 2k^2(1-\delta)^2,$$

proving (3.12) for $b = 2$.

The case $b \geqslant 3$ can now be treated by backward induction, using very similar calculations to those in (3.13) and (3.14). We sketch these arguments by considering the case $b = 3$.

We want to establish (3.12) for $b = 3$. In view of (3.17), we further bound R_1 or, more precisely, the right side of (3.15). Proceeding as in (3.13), we have

$$\begin{aligned} &(1-\delta)^2 \big| E\big[(Z_i - E\{Z_i\}) S_{j,k} X_{j-l^{(1)}}\big]\big| \\ &\qquad \leqslant k(1-\delta)^2 \max_{l=1,\dots,k} \big| E\big[(Z_i - E\{Z_i\}) X_{j-l} X_{j-l^{(1)}}\big]\big| \\ &\qquad = k(1-\delta)^2 \big| E\big[(Z_i - E\{Z_i\}) X_{j-l^{(2)}} X_{j-l^{(1)}}\big]\big| \end{aligned} \tag{3.18}$$

Here, $l^{(2)}$ is the index for which the maximum in the second line is attained. We distinguish two cases: $l^{(2)} \neq l^{(1)}$ and $l^{(2)} = l^{(1)}$.

We begin with the case $l^{(2)} \neq l^{(1)}$, where without loss of generality we assume $l^{(2)} < l^{(1)}$ (otherwise swap their roles). Notice that we then have $j - l^{(2)} > i$ since

$$j - l^{(2)} \geqslant j - k > i$$

for $b = 3$. Hence, $X_i, S_{i,k}, X_{j-l^{(1)}}$ are $\mathscr{F}_{j-l^{(2)}-1}$-measurable. This yields (in analogy to (3.14))

$$\begin{aligned}(3.19)\qquad k(1-\delta)^2 & \big|E\big[E\big[(Z_i - E\{Z_i\})X_{j-l^{(2)}}X_{j-l^{(1)}} \mid \mathscr{F}_{j-l^{(2)}-1}\big]\big]\big| \\ &= k(1-\delta)^2\big|E\big[(Z_i - E\{Z_i\})E(X_{j-l^{(2)}}|\mathscr{F}_{j-l^{(2)}-1})X_{j-l^{(1)}}\big]\big| \\ &\leqslant (1-\delta)^3\big|E\big[(Z_i - E\{Z_i\})S_{j-l^{(2)},k}X_{j-l^{(1)}}\big]\big| \\ &\quad + k(1-\delta)^3\big|E\big[(Z_i - E\{Z_i\})X_{j-l^{(1)}}\big]\big| =: R'_1 + R'_2.\end{aligned}$$

Again, the terms R'_1 and R'_2 are defined in the obvious way. By simple calculations, we can bound the expectations on the right of (3.19) by k^2 and k respectively, which entails

$$R'_1, R'_2 \leqslant k^2(1-\delta)^3. \tag{3.20}$$

Now, using inequality (3.17) in the first step, the decomposition $R_1 \leqslant R'_1 + R'_2$ in the second and the bounds (3.16), (3.20) in the third yields

$$|\text{Cov}(Z_i, Z_j)| \leqslant R_1 + R_2 \leqslant R'_1 + R'_2 + R_2 \leqslant 3k^2(1-\delta)^3.$$

This proves (3.12) for $b = 3$ in the case $l^{(2)} \neq l^{(1)}$.

Next, let us turn to the case $l^{(2)} = l^{(1)}$. Again, we want to bound R_1. This time, we have $X_{j-l^{(1)}}X_{j-l^{(2)}} = X^2_{j-l^{(1)}} = X_{j-l^{(1)}}$, so that the right side of (3.18) can be expressed as

$$k(1-\delta)^2\big|E\big[(Z_i - E\{Z_i\})X_{j-l^{(1)}}\big]\big|.$$

We can now further bound the expectation using the inductive formula (3.21) (below) for $j' = j - l^{(1)}$ and $b' = 2$ (note again that $j - l^{(1)} \geqslant j - k > i$ since $b = 3$). This gives

$$\big|E\big[(Z_i - E\{Z_i\})X_{j-l^{(1)}}\big]\big| \leqslant k(1-\delta),$$

entailing $R_1 \leqslant k^2(1-\delta)^3$ and together with (3.16) that

$$|\text{Cov}(Z_i, Z_j)| \leqslant R_1 + R_2 \leqslant 2k^2(1-\delta)^3.$$

This means that (3.12) holds for $b = 3$ in the case $l^{(2)} = l^{(1)}$.

We complete the proof by showing that for any $b' \geqslant 2$ and $j' > i$ satisfying $(b'-1)k \leqslant |i - j'| < b'k$ we have

$$\big|E\big[(Z_i - E\{Z_i\})X_{j'}\big]\big| \leqslant (1-\delta)^{b'-1}k. \tag{3.21}$$

By similar arguments to those before, we upper bound the left side as follows:

$$
\begin{aligned}
(3.22)\qquad \big|E\big[E\big[(Z_i - E\{Z_i\})X_{j'} \,\big|\, \mathscr{F}_{j'-1}\big]\big]\big| \\
&= \frac{\omega_{j'}}{k}\big|E\big[(Z_i - E\{Z_i\})S_{j',k}\big]\big| \\
&\leqslant (1-\delta) \max_{l=1,\dots,k}\big|E\big[(Z_i - E\{Z_i\})X_{j'-l}\big]\big| \\
&= (1-\delta)\big|E\big[(Z_i - E\{Z_i\})X_{j'-l^{(3)}}\big]\big|.
\end{aligned}
$$

Here, $l^{(3)}$ is the index where the maximum in the third line is attained. Notice that $j' - l^{(3)} \geqslant j' - k$. The proof now follows by backward induction on b' (with a total of $b' - 1$ steps), while noticing that for any pair (i, j) we have

$$
\big|E\big[(Z_i - E\{Z_i\})X_j\big]\big| \leqslant k.
$$

This last inequality follows since the random variable inside the expectation is absolutely bounded by k. ■

LEMMA 3.10. *Under model* (1.1) *and Assumption* 2.1*,*

$$
\left|\frac{1}{n}\sum_{i=k+1}^{n}(\omega_i S_{i,k})^2 - \frac{1}{n}\sum_{i=k+1}^{n}\omega_i^2\mu_i\right| \xrightarrow{P} 0,
$$

where $\mu_i = k^2p^2 + \mathbb{V}(S_{i,k})$.

Proof. Define

$$
(3.23)\qquad W_i = \omega_i S_{i,k}\left(X_i - (1-\omega_i)p - \frac{\omega_i}{k}S_{i,k}\right), \qquad i \geqslant k+1.
$$

Since $|W_i| \leqslant 3k$ and $E(W_i \,|\, \mathscr{F}_{i-1}) = 0$, $\{W_i\}$ is a bounded martingale difference sequence. Set $Q_n = W_n/n$ and observe that for a constant D,

$$
\sum_{i=k+1}^{n} E(Q_i^2 \,|\, \mathscr{F}_{i-1}) \leqslant D\sum_{i=k+1}^{n}\frac{1}{i^2} < \infty.
$$

By Lemma 3.1, $\sum_{i=k+1}^{n} Q_n$ converges almost surely. Using the Kronecker lemma, Lemma 3.3, it follows that

$$
\frac{1}{n}\sum_{i=k+1}^{n} iQ_i = \frac{1}{n}\sum_{i=k+1}^{n} W_i \xrightarrow{\text{a.s.}} 0.
$$

From (3.23), we get

$$
\frac{1}{n}\sum_{i=k+1}^{n}\omega_i X_i S_{i,k} - \frac{p}{n}\sum_{i=k+1}^{n}\omega_i(1-\omega_i)S_{i,k} - \frac{1}{kn}\sum_{i=k+1}^{n}(\omega_i S_{i,k})^2 \to 0
$$

with probability 1. Lemmas 3.8 and 3.9 yield

$$\frac{1}{n}\sum_{i=k+1}^{n}\omega_i^2\mathbb{V}(S_{i,k})+\frac{k^2p^2}{n}\sum_{i=k+1}^{n}\omega_i^2-\frac{1}{n}\sum_{i=k+1}^{n}(\omega_iS_{i,k})^2\xrightarrow{\text{P}}0.$$

This completes the proof. ∎

Let $\{M_n,\mathscr{F}_n,\ n\geqslant 1\}$ be a martingale difference sequence defined by $M_1=X_1-p$ and

$$M_i=\begin{cases}X_i-(1-\omega_i)p-\dfrac{\omega_i}{i-1}S_{i-1} & \text{for } 2\leqslant i\leqslant k+1,\\ X_i-(1-\omega_i)p-\dfrac{\omega_i}{k}S_{i,k} & \text{for } k+2\leqslant i\leqslant n.\end{cases}\tag{3.24}$$

It is easy to verify that $|M_n|\leqslant 3$. Hence $\{M_n,\mathscr{F}_n,\ n\geqslant 1\}$ is a bounded martingale difference sequence.

Proof of Theorem 2.2. To establish the CLT using martingale theory, we must verify conditions analogous to those required in Lemma 3.2. Consider the variable $\Delta_n=\sum_{i=1}^{n}M_i$, where the M_i are bounded martingale differences defined in (3.24). We use the decomposition

$$\sum_{i=1}^{n}E(M_i^2\,|\,\mathscr{F}_{i-1})=\sum_{i=1}^{k+1}E(M_i^2\,|\,\mathscr{F}_{i-1})+\sum_{i=k+2}^{n}E(M_i^2\,|\,\mathscr{F}_{i-1}).$$

Since

$$\frac{1}{n}\sum_{i=1}^{k+1}E(M_i^2\,|\,\mathscr{F}_{i-1})=\frac{O(k)}{n},$$

we focus on the second sum. Since

$$\begin{aligned}M_i^2&=X_i^2+(1-\omega_i)^2p^2+\frac{\omega_i^2}{k^2}S_{i,k}^2\\&\quad-2X_i(1-\omega_i)p-2X_i\frac{\omega_i}{k}S_{i,k}+2(1-\omega_i)p\frac{\omega_i}{k}S_{i,k},\end{aligned}$$

according to (1.1), for $i\geqslant k+2$,

$$\begin{aligned}E(M_i^2\,|\,\mathscr{F}_{i-1})&=\left[(1-\omega_i)p+\frac{\omega_i}{k}S_{i,k}\right]+(1-\omega_i)^2p^2+\frac{\omega_i^2}{k^2}S_{i,k}^2\\&\quad-2(1-\omega_i)p\left[(1-\omega_i)p+\frac{\omega_i}{k}S_{i,k}\right]\\&\quad-2\frac{\omega_i}{k}S_{i,k}\left[(1-\omega_i)p+\frac{\omega_i}{k}S_{i,k}\right]\\&\quad+2(1-\omega_i)p\frac{\omega_i}{k}S_{i,k}\\&=(1-\omega_i)p-(1-\omega_i)^2p^2\\&\quad+\frac{\omega_i}{k}S_{i,k}-\frac{\omega_i^2}{k^2}S_{i,k}^2-2(1-\omega_i)p\frac{\omega_i}{k}S_{i,k}.\end{aligned}$$

We therefore obtain

$$\text{(3.25)}\quad \frac{1}{n}\sum_{i=1}^{n} E(M_i^2|\mathscr{F}_{i-1}) = \frac{m_n}{n} + \frac{1}{nk}\sum_{i=k+2}^{n}\omega_i S_{i,k} - \frac{1}{k^2 n}\sum_{i=k+2}^{n}(\omega_i S_{i,k})^2 - \frac{2p}{nk}\sum_{i=k+2}^{n}\omega_i(1-\omega_i)S_{i,k} + \frac{O(k)}{n},$$

where

$$m_n = p\sum_{i=k+2}^{n}(1-\omega_i) - p^2\sum_{i=k+2}^{n}(1-\omega_i)^2.$$

Lemmas 3.7 and 3.8 and relations (3.10) and (3.25) give

$$\frac{1}{n}\sum_{i=1}^{n} E(M_i^2 \mid \mathscr{F}_{i-1}) \xrightarrow{\mathrm{P}} p(1-p) - \beta_k,$$

where

$$\beta_k = \lim_{n\to\infty}\frac{1}{k^2 n}\sum_{i=k+1}^{n}\omega_i^2\mathbb{V}(S_{i,k}) = \frac{V_\infty}{k^2}.$$

Since $\{M_n, n \geqslant 1\}$ is bounded martingale difference sequence, the conditional Lindeberg condition is also satisfied, i.e., for every $\epsilon > 0$,

$$\text{(3.26)}\quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} E\left[M_i^2 I\left(\left|\frac{M_i}{\sqrt{n}}\right| \geqslant \epsilon\right)\,\middle|\,\mathscr{F}_{i-1}\right] \xrightarrow{\mathrm{P}} 0.$$

Using Lemma 3.2, we get

$$\text{(3.27)}\quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_i = \frac{\Delta_n}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0,\sigma_k^2),$$

where $\sigma_k^2 := p(1-p) - k^{-2}V_\infty$ is as in Lemma 2.1. Moreover,

$$\text{(3.28)}\quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_i = \sqrt{n}\left\{\frac{S_n - np}{n} + \frac{p}{n}\sum_{i=2}^{n}\omega_i - \frac{1}{n}\sum_{i=2}^{k+1}\omega_i\frac{S_{i-1}}{i-1} - \frac{1}{kn}\sum_{i=k+2}^{n}\omega_i S_{i,k}\right\} = \sqrt{n}\left\{\frac{S_n - np}{n} + \frac{p}{n}\sum_{i=1}^{n}\omega_i - \frac{1}{kn}\sum_{i=1}^{n}\omega_i S_{i,k} + \frac{O(k)}{n}\right\}.$$

Using Theorem 2.1 and Lemma 3.7, we get

$$\left|\frac{1}{n}\sum_{i=1}^{n}\omega_i S_{i,k} - \frac{kS_n}{n^2}\sum_{i=1}^{n}\omega_i\right| \xrightarrow{\text{a.s.}} 0.$$

Thus, (3.28) gives

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_i \overset{\text{a.s.}}{=} \frac{S_n - np}{\sqrt{n}}\left(1 - \frac{1}{n}\sum_{i=1}^{n}\omega_i\right) \quad \text{as } n \to \infty.$$

It follows that

$$\left(1 - \frac{1}{n}\sum_{i=1}^{n}\omega_i\right)\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N(0, \sigma_k^2),$$

with σ_k^2 defined in Lemma 2.1. This completes the proof of (2.1). Relation (2.2) follows because by Assumption 2.1, $n^{-1}\sum_{i=1}^{n}\omega_i \to \omega_\infty \in (0,1)$. ■

Proof of Corollary 2.1. When $\omega_i = \omega$ and $k = 1$, then

$$\sigma_1^2 = p(1-p) - \lim_{n\to\infty}\frac{\omega^2}{n}\sum_{i=2}^{n}\mathbb{V}(S_{i,1}),$$

where $\mathbb{V}(S_{i,1}) = \mathbb{V}(X_{i-1}) = p(1-p)$. This gives $\sigma_1^2 = p(1-p)(1-\omega^2)$. Using Theorem 2.2, we get

$$(1-\omega)\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N(0, \sigma_1^2),$$

completing the proof of (2.3). If $k > 1$, then

$$\sigma_k^2 = p(1-p) - \frac{\omega^2}{k^2}H_\infty$$

and Theorem 2.2 yields

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{\sigma_k^2}{(1-\omega)^2}\right),$$

where H_∞ (see [7]) is given by

$$H_\infty = p(1-p)\frac{k - B(k, 2\omega)^{-1}}{1 - 2\omega}. \blacksquare$$

Proof of Corollary 2.2. Observe that

$$U_n = \frac{n}{\sum_{i=1}^{n}(X_i - p)^2}\left(\frac{S_n - np}{\sqrt{n}}\right)^2.$$

By Theorem 2.1,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - p)^2 \xrightarrow{\text{a.s.}} p(1-p)$$

and Theorem 2.2 gives

$$\left(\frac{S_n - np}{\sqrt{n}}\right)^2 \xrightarrow{\mathrm{d}} \frac{\sigma_k^2}{(1-\omega)^2}\chi^2_{(1)}.$$

Using Slutsky's theorem, we thus get

$$U_n \xrightarrow{\mathrm{d}} \frac{\sigma_k^2}{p(1-p)(1-\omega)^2}\chi^2_{(1)}. \blacksquare$$

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