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# NEW EASY TO COMPUTE FORMULAS FOR THE MOMENTS OF RANDOM VARIABLES APPEARING IN THE COUPON COLLECTOR PROBLEM

BY

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**Abstract.** Assuming that there are $N$ types of coupons, where the probability that the $i$th coupon appears is $p_i \geqslant 0$ for $i = 1, \ldots, N$, with $\sum_{i=1}^{N} p_i = 1$, we consider the variable $T_k$ which represents the number of acquisitions needed to obtain $k \leqslant N$ different coupons, and the variable $Y_n$ which represents the number of different coupons obtained in $n$ acquisitions. In the coupon collector problem it is of interest to obtain the expected value of these random variables, as well as their $r$th moments. We provide new expressions for the $r$th moments of $T_k$ and $Y_n$, and we give expressions for their moment generating functions. Unlike known formulas, our formula for the $r$th moment of $T_k$ is given in terms of recursive expressions and that of $Y_n$ is given in terms of finite sums, so that they can be easily implemented computationally. Furthermore, our formulas allow obtaining simplified expressions of the first few moments of the variables.

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## 1. INTRODUCTION

The coupon collector problem (CCP) is a classical problem in combinatorial probability. This problem consists in the following: There are $N$ different types of coupons, such as, for example, baseball cards, and we want to obtain the full collection.

The CCP is an ancient problem dating back to De Moivre's treatise *De Mensura Sortis* of 1712 and Laplace's work *Théorie Analytique des Probabilités* of 1812 (see [6]). Von Schelling (1954) [15] obtained the expected time to complete a collection in the case of non-equiprobable coupons, and Newman and Chepp

(1960) [13] calculated the expected time to complete two collections of coupons in the case of equiprobable coupons (see [8]).

Some more recent advances, considering non-equiprobable coupons, include: the study of the moments and the distribution of the time required to collect a specific number of types of coupons, as well as the distribution that stochastically minimizes this time (see for example [2, 3]); the asymptotics of the first two moments and the limit distribution of the number of coupons required to complete $m$ collections of $N$ coupons when $N \to \infty$ (see [6]); the expected time to complete a collection of coupons when coupons arrive in groups of constant size and independently (see [7]); several formulas for the expected number of coupons required to complete one or more collections of $N$ coupons (see [8]); the distribution function, expectation and variance of the number of different coupons that are obtained in $n$ acquisitions, as well as the expected number of coupons required to obtain $k$ different coupons from a subset of the entire collection of coupons (see [16]).

It is important to mention that the coupon collector problem has applications in several areas of science, such as engineering, computing and ecology [4, 11, 16], ranked data analysis [1], and quality control problems [12].

Let us consider $N$ different types of coupons, denoted by $1, \dots, N$. The coupons arrive one by one in sequence and independently. The probability of randomly acquiring, at any stage, a coupon of type $i$ is $p_i \geqslant 0$, $i = 1, \dots, N$, with $\sum_i^N p_i = 1$. In this context, two variables of interest that arise in the coupon collector problem are the random variable that represents the number of acquisitions needed to obtain $k \leqslant N$ different coupons, which we denote by $T_k$; and the random variable that represents the number of different coupons obtained in $n$ acquisitions, which we denote by $Y_n$.

Letting $C = \{1, \dots, N\}$, for any nonempty set $A \subset C$ we define

$$p_A := \sum_{i \in A} p_i,$$

where $p_\emptyset := 0$. Also, we denote by $|A|$ the number of elements in the set $A$.

In [2, Theorem 1] it was shown that for any integer $n \geqslant 0$,

(1.1)
$$P(T_k > n) = \sum_{A \subset C,\, |A| \leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1|} p_A^n, \qquad k = 1, \dots, N,$$

and that for $r \geqslant 1$, the $r$th moment of $T_k$ is given by

$$E(T_k^r) = \sum_{l=0}^{r-1} \binom{r}{l} \sum_{n=0}^{\infty} n^l P(T_k > n). \tag{1.2}$$

Letting $r = 1$ in (1.2) it follows that the first moment of $T_k$ is given by

$$E(T_k) = \sum_{A \subset C,\, |A| \leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} \frac{1}{1-p_A}. \tag{1.3}$$

However, in [2], expression (1.2) was not evaluated for $r > 1$. See also [9, 16] for expression (1.3).

In the present paper we provide a new expression for the $r$th moment of $T_k$ (see (2.8)), which is easily computed since it is in terms of recursive expressions, and taking $r = 1$, we recover (1.3). Also, taking $r = 2$ and $r = 3$, we obtain simplified expressions of the second and third moments of $T_k$. Moreover, we provide an expression of the moment generating function of $T_k$ (see (2.11)).

As for the random variable $Y_n$, in [16] its distribution function was given:

$$(1.4)\qquad P(Y_n \leqslant k) = \sum_{A\subset C,\, |A|\leqslant k} (-1)^{k-|A|} \binom{N-|A|-1}{k-|A|} p_A^n, \qquad k = 1, \dots, N,$$

and also its first and second moments:

$$(1.5)\qquad E(Y_n) = N - \sum_{i=1}^{N} (1-p_i)^n,$$

$$(1.6)\qquad E(Y_n^2) = 2 \sum_{1\leqslant i<j\leqslant N} (1-p_i-p_j)^n - (2N-1) \sum_{i=1}^{N} (1-p_i)^n + N^2.$$

In the present work we give an expression for the $r$th moment of $Y_n$ in terms of finite sums (see (3.3) and also (3.6) if $1 \leqslant r \leqslant N-1$), from which we recover (1.5) and (1.6) for $r = 1$ and $r = 2$, respectively. Additionally, taking $r = 3$, we obtain a simplified expression of the third moment of $Y_n$. Furthermore, we provide an expression for the moment generating function of $Y_n$ (see (3.7)).

The expressions of the moments of these random variables can be easily implemented computationally, so the moments of any order can be calculated.

Next we describe how this work is divided. In Section 2 we find the probability mass function of $T_k$, and through it we obtain an expression for its $r$th moment and its moment generating function. In Section 3, through the distribution of $T_k$, we obtain the probability mass function of $Y_n$, which we use to obtain an expression for its $r$th moment and its moment generating function.

## 2. NUMBER OF ACQUISITIONS NEEDED TO OBTAIN $k$ DIFFERENT COUPONS

Recall that the random variable $T_k$ represents the number of acquisitions needed to obtain $k \leqslant N$ different coupons.

Using (1.1), we find that the probability mass function of $T_k$ is

$$(2.1)\qquad \begin{aligned} P(T_k = n) &= P(T_k > n-1) - P(T_k > n) \\ &= \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} p_A^{n-1}(1-p_A) \end{aligned}$$

for $n = 1, 2, \dots$.

In order to have finite moments and the existence of the moment generating function of $T_k$ we need to assume that $p_i > 0$ for $i = 1, \dots, N$ with $\sum_{i=1}^{N} p_i = 1$. To obtain a general formula for the moments of $T_k$, we will use the equality

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r+1} q^n &= \sum_{j=0}^{r} \binom{r}{j} \Big(\sum_{k=0}^{\infty} k^j q^k\Big)\Big(\sum_{n=1}^{\infty} n^{r-j} q^n\Big) \\ &= \Big(\sum_{k=0}^{\infty} q^k\Big)\Big(\sum_{n=1}^{\infty} n^r q^n\Big) + \sum_{j=1}^{r} \binom{r}{j} \Big(\sum_{k=1}^{\infty} k^j q^k\Big)\Big(\sum_{n=1}^{\infty} n^{r-j} q^n\Big),\end{aligned} \tag{2.2}$$

valid for any real $q$ with $|q| < 1$ and any nonnegative integer $r$ (see [14, p. 199]).

Now, for any real $q$ with $|q| < 1$ and any nonnegative integer $r$, let

$$D(r, q) := \sum_{n=1}^{\infty} n^r q^n. \tag{2.3}$$

It is known that $D(0, q) = q/(1-q)$ and $D(1, q) = q/(1-q)^2$. Furthermore, from (2.2) we find that

$$D(r+1, q) = \frac{D(r, q)}{1-q} + \sum_{j=1}^{r} \binom{r}{j} D(j, q) D(r-j, q) \tag{2.4}$$

for any positive integer $r$. Observe that formula (2.4) is also valid for $r = 0$ if $\sum_{j=1}^{0}$ is understood to be zero.

Now, define

$$G(r, q) = \begin{cases} \frac{1-q}{q} D(r, q) & \text{if } 0 < |q| < 1, \\ 1 & \text{if } q = 0, \end{cases} \tag{2.5}$$

for all $r = 0, 1, 2, \dots$. It follows that if $0 < |q| < 1$, then

$$G(0, q) = 1, \qquad G(1, q) = \frac{1}{1-q}, \tag{2.6}$$

and from (2.4), for $r \geqslant 1$,

$$\begin{aligned}G(r+1, q) &= \frac{1-q}{q} D(r+1, q) \\ &= \frac{1}{1-q} G(r, q) + \frac{q}{1-q} \sum_{j=1}^{r} \binom{r}{j} G(j, q) G(r-j, q).\end{aligned} \tag{2.7}$$

Using the above expressions, we obtain the following result for the moments of the random variable $T_k$.

THEOREM 2.1. *For any positive integer $r$, the $r$th moment of the random variable $T_k$ is given by*

$$E(T_k^r) = \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} G(r, p_A). \tag{2.8}$$

*Proof.* From (2.1) we get

$$\begin{aligned} E(T_k^r) &= \sum_{n=1}^{\infty} n^r P(T_k = n) \\ &= \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} (1-p_A) \sum_{n=1}^{\infty} n^r p_A^{n-1} \\ &= (-1)^{k-1} \binom{N-1}{k-1} \\ &\quad + \sum_{A\subset C,\, 1\leqslant|A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} (1-p_A) \sum_{n=1}^{\infty} n^r p_A^{n-1}. \end{aligned}$$

Thus, (2.8) follows using (2.3) and (2.5). ■

Using (2.6), formula (1.3) for the first moment of $T_k$ follows from (2.8) by taking $r = 1$. Now, letting $r = 2$ and $r = 3$ in (2.8) we obtain, using (2.6) and (2.7), the next simplified expressions for the second and third moments of $T_k$ (which we have not found in the literature):

$$E(T_k^2) = \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} \frac{1+p_A}{(1-p_A)^2}, \tag{2.9}$$

$$E(T_k^3) = \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} \frac{1+4p_A+p_A^2}{(1-p_A)^3}. \tag{2.10}$$

Alternatively, the moments of $T_k$ can be obtained by differentiating its moment generating function, whose expression we provide in the following theorem.

THEOREM 2.2. *The moment generating function of the random variable $T_k$ is given by*

$$M_{T_k}(t) = \sum_{A\subset C,\, |A|\leqslant k-1} (-1)^{k-|A|-1} \binom{N-|A|-1}{k-|A|-1} \frac{(1-p_A)e^t}{1-e^t p_A} \tag{2.11}$$

*for all $t < \min_{A\subset C,\, 1\leqslant|A|\leqslant k-1}\{-\log p_A\}$.*

*Proof.* Using (2.1) we have

$$
\begin{aligned}
(2.12)\qquad M_{T_k}(t) &= \sum_{n=1}^{\infty} e^{tn} P(T_k = n)\\
&= \sum_{A\subset C,\,|A|\leqslant k-1} (-1)^{k-|A|-1}\binom{N-|A|-1}{k-|A|-1}(1-p_A)\sum_{n=1}^{\infty} e^{tn} p_A^{n-1}\\
&= (-1)^{k-1}\binom{N-1}{k-1}e^t\\
&\quad + \sum_{A\subset C,\,1\leqslant|A|\leqslant k-1} (-1)^{k-|A|-1}\binom{N-|A|-1}{k-|A|-1}(1-p_A)e^t\sum_{n=1}^{\infty} e^{t(n-1)} p_A^{n-1}.
\end{aligned}
$$

Now, if $t < \min_{A\subset C,\,1\leqslant|A|\leqslant k-1}\{-\log p_A\}$ then $0 < e^t p_A < 1$ and thus

$$
\sum_{n=1}^{\infty} e^{t(n-1)} p_A^{n-1} = \frac{1}{1-e^t p_A} \tag{2.13}
$$

for all $A \subset C$ with $1 \leqslant |A| \leqslant k-1$. Therefore, (2.11) follows from (2.12) and (2.13). ■

As mentioned before, we can recover expressions (1.3) and (2.9) also using the moment generating function of $T_k$, given in Theorem 2.2. To do this, we define

$$
h(t) = \frac{e^t}{1-e^t p_A}.
$$

Then

$$
\lim_{t\to 0} h'(t) = \frac{1}{(1-p_A)^2} \quad \text{and} \quad \lim_{t\to 0} h''(t) = \frac{1+p_A}{(1-p_A)^3}.
$$

Hence

$$
E(T_k) = \lim_{t\to 0} M'_{T_k}(t) = \sum_{A\subset C,\,|A|\leqslant k-1} (-1)^{k-|A|-1}\binom{N-|A|-1}{k-|A|-1}\frac{1}{1-p_A},
$$

which is the same expression as in (1.3), and

$$
E(T_k^2) = \lim_{t\to 0} M''_{T_k}(t) = \sum_{A\subset C,\,|A|\leqslant k-1} (-1)^{k-|A|-1}\binom{N-|A|-1}{k-|A|-1}\frac{1+p_A}{(1-p_A)^2},
$$

as in (2.9).

REMARK 2.1. Note that the expressions for the derivatives of $h(t)$ become more and more complex, hence it is difficult to obtain high-order moments of $T_k$ by differentiating the moment generating function. However, formula (2.8) makes it easy to compute the moments of $T_k$ of any order, since it uses recursive expressions. Furthermore, (2.8) can be implemented computationally for given values of $r, N, k$ and $(p_1, \ldots, p_N)$.

REMARK 2.2. In [2] a slightly more general model is considered, namely, one supposes that $\sum_{i=1}^N p_i \leqslant 1$ and defines $p_0 = 1 - \sum_{i=1}^N p_i$. This means that there is a null coupon, denoted by 0, which is drawn with probability $p_0$, but which does not belong to the collection. Under this assumption, $T_k$ is the number of coupons that need to be drawn from the set $\{0, 1, \ldots, N\}$, with replacement, until we first obtain a collection composed of $k$ different coupons, $1 \leqslant k \leqslant N$, among $\{1, \ldots, N\}$. In this case, for the tail of the distribution of $T_k$ we have (1.1) with $p_A$ replaced by $p_0 + p_A$ (see [2, Theorem 1]). Therefore, by analogous arguments to those of the proofs of our Theorems 2.1 and 2.2, we find that the $r$th moment and the moment generating function of $T_k$ are given by (2.8) and (2.11), respectively, with $p_A$ replaced by $p_0 + p_A$.

### 3. NUMBER OF DIFFERENT COUPONS OBTAINED IN $n$ ACQUISITIONS

Recall that the random variable $Y_n$ represents the number of different coupons obtained in $n$ acquisitions. Using

$$\begin{aligned} P(Y_n = k) &= P(Y_n \leqslant k) - P(Y_n \leqslant k-1) \\ &= P(T_{k+1} > n) - P(T_k > n), \end{aligned} \tag{3.1}$$

where $P(T_{k+1} > n) = 1$ for $k = N$, we find that the probability mass function of $Y_n$ is

$$P(Y_n = k) = \sum_{A \subset C,\, |A| \leqslant k} (-1)^{k-|A|} \binom{N-|A|}{k-|A|} p_A^n \tag{3.2}$$

for $k = 0, 1, \ldots, N$ (see [16, formula (3.3)]). From (3.2) it follows that the distribution function of $Y_n$ is given by (1.4).

In our next theorem we provide a general formula for the moments of $Y_n$.

THEOREM 3.1. *For any positive integer $r$, the $r$th moment of the random variable $Y_n$ is given by*

$$E(Y_n^r) = \sum_{k=1}^{N} \left( \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} (i+k)^r \right) \sum_{A \subset C,\, |A|=k} p_A^n. \tag{3.3}$$

*Proof.* Using (3.2) we have

$$\begin{aligned} E(Y_n^r) &= \sum_{k=1}^{N} k^r P(Y_n = k) \\ &= \sum_{A \subset C,\, |A| \leqslant N} \sum_{k=|A|}^{N} k^r (-1)^{k-|A|} \binom{N-|A|}{k-|A|} p_A^n, \end{aligned}$$

which, letting $i = k - |A|$, gives

$$\begin{aligned}
E(Y_n^r) &= \sum_{A\subset C,\,|A|\leqslant N} \sum_{i=0}^{N-|A|} (i+|A|)^r(-1)^i \binom{N-|A|}{i} p_A^n \\
&= \sum_{k=1}^{N} \sum_{A\subset C,\ |A|=k} \sum_{i=0}^{N-k} (-1)^i (i+k)^r \binom{N-k}{i} p_A^n \\
&= \sum_{k=1}^{N} \Big( \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} (i+k)^r \Big) \sum_{A\subset C,\,|A|=k} p_A^n. \quad \blacksquare
\end{aligned}$$

Next, we will obtain a simplified expression of (3.3) for $1 \leqslant r \leqslant N-1$. Let

$$R_{k,r} := \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} (i+k)^r.$$

We have the following.

LEMMA 3.1. *Let $k, N, r$ be nonnegative integers. If $k \leqslant N - r - 1$, then $R_{k,r} = 0$.*

*Proof.* From the binomial theorem it follows that

$$\begin{aligned}
R_{k,r} &= \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} \sum_{m=0}^{r} \binom{r}{m} i^m k^{r-m} \\
&= \sum_{m=0}^{r} \binom{r}{m} k^{r-m} \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} i^m.
\end{aligned} \tag{3.4}$$

By the well known identity for nonnegative integers $a$ and $b$ (see [5, 10])

$$\sum_{i=0}^{a} (-1)^i \binom{a}{i} i^b = \begin{cases} 0 & \text{if } b \leqslant a-1, \\ (-1)^a a! & \text{if } b = a, \end{cases} \tag{3.5}$$

we have

$$\sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} i^m = 0 \quad \text{if } N-k-1 \geqslant m.$$

Note that if $k \leqslant N - r - 1$, then

$$N - k - 1 \geqslant r \geqslant m, \quad \forall m = 0, 1, \ldots, r.$$

Hence, from (3.4),

$$R_{k,r} = 0, \quad \forall k \leqslant N - r - 1. \quad \blacksquare$$

Thus Theorem 3.1 and Lemma 3.1 give us the following result.

COROLLARY 3.1. *If $1 \leqslant r \leqslant N-1$, then the $r$th moment of $Y_n$ is given by*

$$E(Y_n^r) = \sum_{k=N-r}^{N} \left( \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} (i+k)^r \right) \sum_{A \subset C,\, |A|=k} p_A^n. \tag{3.6}$$

Formulas (1.5) and (1.6) are obtained by taking $r = 1$ and $r = 2$ in (3.6); and letting $r = 3$ we obtain the next simplified expression for the third moment of $Y_n$ (which we have not found in the literature):

$$\begin{aligned} E(Y_n^3) = -6 \sum_{1 \leqslant i<j<k \leqslant N} (1-p_i-p_j-p_k)^n + (6N-6) \sum_{1 \leqslant i<j \leqslant N} (1-p_i-p_j)^n \\ - (3N^2-3N+1) \sum_{i=1}^{N} (1-p_i)^n + N^3. \end{aligned}$$

Alternatively, the moments of $Y_n$ can be obtained by differentiating its moment generating function, whose expression we provide in the following theorem; however, the derivatives become more complex when the order increases.

THEOREM 3.2. *The moment generating function of the random variable $Y_n$ is given by*

$$M_{Y_n}(t) = \sum_{k=1}^{N} e^{tk} (1-e^t)^{N-k} \sum_{A \subset C,\, |A|=k} p_A^n, \qquad -\infty < t < \infty. \tag{3.7}$$

*Proof.* Using (3.2) we have

$$M_{Y_n}(t) = \sum_{k=1}^{N} e^{tk} P(Y_n = k) = \sum_{A \subset C,\, |A| \leqslant N} \left( \sum_{k=|A|}^{N} e^{tk} (-1)^{k-|A|} \binom{N-|A|}{k-|A|} \right) p_A^n.$$

Letting $i = k - |A|$ and applying the binomial theorem gives

$$\begin{aligned} M_{Y_n}(t) &= \sum_{A \subset C,\, |A| \leqslant N} \left( \sum_{i=0}^{N-|A|} e^{t(i+|A|)} (-1)^i \binom{N-|A|}{i} \right) p_A^n \\ &= \sum_{A \subset C,\, |A| \leqslant N} e^{t|A|} \left( \sum_{i=0}^{N-|A|} (-e^t)^i \binom{N-|A|}{i} \right) p_A^n \\ &= \sum_{A \subset C,\, |A| \leqslant N} e^{t|A|} (1-e^t)^{N-|A|} p_A^n \\ &= \sum_{k=1}^{N} \sum_{A \subset C,\, |A|=k} e^{tk} (1-e^t)^{N-k} p_A^n \\ &= \sum_{k=1}^{N} e^{tk} (1-e^t)^{N-k} \sum_{A \subset C,\, |A|=k} p_A^n. \quad \blacksquare \end{aligned}$$

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