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UNIFORM CONVERGENCE RATES OF SKEW-NORMAL EXTREMES

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Abstract. Let $M_n = \max(X_1, \ldots, X_n)$ denote the partial maximum of an independent and identically distributed skew-normal random sequence. In this paper, the rate of uniform convergence of skew-normal extremes is derived. It is shown that with optimal normalizing constants the convergence rate of $a_n^{-1}(M_n - b_n)$ to its ultimate extreme value distribution is proportional to $\frac{1}{\log n}$.

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1. INTRODUCTION

Skew-normal distribution introduced by Azzalini (1985) is an effective tool to model skewed data compared to the normal distribution. Applications of the skew-normal distribution include areas such as climatology, biomedical sciences, economics and finance. Kim and Mallick (2004) presented a model based on the skew-normal distribution for the prediction of weekly rainfall in Korea. Counsell et al. (2011) applied the skew-normal distribution to deal with data from a clinical psychiatry research environment. Considering the effect of skewness and coskewness on asset valuation, Carmichael and Coën (2013) derived restrictions imposed by the Euler equation of optimal portfolio diversification. Zeller et al. (2016) established a mixture regression model by assuming that the random errors follow a scale mixture of skew-normal distributions. With a skew-normal prior distribution for the spatial latent variables, Hosseini et al. (2011) proposed approximate Bayesian methods for inference and spatial prediction in a spatial generalized linear mixed model. For more applications and case studies involving the skew-normal distribution, see Genton (2004).

Recently, probability properties such as tail behavior and asymptotics of skew-normal extremes have been studied. Chang and Genton (2007) showed that F_λ be-

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longs to the domain of attraction of the Gumbel extreme value distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, where F_λ is the cumulative distribution function (cdf) of the standard skew-normal random variable with parameter $\lambda \in \mathbb{R}$ (written as $\mathbb{SN}(\lambda)$). The probability density function (pdf) of $\mathbb{SN}(\lambda)$, f_λ, is

$$f_\lambda(x) = 2\phi(x)\Phi(\lambda x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the pdf and the cdf of a standard normal random variable. For $\mathbb{SN}(0)$, the standard normal random variable, Mills' ratio and extreme value distribution are known; see Leadbetter et al. (1983). Higher-order expansions of the distribution and moments of the extremes of $\mathbb{SN}(0)$, were studied by Nair (1981). For $\lambda \neq 0$, Mills' ratios, the distributional tail representation and the higher-order expansions of the extremes of $\mathbb{SN}(\lambda)$ were studied by Liao et al. (2014b). The higher-order expansions of moments of the extremes of $\mathbb{SN}(\lambda)$, $\lambda \neq 0$, were studied by Liao et al. (2013a). Liao et al. (2013b, 2014a) also considered the tail behaviors and higher-order expansions of the distribution of extremes for the log-skew-normal distribution.

The aim of this paper is to derive the uniform convergence rates of skew-normal extremes for $\lambda \neq 0$. For $\mathbb{SN}(0)$, the standard normal random variable, Hall (1979) derived the optimal uniform convergence rate of $\Phi^n(\widetilde{a}_n x + \widetilde{b}_n)$ to $\Lambda(x)$, i.e.,

$$\frac{\mathbb{C}_1}{\log n} < \sup_{x\in\mathbb{R}} |\Phi^n(\widetilde{a}_n x + \widetilde{b}_n) - \Lambda(x)| < \frac{\mathbb{C}_2}{\log n} \tag{1.2}$$

for some positive constants $\mathbb{C}_1$ and $\mathbb{C}_2$, where the normalizing constant $\widetilde{b}_n$ is the solution of

$$2\pi\widetilde{b}_n^2 \exp(\widetilde{b}_n^2) = n^2$$

and $\widetilde{a}_n = \widetilde{b}_n^{-1}$. For $\mathbb{SN}(0)$, Gasull et al. (2015a) gave more effective normalizing constants a_n and b_n through the Lambert W function. Gasull et al. (2015b) illustrated another application of the Lambert W function to decide on normalizing constants for gamma and other Weibull-like distributions. For other work related to convergence rates of distributions of normalized order statistics, see Liao and Peng (2012) for the log-normal distribution, and Peng et al. (2010) and Vasudeva et al. (2014) for the general error distribution.

In order to derive the uniform convergence rates of skew-normal extremes, we choose the optimal normalizing constants according to the sign of λ with $\lambda \neq 0$. For $\lambda > 0$, let b_n be the solution of

$$\sqrt{\pi/2}\, b_n \exp(b_n^2/2) = n, \tag{1.3}$$

and set

$$a_n = b_n^{-1}. \tag{1.4}$$

For $\lambda < 0$, let $b_n > 0$ be the solution of

$$\pi|\lambda|(1+\lambda^2)b_n^2\exp((1+\lambda^2)b_n^2/2) = n, \tag{1.5}$$

and set

$$a_n = (1+\lambda^2)^{-1}b_n^{-1}. \tag{1.6}$$

The rest of this paper is organized as follows. Section 2 gives the main results. Some auxiliary lemmas and all proofs are presented in Section 3.

2. MAIN RESULTS

In this section, we provide the main results. Theorem 2.1 shows that the limit distribution of normalized maxima for the skew-normal distribution is the Gumbel extreme value distribution $\Lambda(x)$. Theorems 2.2 and 2.3 determine the rates of uniform convergence of skew-normal extremes. Note that the choice of normalizing constants are determined according to the sign of λ.

THEOREM 2.1. *Let M_n denote the partial maximum of independent and identical $\mathbb{SN}(\lambda)$ random variables with the pdf f_λ given by* (1.1). *Then*

$$\mathbf{P}(M_n \leqslant a_n x + b_n) \to \Lambda(x), \quad x \in \mathbb{R} \tag{2.1}$$

as $n \to \infty$, where the normalizing constants b_n and a_n are given by (1.3) *and* (1.4) *for $\lambda > 0$, and by* (1.5) *and* (1.6) *for $\lambda < 0$.*

THEOREM 2.2. *For $\lambda > 0$, there exist positive constants $\mathbb{C}$ and $\mathbb{C}_\lambda$, independent of n, such that*

$$\frac{\mathbb{C}}{\log n} < \sup_{x\in\mathbb{R}} |F_\lambda^n(a_n x + b_n) - \Lambda(x)| < \frac{\mathbb{C}_\lambda}{\log n} \tag{2.2}$$

for all $n \geqslant 9$, where b_n and a_n are given by (1.3) *and* (1.4)*, respectively.*

THEOREM 2.3. *For $\lambda < 0$, there exist positive constants $\mathbb{C}'_\lambda$ and $\mathbb{C}''_\lambda$, independent of n, such that*

$$\frac{\mathbb{C}'_\lambda}{\log n} < \sup_{x\in\mathbb{R}} |F_\lambda^n(a_n x + b_n) - \Lambda(x)| < \frac{\mathbb{C}''_\lambda}{\log n} \tag{2.3}$$

for all $n \geqslant n_0(\lambda)$, where b_n and a_n are given by (1.5) *and* (1.6)*, respectively, and $n_0(\lambda)$ is a constant.*

REMARK 2.1. As noted on pages 39–40 in Leadbetter et al. (1983) for the normal case, the convergence rate of the distribution of the normalized maximum to its ultimate extreme value distribution $\Lambda(x)$ may be different for different choices of the normalizing constants. For the skew-normal distribution, Proposition 3 and Theorem 1 in Liao et al. (2014b) showed that Theorem 2.1 holds with normalizing constants α_n and β_n given by Proposition 3 of Liao et al. (2014b), and its pointwise convergence rate is proportional to $\frac{(\log\log n)^2}{\log n}$. Theorem 2 of Liao et al. (2014b) showed that the pointwise convergence rate can be improved by using another pair of normalizing constants. Proposition 1 of Liao et al. (2014b) inspired us to choose the normalizing constants a_n and b_n given by (1.3)–(1.4) or (1.5)–(1.6) according to the sign of λ. Theorems 2.2 and 2.3 provide the uniform convergence rate of $F_\lambda^n(a_n x + b_n)$ to $\Lambda(x)$, which is proportional to $\frac{1}{\log n}$.

3. PROOFS

In order to prove the main results, we first give some auxiliary lemmas. The first one is about the distributional tail representation of the skew-normal distribution, due to Lemma 3.1 in Xiong and Peng (2020). The remaining lemmas provide inequalities on distributional tails of the normal and skew-normal distributions.

LEMMA 3.1. *Let F_λ and f_λ denote, respectively, the cdf and the pdf of the $\mathbb{SN}(\lambda)$ distribution. For large x, we have*

(i) *for $\lambda > 0$,*

$$1 - F_\lambda(x) = \frac{2\phi(x)}{x}[1 - x^{-2} + O(x^{-4})]; \tag{3.1}$$

(ii) *for $\lambda < 0$,*

$$1 - F_\lambda(x) = \frac{e^{-(1+\lambda^2)x^2/2}}{-\pi\lambda(1+\lambda^2)x^2}\left[1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}x^{-2} + O(x^{-4})\right]. \tag{3.2}$$

LEMMA 3.2. *Let $\phi(x)$ and $\Phi(x)$ denote, respectively, the pdf and the cdf of a standard normal random variable. For all $x > 0$, we have*

$$\frac{\phi(x)}{x}(1 - x^{-2}) < 1 - \Phi(x) < \frac{\phi(x)}{x}. \tag{3.3}$$

Proof. The proof is straightforward by integration by parts: see (6)–(9) in Hall (1979).

LEMMA 3.3. *Let F_λ denote the cdf of $\mathbb{SN}(\lambda)$ and let $\phi(x)$ denote the pdf of a standard normal random variable. For all $x > 0$, we have*

(i) *for $\lambda > 0$,*

$$\frac{2\phi(x)}{x}\left[1 - \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)x^{-2}\right] < 1 - F_\lambda(x) < \frac{2\phi(x)}{x}; \tag{3.4}$$

(ii) *for* $\lambda < 0$,

$$\frac{2\phi(x)\phi(\lambda x)}{|\lambda|(1+\lambda^2)x^2}\left[1 - \frac{(1+\lambda^2)^2}{\lambda^2}x^{-2}\right] < 1 - F_\lambda(x) < \frac{2\phi(x)\phi(\lambda x)}{|\lambda|(1+\lambda^2)x^2}. \tag{3.5}$$

Proof. In the case of $\lambda > 0$, for any $x > 0$ we have

$$\begin{aligned} 1 - F_\lambda(x) &= \int_x^\infty 2\phi(t)\Phi(\lambda t)\,dt \\ &< \int_x^\infty 2\phi(t)\,dt = 2[1-\Phi(x)] \\ &< \frac{2\phi(x)}{x}. \end{aligned} \tag{3.6}$$

By integration by parts and Lemma 3.2, we have

$$\begin{aligned} 1 - F_\lambda(x) &= \frac{f_\lambda(x)}{x} - \int_x^\infty 2\phi(t)\Phi(\lambda t)t^{-2}\,dt + \frac{\lambda}{\pi}\int_x^\infty t^{-1}e^{-(1+\lambda^2)t^2/2}\,dt \\ &> \frac{f_\lambda(x)}{x} - \int_x^\infty 2\phi(t)\Phi(\lambda t)t^{-2}\,dt > \frac{f_\lambda(x)}{x} - \int_x^\infty 2\phi(t)t^{-2}\,dt \\ &> \frac{f_\lambda(x)}{x} - x^{-2}\int_x^\infty 2\phi(t)\,dt = \frac{2\phi(x)\Phi(\lambda x)}{x} - 2x^{-2}[1-\Phi(x)] \\ &> \frac{2\phi(x)}{x}\left[1 - \frac{\phi(\lambda x)}{\lambda x}\right] - 2x^{-2}\cdot\frac{\phi(x)}{x} \\ &= \frac{2\phi(x)}{x}\left\{1 - \left[1 + \frac{x\phi(\lambda x)}{\lambda}\right]x^{-2}\right\} \\ &> \frac{2\phi(x)}{x}\left[1 - \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)x^{-2}\right]. \end{aligned} \tag{3.7}$$

The last inequality was obtained by bounding the function $x\exp(-\lambda^2x^2/2)$. Combining (3.6) with (3.7), we can derive (3.4).

In the case of $\lambda < 0$, for any $x > 0$, by Lemma 3.2, we have

$$\begin{aligned} 1 - F_\lambda(x) &= \int_x^\infty 2\phi(t)\Phi(\lambda t)\,dt = \int_x^\infty 2\phi(t)[1-\Phi(|\lambda|t)]\,dt \\ &< \int_x^\infty 2\phi(t)\cdot\frac{\phi(|\lambda|t)}{|\lambda|t}\,dt = \frac{1}{\pi|\lambda|}\int_x^\infty t^{-1}e^{-(1+\lambda^2)t^2/2}\,dt \\ &= \frac{2\phi(x)\phi(\lambda x)}{|\lambda|(1+\lambda^2)x^2} - \frac{2}{\pi|\lambda|(1+\lambda^2)}\int_x^\infty t^{-3}e^{-(1+\lambda^2)t^2/2}\,dt \\ &< \frac{2\phi(x)\phi(\lambda x)}{|\lambda|(1+\lambda^2)x^2}. \end{aligned} \tag{3.8}$$

By integration by parts and Lemma 3.2, we have

$$
\begin{aligned}
(3.9)\qquad & 1-F_\lambda(x) \\
&= \frac{f_\lambda(x)}{x} - \frac{f_\lambda(x)}{x^3} + 6\int_x^\infty \phi(t)\Phi(\lambda t)t^{-4}\,dt + \frac{|\lambda|}{\pi}\int_x^\infty t^{-3}e^{-(1+\lambda^2)t^2/2}\,dt \\
&\quad - \frac{f_\lambda(x)}{x}\cdot\frac{|\lambda|}{1+\lambda^2}\cdot\frac{\phi(\lambda x)}{\Phi(\lambda x)}x^{-1} + \frac{2|\lambda|}{(1+\lambda^2)\pi}\int_x^\infty t^{-3}e^{-(1+\lambda^2)t^2/2}\,dt \\
&> \frac{f_\lambda(x)}{x} - \frac{f_\lambda(x)}{x^3} - \frac{f_\lambda(x)}{x}\cdot\frac{|\lambda|}{1+\lambda^2}\cdot\frac{\phi(\lambda x)}{\Phi(\lambda x)}x^{-1} \\
&= \frac{2\phi(x)}{x}\left[\Phi(\lambda x)(1-x^{-2}) - \frac{|\lambda|}{1+\lambda^2}\cdot\frac{\phi(\lambda x)}{x}\right] \\
&> \frac{2\phi(x)}{x}\left[\frac{\phi(\lambda x)}{|\lambda|x}(1-\lambda^{-2}x^{-2})(1-x^{-2}) - \frac{|\lambda|}{1+\lambda^2}\cdot\frac{\phi(\lambda x)}{x}\right] \\
&> \frac{2\phi(x)\phi(\lambda x)}{|\lambda|x^2}\left[(1-\lambda^{-2}x^{-2})(1-x^{-2}) - \frac{\lambda^2}{1+\lambda^2}\right] \\
&> \frac{2\phi(x)\phi(\lambda x)}{|\lambda|(1+\lambda^2)x^2}\left[1-\frac{(1+\lambda^2)^2}{\lambda^2}x^{-2}\right].
\end{aligned}
$$

Combining (3.8) with (3.9), we can derive (3.5). ■

Proof of Theorem 2.1. We first consider the case of $\lambda > 0$. If n is sufficiently large then $a_n x + b_n > 0$ with b_n and a_n satisfying (1.3) and (1.4). So, by Lemma 3.1, we have

$$
\begin{aligned}
n[1-F_\lambda(a_n x+b_n)] &\sim n\cdot\frac{2\phi(a_n x+b_n)}{a_n x+b_n} \\
&= (1+a_n^2 x)^{-1}e^{-a_n^2x^2/2}\cdot e^{-x} \\
&\to e^{-x} \quad \text{as } n\to\infty.
\end{aligned}
$$

Then the result follows from Theorem 1.5.1 of Leadbetter et al. (1983).

Similarly, for $\lambda < 0$, if n is sufficiently large then $a_n x + b_n > 0$ with b_n and a_n satisfying (1.5) and (1.6). So, by Lemma 3.1 we have

$$
\begin{aligned}
n[1-F_\lambda(a_n x+b_n)] &\sim n\cdot\frac{e^{-(1+\lambda^2)(a_n x+b_n)^2/2}}{\pi|\lambda|(1+\lambda^2)(a_n x+b_n)^2} \\
&= [1+(1+\lambda^2)a_n^2 x]^{-2}e^{-(1+\lambda^2)a_n^2x^2/2}\cdot e^{-x} \\
&\to e^{-x} \quad \text{as } n\to\infty.
\end{aligned}
$$

Then the result follows from Theorem 1.5.1 of Leadbetter et al. (1983). ■

Proof of Theorem 2.2. First note that for $\lambda > 0$, sufficiently large n implies $a_n x + b_n > 0$ with b_n and a_n satisfying (1.3) and (1.4). Writing $z_n = a_n x + b_n$, for large n and $k \in \mathbb{R}$, we have

$$z_n^k = b_n^k (1 + a_n^2 x)^k = b_n^k [1 + k a_n^2 x + O(a_n^4)]. \tag{3.10}$$

Applying (3.10) for $k = -1$, $k = 2$ and $k = -2$, we obtain

$$\frac{2\phi(z_n)}{z_n} = n^{-1} e^{-x} \big[1 - a_n^2 \big(x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big] \tag{3.11}$$

and

$$\begin{aligned} 1 - z_n^{-2} &= 1 - b_n^{-2}[1 - 2a_n^2 x + O(a_n^4)] \\ &= 1 - a_n^2 + O(a_n^4). \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we have

$$\frac{2\phi(z_n)}{z_n}[1 - z_n^{-2}] = n^{-1} e^{-x} \big[1 - a_n^2 \big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]. \tag{3.13}$$

Therefore, by Lemma 3.1 and the fact that $\log(1-x) = -x[1 + O(x)]$ as $x \to 0$, we have

$$\begin{aligned} &F_\lambda^n(z_n) - \Lambda(x) \\ &\quad = \left\{1 - \frac{2\phi(z_n)}{z_n}[1 - z_n^{-2} + O(z_n^{-4})]\right\}^n - \Lambda(x) \\ &\quad = \big\{1 - n^{-1} e^{-x} \big[1 - a_n^2\big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]\big\}^n - \Lambda(x) \\ &\quad = \exp\big\{n \log\big\{1 - n^{-1} e^{-x}\big[1 - a_n^2\big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]\big\}\big\} - \Lambda(x) \\ &\quad = \Lambda(x) \exp\big\{e^{-x}\big[a_n^2\big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]\big\} - \Lambda(x) \\ &\quad = \Lambda(x)\big\{1 + e^{-x}\big[a_n^2\big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]\big\} - \Lambda(x) \\ &\quad = \Lambda(x) e^{-x}\big[a_n^2\big(1 + x + \tfrac{1}{2}x^2\big) + O(a_n^4)\big]. \end{aligned}$$

Further, by (1.3), we have

$$\log \frac{\pi}{2} + 2 \log b_n + b_n^2 = 2 \log n. \tag{3.14}$$

It follows at once that $b_n^2 \sim 2 \log n$. Noting that $a_n = b_n^{-1}$, we can obtain the left hand inequality in (2.2). So, it remains to show that there exists a positive constant $\mathbb{C}_\lambda$ such that

$$\sup_{x \in \mathbb{R}} |F_\lambda^n(z_n) - \Lambda(x)| < \frac{\mathbb{C}_\lambda}{\log n} \quad \text{for all } n \geqslant 9.$$

For $n \geqslant 2$, (3.14) implies that

$$b_n^2 < 2\log n, \tag{3.15}$$

so that

$$2\log b_n < \log 2 + \log\log n. \tag{3.16}$$

Combining (3.16) with (3.14), we obtain

$$b_n^2 > 2\log n - \log \pi - \log\log n,$$

and hence, for $n \geqslant 9$,

$$\begin{aligned} \frac{b_n^2}{\log n} &> 2 - \frac{\log \pi}{\log n} - \frac{\log\log n}{\log n} \\ &> 2 - \frac{\log \pi}{\log 9} - \frac{1}{e} > 1.1, \end{aligned} \tag{3.17}$$

where the second inequality is obtained by bounding $(\log x)^{-1}(\log\log x)$. Since $a_n = b_n^{-1}$, inequality (3.17) implies that $a_n^2 < \frac{1}{1.1\log n}$ for $n \geqslant 9$, and so it suffices to prove that

$$\sup_{x\in\mathbb{R}} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_\lambda a_n^2 \quad \text{for } n \geqslant 9. \tag{3.18}$$

We will prove this by showing

$$\sup_{0\leqslant x<\infty} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{1,\lambda} a_n^2, \tag{3.19}$$

$$\sup_{-c_n<x<0} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{2,\lambda} a_n^2, \tag{3.20}$$

$$\sup_{-\infty<x\leqslant -c_n} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{3,\lambda} a_n^2, \tag{3.21}$$

where $c_n = \log\log b_n^2 > 0$ for $n \geqslant 9$.

The following bounds are needed for the rest of the proof:

$$1.69 < b_9 < 1.70, \tag{3.22}$$

$$\sup_{n\geqslant 9} (1 - a_n^2 c_n)^{-1} < 1.11, \tag{3.23}$$

$$\sup_{n\geqslant 9} a_n^2 \log b_n^2 < 0.37, \tag{3.24}$$

$$\sup_{n\geqslant 9} a_n^2 (\log b_n^2)^2 < 0.55, \tag{3.25}$$

$$\sup_{n\geqslant 9} n^{-1} \log b_n^2 < 0.17, \tag{3.26}$$

$$\sup_{n\geqslant 9} b_n^3 e^{-b_n^2/2} < 1.16. \tag{3.27}$$

Inequality (3.22) follows from (1.3), and (3.26) follows from (3.15). Inequalities (3.23)–(3.25) and (3.27) are obtained by bounding the functions $x^{-1}\log\log x$, $x^{-1}\log x$, $x^{-1}(\log x)^2$ and $x^3e^{-x^2/2}$, respectively.

Let $\Psi_{n,\lambda}(x) = 1 - F_\lambda(z_n)$. Then

$$n\log F_\lambda(z_n) = n\log[1-\Psi_{n,\lambda}(x)] = -n\Psi_{n,\lambda}(x) - R_{n,\lambda}(x), \tag{3.28}$$

where

$$0 < R_{n,\lambda}(x) < \frac{n\Psi^2_{n,\lambda}(x)}{2[1-\Psi_{n,\lambda}(x)]}. \tag{3.29}$$

If $x > -c_n$, by (1.3), (3.23), (3.26) and Lemma 3.3, we have

$$\begin{aligned}\Psi_{n,\lambda}(x) &< \Psi_{n,\lambda}(-c_n) \\ &= 1 - F_\lambda(b_n - a_nc_n) < \frac{2\phi(b_n-a_nc_n)}{b_n-a_nc_n} \\ &= \sqrt{2/\pi}\, b_n^{-1}(1-a_n^2c_n)^{-1}e^{-b_n^2(1-a_n^2c_n)^2/2} \\ &= n^{-1}(1-a_n^2c_n)^{-1}(\log b_n^2)e^{-a_n^2c_n^2/2} \\ &< (1-a_n^2c_n)^{-1}(n^{-1}\log b_n^2) < 0.1887.\end{aligned} \tag{3.30}$$

From (3.29), (3.30) (1.3) and (1.4), we can see that

$$\begin{aligned}R_{n,\lambda}(x) &< \frac{n[(1-a_n^2c_n)^{-1}(n^{-1}\log b_n^2)]^2}{2(1-0.1887)} \\ &= \frac{n^{-1}(1-a_n^2c_n)^{-2}(\log b_n^2)^2}{1.6226} \\ &= \frac{\sqrt{2/\pi}\, b_n^{-1}e^{-1/2b_n^2}(1-a_n^2c_n)^{-2}(\log b_n^2)^2}{1.6226} \\ &= \frac{[\sqrt{2/\pi}\,(1-a_n^2c_n)^{-2}](b_n^3e^{-b_n^2/2})[a_n^2(\log b_n^2)^2]a_n^2}{1.6226} \\ &< 0.39a_n^2,\end{aligned} \tag{3.31}$$

where the last inequality follows from (3.23), (3.25), and (3.27). Hence for $n \geqslant 9$, we have

$$|e^{-R_{n,\lambda}(x)} - 1| = 1 - e^{-R_{n,\lambda}(x)} < R_{n,\lambda}(x) < 0.39a_n^2. \tag{3.32}$$

Let $A_{n,\lambda}(x) = e^{-n\Psi_{n,\lambda}(x)+e^{-x}}$ and $B_{n,\lambda}(x) = e^{-R_{n,\lambda}(x)}$. Inequality (3.32) implies that

$$\begin{aligned}|F_\lambda^n(z_n) - \Lambda(x)| &< \Lambda(x)|A_{n,\lambda}(x) - 1| + |B_{n,\lambda}(x) - 1| \\ &< \Lambda(x)|A_{n,\lambda}(x) - 1| + 0.39a_n^2\end{aligned} \tag{3.33}$$

if $x > -c_n$.

We first show that (3.19) holds. Note that $0 < A_{n,\lambda}(x) \to 1$ as $x \to \infty$ and

$$A'_{n,\lambda}(x) = A_{n,\lambda}(x)e^{-x}[e^{-1/2a_n^2x^2}\Phi(\lambda(z_n)) - 1] \leqslant 0$$

for $x \geqslant 0$. Hence, it follows from (3.22), (1.4) and Lemma 3.3 that

$$\begin{aligned}
(3.34) \qquad \sup_{x\geqslant 0}|A_{n,\lambda}(x) - 1| &= A_{n,\lambda}(0) - 1 \\
&< \exp\left\{-n\frac{2\phi(b_n)}{b_n}\left[1 - \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)b_n^{-2}\right] + 1\right\} - 1 \\
&< \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)b_n^{-2}\exp\left\{\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)b_n^{-2}\right\} \\
&< \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)\exp\left\{1.69^{-2}\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right\}a_n^2.
\end{aligned}$$

The second inequality is obtained by observing that $e^x < 1 + xe^x$ for $x > 0$. The last inequality is obtained by the monotonicity of b_n (i.e. $b_n \geqslant b_9$ for $n \geqslant 9$). Combining (3.33) with (3.34), we complete the proof of (3.19).

Next we prove that (3.20) holds as $-c_n < x < 0$. By (1.4), (3.23) and the fact that $e^x > 1 + x$ for $x \in \mathbb{R}$, we have

$$\begin{aligned}
(3.35) \qquad & - e^{-a_n^2x^2/2}\left[1 - \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)z_n^{-2}\right] + 1 + a_n^2x \\
&< -\left(1 - \tfrac{1}{2}a_n^2x^2\right)\left[1 - \left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right)z_n^{-2}\right] + 1 + a_n^2x \\
&= a_n^2(1 + a_n^2x)^{-2}\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right) + \tfrac{1}{2}a_n^2x^2 \\
&\quad - \tfrac{1}{2}a_n^4x^2(1 + a_n^2x)^{-2}\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right) + a_n^2x \\
&< a_n^2(1 + a_n^2x)^{-2}\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right) + \tfrac{1}{2}a_n^2x^2 \\
&< a_n^2\left[(1 - a_n^2c_n)^{-2}\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right) + \tfrac{1}{2}x^2\right] \\
&< a_n^2\left[1.11^2\left(1 + \frac{1}{\lambda^2\sqrt{2\pi e}}\right) + \tfrac{1}{2}x^2\right].
\end{aligned}$$

Let

$$h_{n,\lambda}(x) = -n\Psi_{n,\lambda}(x) + e^{-x}.$$

From Lemma 3.3, (1.3), (1.4) and (3.35), we have

$$(3.36)\qquad h_{n,\lambda}(x) < -n\frac{2\phi(z_n)}{z_n}\left[1-\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)z_n^{-2}\right]+e^{-x}$$
$$= -n\sqrt{2/\pi}\,b_n^{-1}(1+a_n^2x)^{-1}e^{-z_n^2/2}\left[1-\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)z_n^{-2}\right]+e^{-x}$$
$$= (1+a_n^2x)^{-1}e^{-x}\left\{-e^{-a_n^2x^2/2}\left[1-\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)z_n^{-2}\right]+1+a_n^2x\right\}$$
$$< (1+a_n^2x)^{-1}e^{-x}a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2\right].$$

Further, by Lemma 3.3, (1.3) and (1.4), we have

$$(3.37)\qquad h_{n,\lambda}(x) > -n\frac{2\phi(z_n)}{z_n}+e^{-x} = -n\sqrt{2/\pi}\,b_n^{-1}(1+a_n^2x)^{-1}e^{-z_n^2/2}+e^{-x}$$
$$= (1+a_n^2x)^{-1}e^{-x}(-e^{-a_n^2x^2/2}+1+a_n^2x) > (1+a_n^2x)^{-1}e^{-x}a_n^2|x|.$$

Hence, for $-c_n < x < 0$, it follows from (3.36) and (3.37) that

$$(3.38)\qquad |h_{n,\lambda}(x)| < (1+a_n^2x)^{-1}e^{-x}a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2+|x|\right]$$
$$< (1-a_n^2c_n)^{-1}e^{c_n}a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}c_n^2+c_n\right]$$
$$< (1-a_n^2c_n)^{-1}\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)(a_n^2\log b_n^2\right)+\tfrac{3}{2}a_n^2(\log b_n^2)^2] < \mathbb{C}'_{2,\lambda}.$$

The third inequality holds because $\log x > (\log\log x)^2$ for $x > e$ and $\log x > \log\log x$ for $x > 1$. The last inequality holds by (3.23)–(3.25). Noting that $|e^x-1| < |x|e^{|x|}$ for $x \in \mathbb{R}$ and $e^x > 1+x+\frac{1}{2}x^2$ for $x > 0$, we have

$$(3.39)\qquad \Lambda(x)|A_{n,\lambda(x)}-1| = \Lambda(x)|e^{h_{n,\lambda}(x)}-1| < \Lambda(x)|h_{n,\lambda}(x)|e^{|h_{n,\lambda}(x)|}$$
$$< \Lambda(x)(1+a_n^2x)^{-1}e^{-x}a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2+|x|\right]e^{\mathbb{C}'_{2,\lambda}}$$
$$< \Lambda(x)(1-a_n^2c_n)^{-1}e^{-x}a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2+|x|\right]e^{\mathbb{C}'_{2,\lambda}}$$
$$= a_n^2(1-a_n^2c_n)^{-1}\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2+|x|\right]e^{-e^{-x}-x+\mathbb{C}'_{2,\lambda}}$$
$$< a_n^2(1-a_n^2c_n)^{-1}\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+\tfrac{1}{2}x^2+|x|\right]e^{-x^2/2+\mathbb{C}'_{2,\lambda}-1}$$
$$= a_n^2(1-a_n^2c_n)^{-1}$$
$$\cdot\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)e^{-x^2/2}+\tfrac{1}{2}x^2e^{-x^2/2}+|x|e^{-x^2/2}\right]e^{\mathbb{C}'_{2,\lambda}-1}$$
$$< 1.11a_n^2\left[1.11^2\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)+1+1\right]e^{\mathbb{C}'_{2,\lambda}-1}.$$

Inserting (3.39) into (3.33), we can establish (3.20).

The last step is to show that (3.21) holds. For $-\infty < x \leqslant -c_n$, (1.4) implies

$$0 \leqslant \Lambda(x) \leqslant \Lambda(-c_n) = a_n^2. \tag{3.40}$$

Since $e^x > 1 + x$ for $x \in \mathbb{R}$, we have

$$\begin{aligned}(1-a_n^2c_n)^{-1}e^{-a_n^2c_n^2/2}&\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\\ &> e^{-a_n^2c_n^2/2}\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\\ &> \left(1-\tfrac{1}{2}a_n^2c_n^2\right)\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\\ &> 1-a_n^2\left[\tfrac{1}{2}c_n^2+(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right].\end{aligned} \tag{3.41}$$

Thus, by (1.3), (1.4), (3.23)–(3.25), (3.41) and Lemma 3.3, we have

$$\begin{aligned}&F_\lambda^n(z_n) \leqslant F_\lambda^n(b_n-a_nc_n)\\ &= \{1-[1-F_\lambda(b_n-a_nc_n)]\}^n\\ &< \left\{1-\frac{2\phi(b_n-a_nc_n)}{b_n-a_nc_n}\left[1-\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)(b_n-a_nc_n)^{-2}\right]\right\}^n\\ &= \Bigg\{1-\sqrt{2/\pi}\,b_n^{-1}(1-a_n^2c_n)^{-1}e^{-(b_n-a_nc_n)^2/2}\\ &\qquad\qquad \cdot\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\Bigg\}^n\\ &= \left\{1-n^{-1}e^{c_n}(1-a_n^2c_n)^{-1}e^{-a_n^2c_n^2/2}\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\right\}^n\\ &< \exp\left\{-e^{c_n}(1-a_n^2c_n)^{-1}e^{-a_n^2c_n^2/2}\left[1-a_n^2(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right]\right\}\\ &< \exp\left\{-e^{c_n}\left[1-a_n^2\left(\tfrac{1}{2}c_n^2+(1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right)\right]\right\}\\ &< a_n^2\exp\left\{\tfrac{1}{2}a_n^2(\log b_n^2)^2+[a_n^2(\log b_n^2)](1-a_n^2c_n)^{-2}\left(1+\frac{1}{\lambda^2\sqrt{2\pi e}}\right)\right\}\\ &< \mathbb{C}'_{3,\lambda}a_n^2\end{aligned} \tag{3.42}$$

due to $(1-\frac{x}{n})^n < e^{-x}$ for $x \leqslant n$ and $\log x > (\log\log x)^2$ for $x > e$. Combining (3.40) with (3.42), we see that (3.21) holds. The proof is complete. ■

Proof of Theorem 2.3. For $\lambda < 0$, if n is sufficiently large then $a_nx + b_n > 0$ with b_n and a_n satisfying (1.5) and (1.6). Noting that $z_n = a_nx + b_n$, for large n

and $k \in \mathbb{R}$ we have

$$\begin{aligned} z_n^k &= b_n^k[1 + (1+\lambda^2)a_n^2 x]^k \\ &= b_n^k\left[1 + k(1+\lambda^2)a_n^2 x + \frac{k(k-1)}{2}(1+\lambda^2)^2 a_n^4 x^2 + O(a_n^6)\right]. \end{aligned} \tag{3.43}$$

Applying (3.43) with $k = 2$ and with $k = -2$, we have

$$\frac{e^{-(1+\lambda^2)z_n^2/2}}{-\pi\lambda(1+\lambda^2)z_n^2} = n^{-1}e^{-x}\left[1 - (1+\lambda^2)a_n^2\left(2x + \tfrac{1}{2}x^2\right) + O(a_n^4)\right] \tag{3.44}$$

and

$$1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}z_n^{-2} = 1 - \frac{(1+3\lambda^2)(1+\lambda^2)}{\lambda^2}a_n^2 + O(a_n^4). \tag{3.45}$$

Combining (3.44) with (3.45), we have

$$\begin{aligned} &\frac{e^{-(1+\lambda^2)z_n^2/2}}{-\pi\lambda(1+\lambda^2)z_n^2}\left[1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}z_n^{-2}\right] \\ &\qquad = n^{-1}e^{-x}\left[1 - (1+\lambda^2)a_n^2\left(\frac{1+3\lambda^2}{\lambda^2} + 2x + \tfrac{1}{2}x^2\right) + O(a_n^4)\right]. \end{aligned} \tag{3.46}$$

Therefore, by Lemma 3.1,

$$\begin{aligned} &F_\lambda^n(z_n) - \Lambda(x) \\ &= \left\{1 - \frac{e^{-(1+\lambda^2)z_n^2/2}}{-\pi\lambda(1+\lambda^2)z_n^2}\left[1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)}z_n^{-2} + O(z_n^{-4})\right]\right\}^n - \Lambda(x) \\ &= \left\{1 - n^{-1}e^{-x}\left[1 - (1+\lambda^2)a_n^2\left(\frac{1+3\lambda^2}{\lambda^2} + 2x + \tfrac{1}{2}x^2\right) + O(a_n^4)\right]\right\}^n - \Lambda(x) \\ &= \Lambda(x)e^{-x}\left[(1+\lambda^2)a_n^2\left(\frac{1+3\lambda^2}{\lambda^2} + 2x + \tfrac{1}{2}x^2\right) + O(a_n^4)\right]. \end{aligned}$$

Further, by (1.5),

$$\log[\pi|\lambda|(1+\lambda^2)] + 2\log b_n + \frac{(1+\lambda^2)b_n^2}{2} = \log n, \tag{3.47}$$

implying that $(1+\lambda^2)b_n^2 \sim 2\log n$. Noting that $a_n = (1+\lambda^2)^{-1}b_n^{-1}$, we can obtain the left-hand inequality in (2.3).

It remains to show that there exists a positive constant $\mathbb{C}''_\lambda$ such that

$$\sup_{x\in\mathbb{R}} |F_\lambda(z_n)^n - \Lambda(x)| < \frac{\mathbb{C}''_\lambda}{\log n}$$

for all $n > n_0(\lambda)$.

For $n > n_0(\lambda)$, (3.47) implies

$$b_n^2 < \frac{2}{1+\lambda^2}\log n, \tag{3.48}$$

so that

$$2\log b_n < \log\frac{2}{1+\lambda^2} + \log\log n. \tag{3.49}$$

Combining (3.49) with (3.47), we get

$$(1+\lambda^2)b_n^2 > 2\log n - 2\log(2\pi|\lambda|) - 2\log\log n$$

and

$$\begin{aligned}\frac{(1+\lambda^2)b_n^2}{\log n} &> 2 - \frac{2\log(2\pi|\lambda|)}{\log n} - \frac{2\log\log n}{\log n}\\ &> 2 - \frac{2\log(2\pi|\lambda|)}{\log n_0(\lambda)} - \frac{2}{e} = c_0,\end{aligned} \tag{3.50}$$

where c_0 is a positive constant and the last inequality is obtained by bounding the function $(\log x)^{-1}(\log\log x)$. Since $a_n = (1+\lambda^2)^{-1}b_n^{-1}$, (3.50) implies that $a_n^2 < \frac{1}{c_0(1+\lambda^2)\log n}$ for $n \geqslant n_0(\lambda)$, and so it suffices to prove that

$$\sup_{x\in\mathbb{R}} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_\lambda'' a_n^2 \tag{3.51}$$

for $n \geqslant n_0(\lambda)$.

We will now prove the following inequalities:

$$\sup_{0\leqslant x<\infty} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{1,\lambda}'' a_n^2, \tag{3.52}$$

$$\sup_{-d_n<x<0} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{2,\lambda}'' a_n^2 \tag{3.53}$$

$$\sup_{-\infty<x\leqslant -d_n} |F_\lambda^n(z_n) - \Lambda(x)| < \mathbb{C}_{3,\lambda}'' a_n^2, \tag{3.54}$$

where $d_n = \log\log[(1+\lambda^2)b_n^2]$ and $d_n > 0$ for $n \geqslant n_0(\lambda)$.

The following bounds are needed:

$$c_1 < b_{n_0(\lambda)}, \tag{3.55}$$

$$\sup_{n\geqslant 2}[1 - (1+\lambda^2)a_n^2 d_n]^{-1} < 1.11, \tag{3.56}$$

$$\sup_{n\geqslant 2}(1+\lambda^2)a_n^2\log[(1+\lambda^2)b_n^2] < 0.37, \tag{3.57}$$

$$\sup_{n \geqslant n_0(\lambda)} (1+\lambda^2)a_n^2 \left\{\log[(1+\lambda^2)b_n^2]\right\}^2 < 0.55, \tag{3.58}$$

$$\sup_{n\geqslant 2} n^{-1}\log[(1+\lambda^2)b_n^2] < 0.27, \tag{3.59}$$

$$\sup_{n\geqslant 2}(1+\lambda^2)b_n^2 e^{-(1+\lambda^2)b_n^2/2} < 0.74, \tag{3.60}$$

where c_1 is a positive constant and inequality (3.59) follows from (3.48), and (3.56)–(3.58) and (3.60) are obtained by bounding the functions $x^{-1}\log\log x$, $x^{-1}\log x$, $x^{-1}(\log x)^2$ and $xe^{-x/2}$, respectively.

If $x > -d_n$, by (1.5), (3.29), (3.56), (3.59) and Lemma 3.3 we have

$$\begin{aligned}\Psi_{n,\lambda}(x) &< \Psi_{n,\lambda}(-d_n)\\ &< \frac{2\phi(b_n - a_n d_n)\phi(\lambda(b_n - a_n d_n))}{|\lambda|(1+\lambda^2)(b_n - a_n d_n)^2}\\ &= n^{-1}[1-(1+\lambda^2)a_n^2 d_n]^{-2}\{\log[(1+\lambda^2)b_n^2]\}e^{-(1+\lambda^2)a_n^2 d_n^2/2}\\ &< [1-(1+\lambda^2)a_n^2 d_n]^{-2}\{n^{-1}\log[(1+\lambda^2)b_n^2]\} < 0.332667\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}R_{n,\lambda}(x) &< \frac{n\{[1-(1+\lambda^2)a_n^2 d_n]^{-2}\{n^{-1}\log[(1+\lambda^2)b_n^2]\}\}^2}{2(1-0.332667)}\\ &< \frac{1}{\pi|\lambda|}\{[1-(1+\lambda^2)a_n^2 d_n]^{-4}\}\left[(1+\lambda^2)a_n^2\{\log[(1+\lambda^2)b_n^2]\}^2\right]\\ &\quad\cdot[(1+\lambda^2)b_n^2 e^{-(1+\lambda^2)b_n^2/2}](1+\lambda^2)a_n^2\\ &< \mathbb{C}_4'' a_n^2,\end{aligned} \tag{3.62}$$

where $\Psi_{n,\lambda}(x)$ and $R_{n,\lambda}(x)$ are given by (3.28), i.e., $\Psi_{n,\lambda}(x) = 1 - F_\lambda(z_n)$ and $R_{n,\lambda}(x) = -n\log F_\lambda(z_n) - n\Psi_{n,\lambda}(x)$ with b_n and a_n given by (1.5) and (1.6), respectively. Hence, for $n \geqslant n_0(\lambda)$, we obtain

$$|e^{-R_{n,\lambda}(x)} - 1| = 1 - e^{-R_{n,\lambda}(x)} < R_{n,\lambda}(x) < \mathbb{C}_4'' a_n^2 \tag{3.63}$$

by using the inequality $e^x > 1 + x$ for $x \in \mathbb{R}$.

Let $A_{n,\lambda}(x) = e^{-n\Psi_{n,\lambda}(x)+e^{-x}}$ and $B_{n,\lambda}(x) = e^{-R_{n,\lambda}(x)}$. Then (3.63) implies that

$$\begin{aligned}|F_\lambda^n(z_n) - \Lambda(x)| &< \Lambda(x)|A_{n,\lambda}(x) - 1| + |B_{n,\lambda}(x) - 1|\\ &< \Lambda(x)|A_{n,\lambda}(x) - 1| + \mathbb{C}_4'' a_n^2\end{aligned} \tag{3.64}$$

for $x > -d_n$.

Now we prove (3.52)–(3.54) in turn. To prove (3.52), noting that $0 < A_{n,\lambda}(x) \to 1$ as $x \to \infty$ and by Lemma 3.2, we have

$$\begin{aligned} A'_{n,\lambda}(x) &= A_{n,\lambda}(x)[-e^{-x} + na_n f_\lambda(z_n)] \\ &< A_{n,\lambda}(x)\left[-e^{-x} + na_n 2\phi(z_n)\frac{\phi(|\lambda|(z_n))}{|\lambda|(z_n)}\right] \\ &= A_{n,\lambda}(x)e^{-x}\{e^{-(1+\lambda^2)a_n^2x^2/2}[1+(1+\lambda^2)a_n^2x]^{-1} - 1\} \leqslant 0 \end{aligned}$$

for $x \geqslant 0$. Hence, by (3.55), (1.4) and Lemma 3.3,

$$\begin{aligned} \sup_{x\geqslant 0}|A_{n,\lambda}(x) - 1| &= A_{n,\lambda}(0) - 1 \\ &< \exp\left\{-n\frac{2\phi(b_n)\phi(\lambda b_n)}{|\lambda|(1+\lambda^2)b_n}\left[1 - \frac{(1+\lambda^2)^2}{\lambda^2}b_n^{-2}\right] + 1\right\} - 1 \\ &= \exp\left\{\frac{(1+\lambda^2)^2}{\lambda^2}b_n^{-2}\right\} - 1 \\ &< \frac{(1+\lambda^2)^2}{\lambda^2}b_n^{-2}\exp\left\{\frac{(1+\lambda^2)^2}{\lambda^2}b_n^{-2}\right\} \\ &< \frac{(1+\lambda^2)^4}{\lambda^2}\exp\left\{\frac{(1+\lambda^2)^2}{\lambda^2}c_1^{-2}\right\}a_n^2 \end{aligned} \tag{3.65}$$

due to $e^x < 1 + xe^x$ for $x > 0$ and the monotonicity of b_n (i.e. $b_n \geqslant b_{n_0(\lambda)}$ for $n \geqslant n_0(\lambda)$). Combining (3.64) with (3.65), we see (3.52) holds.

Before proving (3.53), we need the following inequalities. By (1.5) and the fact that $e^x > 1 + x$ for $x \in \mathbb{R}$, we have

$$\begin{aligned} &- e^{-(1+\lambda^2)a_n^2x^2/2}\left[1 - \frac{(1+\lambda^2)^2}{\lambda^2}z_n^{-2}\right] + [1+(1+\lambda^2)a_n^2x]^2 \\ &< -\left[1 - \tfrac{1}{2}(1+\lambda^2)a_n^2x^2\right]\left[1 - \frac{(1+\lambda^2)^2}{\lambda^2}z_n^{-2}\right] + [1+(1+\lambda^2)a_n^2x]^2 \\ &< (1+\lambda^2)a_n^2\left\{\frac{(1+\lambda^2)^3}{\lambda^2}[1+(1+\lambda^2)a_n^2x]^{-2} + \tfrac{1}{2}x^2 + (1+\lambda^2)a_n^2x^2\right\}. \end{aligned} \tag{3.66}$$

Let $h_{n,\lambda}(x) = -n\Psi_{n,\lambda}(x) + e^{-x}$. By (1.5), (1.6), (3.66) and Lemma 3.3,

$$\begin{aligned} h_{n,\lambda}(x) &< -n\frac{2\phi(z_n)\phi(\lambda(z_n))}{|\lambda|(1+\lambda^2)z_n^2}\left[1 - \frac{(1+\lambda^2)^2}{\lambda^2}z_n^{-2}\right] + e^{-x} \\ &< [1+(1+\lambda^2)a_n^2x]^{-2}e^{-x}(1+\lambda^2)a_n^2\left\{\frac{(1+\lambda^2)^3}{\lambda^2}[1+(1+\lambda^2)a_n^2x]^{-2}\right. \\ &\qquad \left. + \tfrac{1}{2}x^2 + (1+\lambda^2)a_n^2x^2\right\} \end{aligned} \tag{3.67}$$

and

$$(3.68)\qquad h_{n,\lambda}(x) > -n\frac{2\phi(z_n)\phi(\lambda(z_n))}{|\lambda|(1+\lambda^2)z_n^2} + e^{-x}$$
$$> [1+(1+\lambda^2)a_n^2 x]^{-2}e^{-x}(1+\lambda^2)a_n^2[2x+(1+\lambda^2)a_n^2x^2].$$

Hence, for $-d_n < x < 0$, we note from (3.67) and (3.68) that

$$(3.69)\qquad |h_{n,\lambda}(x)|$$
$$< [1+(1+\lambda^2)a_n^2x]^{-2}e^{-x}(1+\lambda^2)a_n^2\Big\{\frac{(1+\lambda^2)^3}{\lambda^2}[1+(1+\lambda^2)a_n^2x]^{-2} + \tfrac{1}{2}x^2 + 2(1+\lambda^2)a_n^2x^2 + 2|x|\Big\}$$
$$< [1-(1+\lambda^2)a_n^2d_n]^{-2}e^{d_n}(1+\lambda^2)a_n^2\Big\{\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2d_n]^{-2} + \tfrac{1}{2}d_n^2 + 2(1+\lambda^2)a_n^2d_n^2 + 2d_n\Big\}$$
$$= [1-(1+\lambda^2)a_n^2d_n]^{-2}$$
$$\cdot\Big[\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2d_n]^{-2}\{(1+\lambda^2)a_n^2\log[(1+\lambda^2)b_n^2]\}$$
$$+ \tfrac{1}{2}\{(1+\lambda^2)a_n^2\log[(1+\lambda^2)b_n^2]\}\{\log\log[(1+\lambda^2)b_n^2]\}^2$$
$$+ 2\{(1+\lambda^2)^2a_n^4\log[(1+\lambda^2)b_n^2]\}\{\log\log[(1+\lambda^2)b_n^2]\}^2$$
$$+ 2\{(1+\lambda^2)a_n^2\log[(1+\lambda^2)b_n^2]\}\{\log\log[(1+\lambda^2)b_n^2]\}\Big] < \mathbb{C}''_{5,\lambda}.$$

The last inequality holds because $\log x > (\log\log x)^2$ for $x > e$, $\log x > \log\log x$ for $x > 1$ and (3.56)–(3.58). Noting that $|e^x - 1| < |x|e^{|x|}$ for $x \in \mathbb{R}$ and $e^x > 1 + x + \frac{1}{2}x^2$ for $x > 0$, for $-d_n < x < 0$ we have

$$(3.70)\qquad \Lambda(x)|A_{n,\lambda(x)} - 1| < \Lambda(x)|h_{n,\lambda}(x)|e^{|h_{n,\lambda}(x)|}$$
$$< \Lambda(x)[1+(1+\lambda^2)a_n^2x]^{-2}e^{-x}(1+\lambda^2)a_n^2$$
$$\cdot\Big\{\frac{(1+\lambda^2)^3}{\lambda^2}[1+(1+\lambda^2)a_n^2x]^{-2} + \tfrac{1}{2}x^2 + 2(1+\lambda^2)a_n^2x^2 + 2|x|\Big\}e^{\mathbb{C}''_{5,\lambda}}$$
$$= a_n^2[1+(1+\lambda^2)a_n^2x]^{-2}(1+\lambda^2)\exp\{-e^{-x} - x + \mathbb{C}''_{5,\lambda}\}$$
$$\cdot\Big\{\frac{(1+\lambda^2)^3}{\lambda^2}[1+(1+\lambda^2)a_n^2x]^{-2} + \tfrac{1}{2}x^2 + 2(1+\lambda^2)a_n^2x^2 + 2|x|\Big\}$$
$$< a_n^2[1-(1+\lambda^2)a_n^2d_n]^{-2}(1+\lambda^2)\exp\{-\tfrac{1}{2}x^2 + \mathbb{C}''_{5,\lambda} - 1\}$$
$$\cdot\Big\{\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2d_n]^{-2} + \tfrac{1}{2}x^2 + 2(1+\lambda^2)a_n^2d_n^2 + 2|x|\Big\}$$

$$
\begin{aligned}
&= a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}(1+\lambda^2)e^{\mathbb{C}''_{5,\lambda}-1}\\
&\quad\cdot\left\{\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}e^{-x^2/2}+\tfrac{1}{2}x^2e^{-x^2/2}\right.\\
&\quad\left.+2(1+\lambda^2)a_n^2 d_n^2 e^{-x^2/2}+2|x|e^{-x^2/2}\right\}\\
&< a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}(1+\lambda^2)e^{\mathbb{C}''_{5,\lambda}-1}\\
&\quad\cdot\left\{\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}+1+2(1+\lambda^2)a_n^2[\log((1+\lambda^2)b_n^2)]^2+1\right\}\\
&< \mathbb{C}''_{6,\lambda}a_n^2.
\end{aligned}
$$

Thus, inserting (3.70) into (3.64), we can see that (3.53) holds.

Now it remains to show that (3.54) holds. For $-\infty < x \leqslant -d_n$, noting that $e^x > 1+x$ for $x \in \mathbb{R}$ and the values of a_n and d_n, we have

$$0 \leqslant \Lambda(x) \leqslant \Lambda(-d_n) = (1+\lambda^2)a_n^2 \tag{3.71}$$

and

$$
\begin{aligned}
&e^{-(1+\lambda^2)a_n^2 d_n^2/2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}\\
&\qquad\cdot\left\{1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}\right\}\\
&> e^{-(1+\lambda^2)a_n^2 d_n^2/2}\left\{1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}\right\}\\
&> \left[1-\tfrac{1}{2}(1+\lambda^2)a_n^2 d_n^2\right]\left\{1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}\right\}\\
&= 1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}-\tfrac{1}{2}(1+\lambda^2)a_n^2 d_n^2\\
&\quad+\frac{(1+\lambda^2)^5}{2\lambda^2}a_n^4 d_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}\\
&> 1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}-\tfrac{1}{2}(1+\lambda^2)a_n^2 d_n^2.
\end{aligned}
\tag{3.72}
$$

Thus, by (1.5), (1.6), (3.56)–(3.58), (3.72) and Lemma 3.3, for $-\infty < x \leqslant -d_n$ we have

$$
\begin{aligned}
&F_\lambda^n(z_n) \leqslant F_\lambda^n(b_n - a_n d_n)\\
&< \left\{1-\frac{2\phi(b_n-a_nd_n)\phi(\lambda(b_n-a_nd_n))}{|\lambda|(1+\lambda^2)(b_n-a_nd_n)^2}\left[1-\frac{(1+\lambda^2)^2}{\lambda^2}(b_n-a_nd_n)^{-2}\right]\right\}^n\\
&= \left\{1-n^{-1}e^{d_n}e^{-(1+\lambda^2)a_n^2d_n^2/2}\frac{1-\frac{(1+\lambda^2)^4}{\lambda^2}a_n^2[1-(1+\lambda^2)a_n^2d_n]^{-2}}{[1-(1+\lambda^2)a_n^2d_n]^2}\right\}^n
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
&< \exp\bigg\{-e^{d_n} e^{-(1+\lambda^2)a_n^2 d_n^2/2} \frac{1-\frac{(1+\lambda^2)^4}{\lambda^2} a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2}}{[1-(1+\lambda^2)a_n^2 d_n]^2}\bigg\} \\
&< \exp\bigg\{-e^{d_n}\bigg[1-\frac{(1+\lambda^2)^4}{\lambda^2} a_n^2[1-(1+\lambda^2)a_n^2 d_n]^{-2} - \tfrac{1}{2}(1+\lambda^2)a_n^2 d_n^2\bigg]\bigg\} \\
&= \exp\bigg\{-\log[(1+\lambda^2)b_n^2] \\
&\quad + \frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}(1+\lambda^2)a_n^2 \log[(1+\lambda^2)b_n^2] \\
&\quad + \tfrac{1}{2}(1+\lambda^2)a_n^2 \log[(1+\lambda^2)b_n^2][\log\log((1+\lambda^2)b_n^2)]^2\bigg\} \\
&< \exp\bigg\{-\log[(1+\lambda^2)b_n^2] \\
&\quad + \frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}(1+\lambda^2)a_n^2 \log[(1+\lambda^2)b_n^2] \\
&\quad + \tfrac{1}{2}(1+\lambda^2)a_n^2[\log((1+\lambda^2)b_n^2)]^2\bigg\} \\
&= a_n^2(1+\lambda^2)\exp\bigg\{\frac{(1+\lambda^2)^3}{\lambda^2}[1-(1+\lambda^2)a_n^2 d_n]^{-2}(1+\lambda^2)a_n^2 \log[(1+\lambda^2)b_n^2] \\
&\quad + \tfrac{1}{2}(1+\lambda^2)a_n^2[\log((1+\lambda^2)b_n^2)]^2\bigg\} \\
&< \mathbb{C}''_{7,\lambda} a_n^2.
\end{aligned}
$$

Combining (3.71) with (3.73), we see that (3.54) holds. The proof is complete. ■

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