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THE SKOROKHOD PROBLEM WITH TWO NONLINEAR CONSTRAINTS

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Abstract. In this paper, we study the Skorokhod problem with two constraints, where both constraints are nonlinear. We prove the existence and uniqueness of a solution and also provide an explicit construction for the solution. In addition, a number of properties of the solution are investigated, including continuity under uniform and J_1 metrics and a comparison principle.

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1. INTRODUCTION

The Skorokhod problem is a convenient tool to study equations with reflecting boundary conditions. In 1961, Skorokhod [11] originally constructed the solution of a stochastic differential equation (SDE for short) on the half-line $[0, \infty)$. A nondecreasing function is added in this equation to push the solution upward in a minimal way so that it satisfies the so-called Skorokhod condition. In [3, 4], the authors considered a deterministic version of the Skorokhod problem for continuous functions and for càdlàg functions, respectively. A multidimensional extension of the Skorokhod problem was considered by Tanaka [17].

Due to the wide applications of reflecting Brownian motions including statistical physics [2, 16], queueing theory [10], control theory [5], the Skorokhod problem with two reflecting boundaries, also called the two-sided Skorokhod problem, has attracted a great deal of attention of many researchers. Roughly speaking, the Skorokhod problem with two reflecting boundaries α, β is to find a pair (X, K) of

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functions such that $X_t = S_t + K_t \in [\alpha_t, \beta_t]$ for any $t \geqslant 0$ and K satisfies some necessary conditions, where S, α, β ($\alpha < \beta$) are some given right-continuous functions with left limits. For simplicity, (X, K) is called the solution to the Skorokhod problem on $[\alpha, \beta]$ for S. Kruk et al. [9] presented an explicit formula to make a deterministic function stay in the interval $[0, a]$ (i.e., α, β are two constants) and studied the properties of the solutions. Then Burdzy et al. [1] considered the Skorokhod problem in a time-dependent interval. They obtained the existence and uniqueness of a solution to the so-called extended Skorokhod problem, which is a slight generalization of the Skorokhod problem. Under the assumption that $\inf_t(\beta_t - \alpha_t) > 0$, the solution to the extended Skorokhod problem coincides with the one to the Skorokhod problem. We refer the interested reader to [8, 12, 13, 14, 15] and the references therein for related work in this field.

It is worth pointing out that in the existing literature, the solution of a Skorokhod problem with two reflecting boundaries is required to remain in a (time-dependent) interval. The objective of this paper is to study the Skorokhod problem with two reflecting boundaries behaving in a nonlinear way, that is, we need to make sure that two functions of the solution stay positive and negative, respectively. More precisely, let S be a right-continuous function with left limits on $[0, \infty)$ taking values in $\mathbb{R}$. Given two functions $L, R : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $L < R$, we need to find a pair (X, K) of functions such that

(i) $X_t = S_t + K_t$;

(ii) $L(t, X_t) \leqslant 0 \leqslant R(t, X_t)$;

(iii) $K_{0-} = 0$ and K has the decomposition $K = K^r - K^l$, where K^r, K^l are nondecreasing functions satisfying

$$\int_0^\infty \mathbf{1}_{\{L(s,X_s)<0\}}\, dK^l_s = 0, \qquad \int_0^\infty \mathbf{1}_{\{R(s,X_s)>0\}}\, dK^r_s = 0.$$

By choosing L, R appropriately, the Skorokhod problem with two nonlinear reflecting boundaries may degenerate into the classical Skorokhod problem (see [11]), the Skorokhod problem on $[0, a]$ (see [9]) and the Skorokhod problem in a time-dependent interval (see [1, 12, 13]). Recalling the results in [1], the second component of the solution to the Skorokhod problem on $[\alpha, \beta]$ for S is given by

$$K_t = -\max\Big(\Big[(\underline{S}_0)^+ \wedge \inf_{u\in[0,t]} \bar{S}_u\Big], \sup_{s\in[0,t]} \Big[\underline{S}_s \wedge \inf_{u\in[s,t]} \bar{S}_u\Big]\Big),$$

where $\bar{S} = S - \alpha$ and $\underline{S} = S - \beta$. Motivated by this construction, for the nonlinear case, we first find functions Φ^S, Ψ^S satisfying the following nonlinear reflecting constraints:

$$L(t, S_t + \Phi^S_t) = 0, \quad R(t, S_t + \Psi^S_t) = 0, \quad t \geqslant 0.$$

We show that Φ^S and Ψ^S will take over the roles of $\beta - S$ and $\alpha - S$, respectively and the induced K is the second component of the solution to the Skorokhod problem with two nonlinear reflecting boundaries. The explicit characterization for K allows us to obtain the continuity of the solution with respect to the input function S and the nonlinear functions L, R. We also present some comparison theorems. Roughly speaking, the nondecreasing function K^r aims to push the solution upward so that $R(t, X_t) \geqslant 0$, while the nondecreasing function K^l tries to pull the solution downward to make sure that $L(t, X_t) \leqslant 0$. It is natural to conjecture that the forces K^r and K^l will increase if R becomes smaller or L becomes larger. The Skorokhod problem with two nonlinear reflecting boundaries is a building block for studying doubly mean reflected (backward) SDEs (see [6, 7]).

The paper is organized as follows. We first formulate the Skorokhod problem with two nonlinear reflecting boundaries in detail and provide the existence and uniqueness result in Section 2. Then, in Section 3, we investigate the properties of solutions to Skorokhod problems, such as the comparison property and the continuity property.

2. SKOROKHOD PROBLEM WITH TWO NONLINEAR REFLECTING BOUNDARIES

2.1. Basic notations and problem formulation. Let $D[0,\infty)$ be the set of real-valued right-continuous functions having left limits (usually called *càdlàg functions*); $I[0,\infty)$, $C[0,\infty)$, $BV[0,\infty)$ and $AC[0,\infty)$ are the subsets of $D[0,\infty)$ consisting of the nondecreasing functions, continuous functions, functions of bounded variation and absolutely continuous functions, respectively. For any $K \in BV[0,\infty)$ and $t \geqslant 0$, $|K|_t$ is the total variation of K on $[0,t]$.

DEFINITION 2.1. Let $S \in D[0,\infty)$ and let $L, R : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ with $L \leqslant R$. A pair $(X, K) \in D[0,\infty) \times BV[0,\infty)$ is called a *solution of the Skorokhod problem* for S with nonlinear constraints L, R (briefly, (X, K) *solves* $\mathbb{SP}_L^R(S)$) if

(i) $X_t = S_t + K_t$;

(ii) $L(t, X_t) \leqslant 0 \leqslant R(t, X_t)$;

(iii) $K_{0-} = 0$ and K has a decomposition $K = K^r - K^l$, where K^r, K^l are nondecreasing functions satisfying

$$\int_0^\infty \mathbf{1}_{\{L(s,X_s)<0\}}\, dK_s^l = 0, \qquad \int_0^\infty \mathbf{1}_{\{R(s,X_s)>0\}}\, dK_s^r = 0.$$

REMARK 2.1. (i) The integration in (iii) of Definition 2.1 is carried out including the initial time 0. That is, if $K_0 > 0$, we must have $R(0, X_0)K_0 = 0$; if $K_0 < 0$, we must have $L(0, X_0)K_0 = 0$.

(ii) If $L \equiv -\infty$ and $R(t, x) = x$, then the Skorokhod problem associated with S, L, R turns into the classical Skorokhod problem as in [11].

(iii) If $L(t,x) = x - a$ and $R(t,x) = x$, where a is a positive constant, then the Skorokhod problem associated with S, L, R coincides with the Skorokhod problem on $[0, a]$ studied in [9].

(iv) If $L(t,x) = x - r_t$ and $R(t,x) = x - l_t$, where $r, l \in D[0,\infty)$ with $l \leqslant r$, then the Skorokhod problem associated with S, L, R corresponds to the Skorokhod problem on $[l, r]$ for S as in [1, 12, 13].

(v) If $L = -\infty$ (resp. $R = \infty$), Definition 2.1 is a special case of Definition 2.8 (resp. Definition 2.10) in [6] with $l = 0$ (resp. $u = 0$).

REMARK 2.2. It is worth pointing out that the Skorokhod problem on $[l, r]$ for S as in Remark 2.1(iv) is a building block for the Skorokhod problem with mean minimality condition studied in [6]. More explicitly, consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathsf{P})$. Let $h : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying the linear growth condition in its second argument. Given an adapted process Y with càdlàg trajectories and $l, r \in D[0,\infty)$ with $l \leqslant r$ and $\mathsf{E}[h(0, Y_0)] \in [l_0, r_0]$, a pair (X, K) is a solution of the Skorokhod problem with mean minimality condition and two constraints associated with h, Y, l, r if

(i) $X_t = Y_t + K_t$, $t \geqslant 0$;

(ii) $\mathsf{E}[h(t, X_t)] \in [l_t, r_t]$, $t \geqslant 0$;

(iii) for every $0 \leqslant t \leqslant q$,

$$\begin{aligned} K_q - K_t \geqslant 0 \quad & \text{if } \mathsf{E}[h(s, X_s)] < r_s \text{ for all } s \in (t, q], \\ K_q - K_t \leqslant 0 \quad & \text{if } \mathsf{E}[h(s, X_s)] > l_s \text{ for all } s \in (t, q], \end{aligned}$$

and for every $t \geqslant 0$, $\Delta K_t \geqslant 0$ if $\mathsf{E}[h(t, X_t)] < r_t$ and $\Delta K_t \leqslant 0$ if $\mathsf{E}[h(t, X_t)] > l_t$.

For any $t \geqslant 0$ and $z \in \mathbb{R}$, we define a new map $H(t, \cdot, Y_t) : \mathbb{R} \to \mathbb{R}$ by

$$H(t, z, Y_t) = \mathsf{E}[h(t, Y_t - \mathsf{E}[Y_t] + z)].$$

Under appropriate assumptions on h, there exists a strictly increasing and continuous inverse map $H^{-1}(t, \cdot, Y_t) : \mathbb{R} \to \mathbb{R}$. Then the second component K of the solution to the Skorokhod problem associated with h, Y, l, r coincides with the one to the Skorokhod problem on $[\bar{l}, \bar{r}]$ for $\bar{y}$, where $y_t = \mathsf{E}[Y_t]$, $\bar{l}_t = H^{-1}(t, l_t, Y_t)$ and $\bar{r}_t = H^{-1}(t, r_t, Y_t)$, $t \geqslant 0$.

However, if we propose the following constraints for the resulting process X:

(ii′) $\mathsf{E}[L(t, X_t)] \leqslant 0 \leqslant \mathsf{E}[R(t, X_t)]$, $t \geqslant 0$,

then the construction of the solution to the above Skorokhod problem with two nonlinear constraints needs our Skorokhod problem introduced in Definition 2.1, which is the motivation to study this problem.

In order to solve the Skorokhod problem with two nonlinear reflecting boundaries, we propose the following assumption on the functions L, R.

ASSUMPTION 2.1. *The functions $L, R : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) *For each fixed $x \in \mathbb{R}$, $L(\cdot, x), R(\cdot, x) \in D[0,\infty)$,*

(ii) *For any fixed $t \geqslant 0$, $L(t,\cdot)$, $R(t,\cdot)$ are strictly increasing,*

(iii) *There exist constants $0 < c < C < \infty$ such that for any $t \geqslant 0$ and $x, y \in \mathbb{R}$,*

$$\begin{aligned} c|x-y| &\leqslant |L(t,x) - L(t,y)| \leqslant C|x-y|, \\ c|x-y| &\leqslant |R(t,x) - R(t,y)| \leqslant C|x-y|. \end{aligned}$$

(iv) $\inf_{(t,x)\in[0,\infty)\times\mathbb{R}}(R(t,x) - L(t,x)) > 0$.

REMARK 2.3. Actually, all the results in this paper, including the existence and uniqueness result (Theorem 2.1) and the properties of solutions to the Skorokhod problems (Theorems 3.1–3.2, Propositions 3.3–3.5), still hold if (iv) in Assumption 2.1 is replaced by the weaker condition

$$\inf_{t\leqslant q,\, x\in\mathbb{R}} (R(t,x) - L(t,x)) > 0, \qquad q \geqslant 0.$$

Conditions (ii) and (iii) in Assumption 2.1 imply that for any $t \geqslant 0$,

$$\begin{aligned} \lim_{x\to-\infty} L(t,x) &= -\infty, & \lim_{x\to+\infty} L(t,x) &= +\infty, \\ \lim_{x\to-\infty} R(t,x) &= -\infty, & \lim_{x\to+\infty} R(t,x) &= +\infty. \end{aligned}$$

Now, given $S \in D[0,\infty)$, for any $t \geqslant 0$, the above equation and condition (iii) in Assumption 2.1 imply that the mappings $x \mapsto L(t, S_t + x)$, $x \mapsto R(t, S_t + x)$: $\mathbb{R} \to \mathbb{R}$ are one-to-one. Let Φ_t^S, Ψ_t^S be the unique solutions to the following equations, respectively:

$$L(t, S_t + x) = 0, \qquad R(t, S_t + x) = 0. \tag{2.1}$$

In the following, for simplicity, we always omit the superscript S. We first investigate the properties of Φ and Ψ.

LEMMA 2.1. *Under Assumption 2.1, for any given $S \in D[0,\infty)$, we have $\Phi, \Psi \in D[0,\infty)$ and*

$$\inf_{t\geqslant 0}(\Phi_t - \Psi_t) > 0.$$

Proof. By a similar analysis to the proof of [6, Lemma 2.2], we have $\Phi, \Psi \in D[0,\infty)$. It remains to prove that $\inf_{t\geqslant 0}(\Phi_t - \Psi_t) > 0$. First, it is easy to check that $\Phi_t > \Psi_t$ for any $t \geqslant 0$. Indeed, suppose that there exists some $t \geqslant 0$ such that $\Phi_t \leqslant \Psi_t$. By Assumption 2.1(ii, iv) and the definition of Φ, Ψ, we have

$$0 = L(t, S_t + \Phi_t) < R(t, S_t + \Phi_t) \leqslant R(t, S_t + \Psi_t) = 0,$$

which is a contradiction. Set

$$\alpha = \inf_{(t,x)\in[0,\infty)\times\mathbb{R}} (R(t,x) - L(t,x)) > 0.$$

It follows that

$$-L(t, S_t + \Psi_t) = R(t, S_t + \Psi_t) - L(t, S_t + \Psi_t) > \alpha.$$

Then

$$\alpha < L(t, S_t + \Phi_t) - L(t, S_t + \Psi_t) \leqslant C(\Phi_t - \Psi_t),$$

which implies that $\inf_{t\geqslant 0}(\Phi_t - \Psi_t) \geqslant \alpha/C$. The proof is complete. ∎

REMARK 2.4. Assume that for any $x \in \mathbb{R}$, we have $L(\cdot, x), R(\cdot, x) \in C[0,\infty)$ and L, R satisfy Assumption 2.1(ii)–(iv). Given $S \in C[0,\infty)$, we can show that $\Phi, \Psi \in C[0,\infty)$.

2.2. Existence and uniqueness result. In this subsection, we first establish uniqueness of solutions to the Skorokhod problem with two nonlinear reflecting boundaries. The proof is a relatively straightforward modification of the proof for the extended Skorokhod problem in a time-dependent interval (see [1, Proposition 2.8]).

PROPOSITION 2.1. *Suppose that L, R satisfy Assumption* 2.1. *For any given $S \in D[0,\infty)$, there exists at most one solution to the Skorokhod problem $\mathbb{SP}_L^R(S)$.*

Proof. Let (X, K) and (X', K') solve $\mathbb{SP}_L^R(S)$. At time 0, we have the following three cases.

CASE I: $L(0, S_0) \leqslant 0 \leqslant R(0, S_0)$. In this case, $\Psi_0 \leqslant 0 \leqslant \Phi_0$ and $K_0 = K'_0 = 0$.

CASE II: $L(0, S_0) < 0$. In this case, $\Psi_0 > 0$ and $K_0 = K'_0 = \Psi_0$.

CASE III: $R(0, S_0) > 0$. In this case, $\Phi_0 < 0$ and $K_0 = K'_0 = \Phi_0$.

Therefore, we may conclude that $K_0 = K'_0 = [-(\Phi_0)^-] \vee \Psi_0$. Consequently, $X_0 = X'_0$.

Suppose that there exists some $T > 0$ such that $X_T > X'_T$. Let

$$\tau = \sup\{t \in [0,T] : X_t \leqslant X'_t\}.$$

Since $X_0 = X'_0$, τ is well-defined. We have the following two cases.

CASE 1: $X_\tau \leqslant X'_\tau$. In this case, for any $t \in (\tau, T]$, we have $X_t > X'_t$. It follows that for any $t \in (\tau, T]$,

$$R(t, X_t) > R(t, X'_t) \geqslant 0 \geqslant L(t, X_t) > L(t, X'_t). \tag{2.2}$$

Using condition (iii) in Definition 2.1, we have

$$K_T - K_\tau = -(K^l_T - K^l_\tau) \leqslant 0 \leqslant K'^{,r}_T - K'^{,r}_\tau = K'_T - K'_\tau.$$

Therefore,

$$0 < X_T - X'_T = K_T - K'_T \leqslant K_\tau - K'_\tau = X_\tau - X'_\tau,$$

which contradicts the case assumption.

CASE 2: $X_\tau > X'_\tau$. Recalling that $X_0 = X'_0$, we have $\tau > 0$. Moreover, the definition of τ yields

$$X_{\tau-} \leqslant X'_{\tau-}. \tag{2.3}$$

Furthermore, in this case, (2.2) holds for $t = \tau$. Similar to the proof of Case 1, we have

$$K_\tau - K_{\tau-} = -(K^l_\tau - K^l_{\tau-}) \leqslant 0 \leqslant K'^{,r}_\tau - K'^{,r}_{\tau'} = K'_\tau - K'_{\tau'}.$$

It follows that

$$0 < X_\tau - X'_\tau = K_\tau - K'_\tau \leqslant K_{\tau-} - K'_{\tau-} = X_{\tau-} - X'_{\tau-},$$

which contradicts (2.3).

All the above analysis indicates that $X_T \leqslant X'_T$ for any $T \geqslant 0$. A similar argument implies that $X_T \geqslant X'_T$ for any $T \geqslant 0$. Hence, $X_T = X'_T$, and consequently $K_T = K'_T$ for any $T \geqslant 0$. ∎

REMARK 2.5. The proof of Proposition 2.1 is valid if L, R satisfy (i) and (ii) in Assumption 2.1 and $R(t, x) \geqslant L(t, x)$ for any $(t, x) \in [0, \infty) \times \mathbb{R}$.

Now, we state the main result of this section.

THEOREM 2.1. *Suppose that Assumption* 2.1 *holds. Given* $S \in D[0, \infty)$*, set*

$$K_t = -\max\Big((-\Phi_0)^+ \wedge \inf_{r\in[0,t]}(-\Psi_r), \sup_{s\in[0,t]}\Big[(-\Phi_s) \wedge \inf_{r\in[s,t]}(-\Psi_r)\Big]\Big), \tag{2.4}$$

and $X = S + K$*. Then* (X, K) *is the unique solution to the Skorokhod problem* $\mathbb{SP}^R_L(S)$.

REMARK 2.6. (i) For the case of Remark 2.1(iv), i.e., $L(t,x) = x - r_t$, $R(t,x) = x - l_t$, we have $\Phi_t = r_t - S_t$, $\Psi_t = l_t - S_t$, and thus

$$K_t = -\max\Big((S_0 - r_0)^+ \wedge \inf_{r\in[0,t]}(S_r - l_r),\ \sup_{s\in[0,t]}\Big[(S_s - r_s) \wedge \inf_{r\in[s,t]}(S_r - l_r)\Big]\Big),$$

which coincides with the results in [1, 12]. In particular, let $r_t = e^{at}q$, $l_t = e^{at}p$, $S_t = \big(x_0 + \frac{\beta}{a}\big) - \frac{\beta}{a}e^{at}$ with $a, \beta > 0$, $p \leqslant x_0 \leqslant q$ and $p > -\beta/a$. Then we may calculate that the second component K of the solution to $\mathbb{SP}_L^R(S)$ is given by

$$K_t = \left\{\left(p + \frac{\beta}{a}\right)e^{at} - \left(x_0 + \frac{\beta}{a}\right)\right\}\mathbf{1}_{\{t>t_0\}},$$

where $t_0 = \frac{1}{a}\big(\ln\big(x_0 + \frac{\beta}{a}\big) - \ln\big(p + \frac{\beta}{a}\big)\big)$.

(ii) Let $L(t,x) = x + \alpha\sin(x) - q$, $R(t,x) = x + \alpha\sin(x) - p$ with $p < q$, $|\alpha| < 1$. Then L, R satisfy Assumption 2.1. Given $S \in D[0,\infty)$, for any $t \geqslant 0$, let Φ_t, Ψ_t be the solutions to

$$L(t, S_t + \Phi_t) = 0, \qquad R(t, S_t + \Psi_t) = 0.$$

We define

$$K_t = -\max\Big([(-\Phi)_0^-] \wedge \inf_{r\in[0,t]}(-\Psi_r),\ \sup_{s\in[0,t]}\Big[(-\Phi_s) \wedge \sup_{r\in[s,t]}(-\Psi_r)\Big]\Big).$$

Then K is the second component of the solution to $\mathbb{SP}_L^R(S)$.

The proof of Theorem 2.1 will be divided into several lemmas. We first show that the function K defined by (2.4) is right-continuous with left limits. The proof needs the following observation used in [1, 12]: $\phi \in D[0,\infty)$ if and only if the following two conditions hold for any $\varepsilon > 0$:

(i) for each $\theta_1 \geqslant 0$, there exists some $\theta_2 > \theta_1$ such that

$$\sup_{t_1,t_2\in[\theta_1,\theta_2)} |\phi_{t_1} - \phi_{t_2}| \leqslant \varepsilon;$$

(ii) for each $\theta_2 > 0$, there exists some $0 \leqslant \theta_1 < \theta_2$ such that

$$\sup_{t_1,t_2\in[\theta_1,\theta_2)} |\phi_{t_1} - \phi_{t_2}| \leqslant \varepsilon.$$

LEMMA 2.2. *If* $\Phi, \Psi \in D[0,\infty)$*, then* K *defined by* (2.4) *belongs to* $D[0,\infty)$.

Proof. For any $0 \leqslant s \leqslant t$, set

$$\begin{aligned} H^{\Phi,\Psi}(t) &= (-\Phi_0)^+ \wedge \inf_{r\in[0,t]}(-\Psi_r), \\ R_t^{\Phi,\Psi}(s) &= (-\Phi_s) \wedge \inf_{r\in[s,t]}(-\Psi_r), \\ C^{\Phi,\Psi}(t) &= \sup_{s\in[0,t]} R_t^{\Phi,\Psi}(s). \end{aligned}$$

For simplicity, we always omit the superscript Φ, Ψ. It is easy to check that

$$(-\Phi_t) \wedge (-\Psi_t) \leqslant C(t) \leqslant -\Psi_t. \tag{2.5}$$

For any $0 \leqslant \theta_1 < \theta_2$, let t_1, t_2 be in $[\theta_1, \theta_2)$ with $t_1 \leqslant t_2$, and

$$a := \sup_{s,u\in[\theta_1,\theta_2)} |\Phi_s - \Phi_u| + \sup_{s,u\in[\theta_1,\theta_2)} |\Psi_s - \Psi_u|.$$

Then, for any $s \in [0, t_1]$, we have $R_{t_2}(s) \leqslant R_{t_1}(s)$ and

$$\sup_{s\in(t_1,t_2]} R_{t_2}(s) \leqslant \sup_{s\in(t_1,t_2]} (-\Phi_s) \wedge (-\Psi_s) \leqslant (-\Phi_{t_1}) \wedge (-\Psi_{t_1}) + a.$$

It follows that

$$\begin{aligned} C(t_2) &= \sup_{s\in[0,t_1]} R_{t_2}(s) \vee \sup_{s\in(t_1,t_2]} R_{t_2}(s) \\ &\leqslant \sup_{s\in[0,t_1]} R_{t_1}(s) \vee [(-\Phi_{t_1}) \wedge (-\Psi_{t_1}) + a] \\ &\leqslant \sup_{s\in[0,t_1]} R_{t_1}(s) + a = C(t_1) + a. \end{aligned}$$

On the other hand, noting (2.5), we obtain

$$\begin{aligned} C(t_1) - a &\leqslant C(t_1) \wedge (-\Psi_{t_1} - a) \\ &\leqslant \sup_{s\in[0,t_1]} \Big[R_{t_1}(s) \wedge \inf_{r\in(t_1,t_2]} (-\Psi_r)\Big] \\ &= \sup_{s\in[0,t_1]} R_{t_2}(s) \leqslant \sup_{s\in[0,t_2]} R_{t_2}(s) = C(t_2). \end{aligned}$$

All the above analysis indicates that $|C(t_1) - C(t_2)| \leqslant a$. Moreover, a simple calculation yields

$$\begin{aligned} H(t_2) &\leqslant H(t_1) = H(t_1) \wedge (-\Psi_{t_1} - a + a) \\ &\leqslant H(t_1) \wedge (-\Psi_{t_1} - a) + a \\ &\leqslant H(t_1) \wedge \inf_{s\in(t_1,t_2]} (-\Psi_s) + a = H(t_2) + a, \end{aligned}$$

which indicates that $|H(t_1) - H(t_2)| \leqslant a$. It is easy to check that the following inequality holds for any $x_i, y_i \in \mathbb{R}$, $i = 1, 2$:

$$|x_1 \vee x_2 - y_1 \vee y_2| \leqslant |x_1 - x_2| \vee |y_1 - y_2|.$$

Then we obtain

$$|K_{t_1} - K_{t_2}| \leqslant |H(t_1) - H(t_2)| \vee |C(t_1) - C(t_2)| \leqslant a.$$

Since $t_1 \leqslant t_2$ are arbitrarily chosen in $[\theta_1, \theta_2)$, we deduce that

$$\sup_{t_1,t_2\in[\theta_1,\theta_2)} |K_{t_1} - K_{t_2}| \leqslant \sup_{s,u\in[\theta_1,\theta_2)} |\Phi_s - \Phi_u| + \sup_{s,u\in[\theta_1,\theta_2)} |\Psi_s - \Psi_u|.$$

Consequently, $K \in D[0, \infty)$. ∎

REMARK 2.7. The proof of Lemma 2.2 also shows that on the closed interval $[\theta_1, \theta_2]$, the oscillation of K can be dominated by the oscillation of Φ and Ψ:

$$\sup_{t_1,t_2\in[\theta_1,\theta_2]} |K_{t_1} - K_{t_2}| \leqslant \sup_{s,u\in[\theta_1,\theta_2]} |\Phi_s - \Phi_u| + \sup_{s,u\in[\theta_1,\theta_2]} |\Psi_s - \Psi_u|.$$

Therefore, if $\Phi, \Psi \in C[0, \infty)$, we have $K \in C[0, \infty)$. Under the assumption as in Remark 2.4, for any given $S \in C[0, \infty)$, each component of the solution (X, K) to $\mathbb{SP}_L^R(S)$ is continuous.

Now, we define the following pair of times:

$$\sigma^* = \inf \{t > 0 : \Phi_t \leqslant 0\}, \quad \tau^* = \inf \{t > 0 : \Psi_t \geqslant 0\}. \tag{2.6}$$

REMARK 2.8. (i) Noting that $a := \inf_t(\Phi_t - \Psi_t) > 0$ as shown in Lemma 2.1, three cases are possible:

$$\text{either} \quad \sigma^* = \tau^* = \infty, \quad \sigma^* < \tau^* \quad \text{or} \quad \sigma^* > \tau^*. \tag{2.7}$$

If $\sigma^* = \tau^* = \infty$, we have $\Psi_t < 0 < \Phi_t$ for any $t \geqslant 0$. Consequently, $K_t = 0$ and $L(t, S_t) < 0 < R(t, S_t)$ for any t. Hence, $(S, 0)$ solves $\mathbb{SP}_L^R(S)$. In the rest of this section, we only consider the other two cases.

(ii) For any $t \in [0, \sigma^* \wedge \tau^*)$, we have $\Psi_t < 0 < \Phi_t$. It follows that $K_t = 0$ when $t \in [0, \sigma^* \wedge \tau^*)$.

If $\tau^* > \sigma^*$, we set $\tau_0 = 0$, $\sigma_0 = \sigma^*$, and for $k \geqslant 1$, we set

$$\tau_k = \inf \Big\{ t > \sigma_{k-1} : \inf_{s\in[\sigma_{k-1},t]} \Phi_s \leqslant \Psi_t \Big\}, \tag{2.8}$$

$$\sigma_k = \inf \Big\{ t > \tau_k : \sup_{s\in[\tau_k,t]} \Psi_s \geqslant \Phi_t \Big\}. \tag{2.9}$$

Since Φ, Ψ are right-continuous, τ_k, σ_k are well-defined.

If $\tau^* < \sigma^*$, we set $\tau_0 = \tau^*$ and define σ_k by (2.9) for all $k \geqslant 0$, and τ_k by (2.8) for all $k \geqslant 1$.

It is easy to check that in both cases $\tau^* < \sigma^*$ and $\tau^* > \sigma^*$, the following two inequalities hold for $k \geqslant 1$:

$$\inf_{s\in[\sigma_{k-1},t]} \Phi_s > \Psi_t \quad \text{for any } t \in [\sigma_{k-1}, \tau_k), \tag{2.10}$$

$$\inf_{s\in[\sigma_{k-1},\tau_k]} \Phi_s \leqslant \Psi_{\tau_k}, \tag{2.11}$$

Furthermore, the following two inequalities hold for any $k \geqslant 1$ if $\tau^* > \sigma^*$, and for any $k \geqslant 0$ if $\tau^* < \sigma^*$:

$$\Phi_t > \sup_{s\in[\tau_k,t]} \Psi_s \quad \text{for any } t \in [\tau_k, \sigma_k), \tag{2.12}$$

$$\Phi_{\sigma_k} \leqslant \sup_{s\in[\tau_k,\sigma_k]} \Psi_s. \tag{2.13}$$

Finally, if $\tau^* > \sigma^*$, we have $\Psi_t \leqslant 0$ for any $t \in [0, \sigma_0]$ and

$$\Phi_{\sigma_0} \leqslant 0. \tag{2.14}$$

If $\tau^* < \sigma^*$, we have $\Phi_t \geqslant 0$ for any $t \in [0, \tau_0]$ and

$$\Psi_{\tau_0} \geqslant 0. \tag{2.15}$$

By (2.10), it is easy to check that $\Phi_s > \Psi_t$ for any $\sigma_{k-1} \leqslant s \leqslant t < \tau_k$, $k \geqslant 1$. It follows that

$$\Phi_s > \sup_{t\in[s,\tau_k)} \Psi_t \quad \text{for any } s \in [\sigma_{k-1}, \tau_k). \tag{2.16}$$

It is clear that

$$0 \leqslant \tau_0 \leqslant \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \cdots .$$

We first show that τ_k, σ_k tend to infinity as $k \to \infty$.

PROPOSITION 2.2. *Under Assumption* 2.1, *we have*

$$\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k = \infty.$$

Proof. The proof is similar to the one of [9, Proposition 3.1]. For the readers' convenience, we give a short argument. By Lemma 2.1, $a := \inf_{t\geqslant 0}(\Phi_t - \Psi_t) > 0$. Suppose that

$$\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k = t^* < \infty.$$

For any $k \geqslant 1$, there exists $\rho_k \in [\sigma_{k-1}, \tau_k]$ such that

$$\inf_{t\in[\sigma_{k-1},\tau_k]} \Phi_t \geqslant \Phi_{\rho_k} - a/2.$$

Recalling (2.11) and the definition of a, we have

$$\Psi_{\tau_k} \geqslant \Psi_{\rho_k} + a/2.$$

Letting $k \to \infty$, it follows that Ψ does not have a left limit at t^*, which is a contradiction. Therefore,

$$\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k = \infty. \quad \blacksquare$$

By (iii) in Definition 2.1, the second component of the solution to the Skorokhod problem with two nonlinear reflecting boundaries is of bounded variation. In the following two propositions, we show that K is piecewise monotone. Therefore, K defined by (2.4) is a bounded variation function.

PROPOSITION 2.3. *Under Assumption* 2.1, *for any* $k \geqslant 1$ *and* $t \in [\sigma_{k-1}, \tau_k)$, *we have*

$$-K_t = \sup_{s\in[\sigma_{k-1},t]} (-\Phi_s).$$

Proof. The proof is similar to the one of [13, Lemma 2.7]. For the readers' convenience, we give a short argument. For any $k \geqslant 1$ and $t \in [\sigma_{k-1}, \tau_k)$, set

$$\begin{aligned}
I_t^1 &= (-\Phi_0)^+ \wedge \inf_{r\in[0,t]} (-\Psi_r),\\
I_t^{2,k-1} &= \sup_{s\in[0,\tau_{k-1}]} \Big[(-\Phi_s) \wedge \inf_{r\in[s,t]} (-\Psi_r)\Big],\\
I_t^{3,k-1} &= \sup_{s\in[\tau_{k-1},\sigma_{k-1})} \Big[(-\Phi_s) \wedge \inf_{r\in[s,t]} (-\Psi_r)\Big],\\
I_t^{4,k-1} &= \sup_{s\in[\sigma_{k-1},t]} \Big[(-\Phi_s) \wedge \inf_{r\in[s,t]} (-\Psi_r)\Big].
\end{aligned}$$

It is obvious that $-K_t = I_t^1 \vee I_t^{2,k-1} \vee I_t^{3,k-1} \vee I_t^{4,k-1}$.

CASE 1: $k \geqslant 2$ if $\tau^* > \sigma^*$ and $k \geqslant 1$ if $\tau^* < \sigma^*$. By (2.13), it follows that

$$\inf_{s\in[\tau_{k-1},\sigma_{k-1}]} (-\Psi_s) \leqslant -\Phi_{\sigma_{k-1}} \leqslant \sup_{s\in[\sigma_{k-1},t]} (-\Phi_s). \tag{2.17}$$

Therefore,

$$\begin{aligned}
I_t^1 &\leqslant \inf_{r\in[0,t]} (-\Psi_r) \leqslant \inf_{s\in[\tau_{k-1},\sigma_{k-1}]} (-\Psi_s) \leqslant \sup_{s\in[\sigma_{k-1},t]} (-\Phi_s),\\
I_t^{2,k-1} &\leqslant \sup_{s\in[0,\tau_{k-1}]} \inf_{r\in[s,t]} (-\Psi_r) \leqslant \inf_{s\in[\tau_{k-1},\sigma_{k-1}]} (-\Psi_s) \leqslant \sup_{s\in[\sigma_{k-1},t]} (-\Phi_s).
\end{aligned}$$

Recalling (2.12), for any $t \in [\tau_{k-1}, \sigma_{k-1})$ we have $-\Phi_t < \inf_{r\in[\tau_{k-1},t]}(-\Psi_r)$. Then we find that for any $t \in [\sigma_{k-1}, \tau_k)$,

$$\begin{aligned} I_t^{3,k-1} &\leqslant \sup_{s\in[\tau_{k-1},\sigma_{k-1})} \Big[\inf_{r\in[\tau_{k-1},t]}(-\Psi_r) \wedge \inf_{r\in[s,t]}(-\Psi_r)\Big] \\ &\leqslant \inf_{r\in[\tau_{k-1},\sigma_{k-1}]}(-\Psi_r) \leqslant \sup_{s\in[\sigma_{k-1},t]}(-\Phi_s), \end{aligned}$$

where we have used (2.17) in the last inequality. It follows from (2.16) that for any $k \geqslant 1$,

$$I_t^{4,k-1} = \sup_{s\in[\sigma_{k-1},t]}(-\Phi_s). \tag{2.18}$$

Thus, in this case, $-K_t = \sup_{s\in[\sigma_{k-1},t]}(-\Phi_s)$.

CASE 2: $k = 1$ if $\tau^* > \sigma^*$. In this case, for any $t \in [0, \sigma_0)$, we have $\Psi_t \leqslant 0 \leqslant \Phi_t$, $\Psi_{\sigma_0} \leqslant 0$, $\Phi_{\sigma_0} \leqslant 0$. Therefore, it is easy to check that for any $t \in [\sigma_0, \tau_1)$,

$$\begin{aligned} &I_t^1 \leqslant (-\Phi_0)^+ = 0, \qquad I_t^{2,0} = (-\Phi_0) \wedge (-\Psi_0) \leqslant 0, \\ &I_t^{3,0} = \sup_{s\in[0,\sigma_0)} [(-\Phi_s) \wedge \inf_{r\in[s,t]}(-\Psi_r)] \\ &\qquad \leqslant \sup_{s\in[0,\sigma_0)} [(-\Phi_s) \wedge \inf_{r\in[s,\sigma_0]}(-\Psi_r)] = \sup_{s\in[0,\sigma_0)}(-\Phi_s) \leqslant 0. \end{aligned}$$

Recalling (2.18), we have $I_t^{4,0} = \sup_{s\in[\sigma_0,t]}(-\Phi_s) \geqslant -\Phi_{\sigma_0} \geqslant 0$. Therefore, in this case, $-K_t = \sup_{s\in[\sigma_0,t]}(-\Phi_s)$. The proof is complete. ■

PROPOSITION 2.4. *Suppose Assumption* 2.1 *holds. If* $k \geqslant 1$ *or* $\tau^* < \sigma^*$ *and* $k = 0$*, then for any* $t \in [\tau_k, \sigma_k)$*, we have*

$$-K_t = \inf_{s\in[\tau_k,t]}(-\Psi_s).$$

Proof. The proof is similar to the one of [13, Lemma 2.8]. For the readers' convenience, we give a short argument. Let $t \in [\tau_k, \sigma_k)$. Set

$$I_t^{5,k} = \sup_{s\in[\tau_k,t]} \Big[(-\Phi_s) \wedge \inf_{r\in[s,t]}(-\Psi_r)\Big].$$

Then $-K_t = I_t^1 \vee I_t^{2,k} \vee I_t^{5,k}$, where I^1, $I^{2,k}$ are defined in the proof of Proposition 2.3. It is easy to check that

$$\begin{aligned} I_t^1 &\leqslant \inf_{s\in[0,t]}(-\Psi_s) \leqslant \inf_{s\in[\tau_k,t]}(-\Psi_s), \\ I_t^{2,k} &\leqslant \sup_{s\in[0,\tau_k]} \inf_{r\in[s,t]}(-\Psi_r) \leqslant \inf_{s\in[\tau_k,t]}(-\Psi_s). \end{aligned}$$

By (2.12), we have

$$I_t^{5,k} \leqslant \sup_{s\in[\tau_k,t]}\Big[\inf_{r\in[\tau_k,s]}(-\Psi_r)\wedge\inf_{r\in[s,t]}(-\Psi_r)\Big] \leqslant \inf_{s\in[\tau_k,t]}(-\Psi_s).$$

The above analysis indicates that $-K_t \leqslant \inf_{s\in[\tau_k,t]}(-\Psi_s)$.

Now, we are in a position to prove the reverse inequality. It is sufficient to prove that for $k \geqslant 1$,

$$I_t^{2,k} \geqslant \inf_{s\in[\tau_k,t]}(-\Psi_s) \quad \text{for } t\in[\tau_k,\sigma_k) \tag{2.19}$$

and for $\tau^* < \sigma^*$ and $k=0$,

$$I_t^1 = \inf_{r\in[\tau_0,t]}(-\Psi_r) \quad \text{for } t\in[\tau_0,\sigma_0). \tag{2.20}$$

We first prove (2.19). For any fixed $\varepsilon>0$ and $k\geqslant 1$, there exists some $\rho\in[\sigma_{k-1},\tau_k]$ such that

$$\inf_{s\in[\sigma_{k-1},\tau_k]}\Phi_s \geqslant \Phi_\rho - \varepsilon.$$

Together with (2.11), we have

$$-\Phi_\rho \geqslant -\Psi_{\tau_k} - \varepsilon.$$

Recalling (2.16), we obtain

$$\begin{aligned} I_t^{2,k} &\geqslant (-\Phi_\rho)\wedge\inf_{r\in[\rho,t]}(-\Psi_r)\\ &\geqslant (-\Phi_\rho)\wedge\inf_{r\in[\rho,\tau_k)}(-\Psi_r)\wedge\inf_{r\in[\tau_k,t]}(-\Psi_r)\\ &= (-\Phi_\rho)\wedge\inf_{r\in[\tau_k,t]}(-\Psi_r)\\ &\geqslant (-\Psi_{\tau_k}-\varepsilon)\wedge\inf_{r\in[\tau_k,t]}(-\Psi_r) \geqslant \inf_{r\in[\tau_k,t]}(-\Psi_r)-\varepsilon. \end{aligned}$$

Since ε can be arbitrarily small, (2.19) holds true for any $k\geqslant 1$.

It remains to prove that (2.20) holds when $\tau^*<\sigma^*$ and $k=0$. Indeed, since $\sigma^*>0$, we have $\Phi_0>0$. Moreover, as $\tau_0=\tau^*$, we have $\sup_{r\in[0,\tau_0)}\Psi_r\leqslant 0$ and $\sup_{r\in[\tau_0,t]}\Psi_r\geqslant\Psi_{\tau_0}\geqslant 0$. Then, we may check that

$$I_t^1 = 0\wedge\inf_{r\in[0,\tau_0)}(-\Psi_r)\wedge\inf_{r\in[\tau_0,t]}(-\Psi_r) = \inf_{r\in[\tau_0,t]}(-\Psi_r).$$

The proof is complete. ∎

Combining Remark 2.8 and Propositions 2.3 and 2.4, we have the following representation for K, which is a generalization of [12, Theorem 2.2] and [13, Theorem 2.6].

THEOREM 2.2. *Under Assumption* 2.1, *let* K *be defined by* (2.4). *If* $\tau^* > \sigma^*$, *then*

$$-K_t = \begin{cases} 0, & t \in [0, \sigma_0), \\ \sup_{s\in[\sigma_{k-1},t]}(-\Phi_s), & t \in [\sigma_{k-1}, \tau_k),\ k \geqslant 1, \\ \inf_{s\in[\tau_k,t]}(-\Psi_s), & t \in [\tau_k, \sigma_k),\ k \geqslant 1. \end{cases} \tag{2.21}$$

If $\tau^* < \sigma^*$, *then*

$$-K_t = \begin{cases} 0, & t \in [0, \tau_0), \\ \inf_{s\in[\tau_k,t]}(-\Psi_s), & t \in [\tau_k, \sigma_k),\ k \geqslant 0, \\ \sup_{s\in[\sigma_{k-1},t]}(-\Phi_s), & t \in [\sigma_{k-1}, \tau_k),\ k \geqslant 1. \end{cases} \tag{2.22}$$

REMARK 2.9. It is easy to check that, for any $k \geqslant 0$, $K_{\sigma_k} = \Phi_{\sigma_k}$ and if $k \geqslant 1$ or $\tau^* < \sigma^*$ and $k = 0$, then $K_{\tau_k} = \Psi_{\tau_k}$.

Now, we show that K can be represented as the difference of two nondecreasing functions which only increase when $R(\cdot, X_\cdot)$, $L(\cdot, X_\cdot)$ hit 0.

THEOREM 2.3. *Suppose that Assumption* 2.1 *holds. Let* $X = S + K$. *Then the following hold:*

(1) $K \in BV[0, \infty)$;

(2) $L(t, X_t) \leqslant 0 \leqslant R(t, X_t)$ *for any* $t \geqslant 0$;

(3) $|K|_t = \int_0^t \mathbf{1}_{\{L(s,X_s)=0 \text{ or } R(s,X_s)=0\}}\, d|K|_s$;

(4) $K_t = \int_0^t \mathbf{1}_{\{R(s,X_s)=0\}}\, d|K|_s - \int_0^t \mathbf{1}_{\{L(s,X_s)=0\}}\, d|K|_s$.

Proof. (1) is a direct consequence of Theorem 2.2. Formula (2.4) can be written as

$$K_t = \min\Big([-(\Phi_0)^-] \vee \sup_{r\in[0,t]} \Psi_r,\ \inf_{s\in[0,t]}\Big[\Phi_s \vee \sup_{r\in[s,t]} \Psi_r\Big]\Big).$$

Recalling that $\Phi \geqslant \Psi$, it follows that

$$\inf_{s\in[0,t]}\Big[\Phi_s \vee \sup_{r\in[s,t]} \Psi_r\Big] \leqslant \Phi_t \vee \Psi_t = \Phi_t,$$

which implies $K_t \leqslant \Phi_t$. On the other hand, it is easy to check that

$$\begin{aligned} &[-(\Phi_0)^-] \vee \sup_{r\in[0,t]} \Psi_r \geqslant \Psi_t, \\ &\Phi_s \vee \sup_{r\in[s,t]} \Psi_r \geqslant \Psi_t \quad \text{for any } s \in [0, t]. \end{aligned}$$

Consequently, $K_t \geqslant \Psi_t$. Therefore, by the definition for Φ, Ψ, we have

$$L(t, X_t) = L(t, S_t + K_t) \leqslant L(t, S_t + \Phi_t) = 0,$$
$$R(t, X_t) = R(t, S_t + K_t) \geqslant R(t, S_t + \Psi_t) = 0.$$

Motivated by the proof of [9, Theorem 3.4], we only prove (3), (4) for the case $\tau^* > \sigma^*$ since the case $\tau^* < \sigma^*$ can be proved similarly. Since $K_t = 0$ when $t \in [0, \sigma_0)$, we focus on $t \geqslant \sigma_0$. We claim that for $t \geqslant \sigma_0$,

(2.23) $\quad R(t, X_t) = 0$ implies that $t \in [\tau_k, \sigma_k)$ for some $k \geqslant 1$,

(2.24) $\quad L(t, X_t) = 0$ implies that $t \in [\sigma_{k-1}, \tau_k)$ for some $k \geqslant 1$.

We first prove (2.23). Suppose that $t \in [\sigma_{k-1}, \tau_k)$ for some $k \geqslant 1$. Recalling (2.10) and (2.21), we have

$$-\Psi_t > \sup_{s \in [\sigma_{k-1}, t]} (-\Phi_s) = -K_t.$$

It follows that

$$R(t, X_t) > R(t, S_t + \Psi_t) = 0,$$

which implies that (2.23) holds.

Now, suppose that $t \in [\tau_k, \sigma_k)$ for some $k \geqslant 1$. By (2.12) and (2.21), we obtain

$$-\Phi_t < \inf_{s \in [\tau_k, t]} (-\Psi_s) = -K_t.$$

Consequently,

$$L(t, X_t) < L(t, S_t + \Phi_t) = 0.$$

Therefore, (2.24) holds. Thus, if assertion (3) is true, by (2.21), (2.23) and (2.24), we deduce that (4) is true.

Now, it remains to prove (3). Set

$$A := \{t \geqslant \sigma_0 : L(t, X_t) < 0 < R(t, X_t)\}.$$

It suffices to prove that $\int_A d|K|_t = 0$. For $t \in A$, we define

$$l_t = L(t, X_t), \qquad r_t = R(t, X_t),$$
$$\alpha_t = \inf\{s \in [\sigma_0, t] : (s, t] \subset A\},$$
$$\beta_t = \sup\{s \in [t, \infty) : [t, s) \subset A\}.$$

By the right-continuity of l and r, we have $\beta_t \notin A$, while α_t may or may not belong to A. Moreover, we have $\alpha_t \leqslant t < \beta_t$, which implies that (α_t, β_t) is nonempty. The above analysis implies that A has the representation

$$A = \Big(\bigcup_{t \in I} (\alpha_t, \beta_t)\Big) \cup \{\alpha_t : t \in J\},$$

where I is a countable subset of $[0,\infty)$ and $J \subset I$.

We first show that $\int_{(\alpha_t,\beta_t)} d|K|_s = 0$ for any $t \in I$. Note that for any $s \in (\alpha_t,\beta_t)$, we have $l_s < 0 < r_s$. Recalling the definition of Φ, Ψ and Remark 2.9, for any $k \geqslant 1$ we have

$$r_{\tau_k} = R(\tau_k, S_{\tau_k} + \Psi_{\tau_k}) = 0, \quad l_{\sigma_{k-1}} = L(\sigma_{k-1}, S_{\sigma_{k-1}} + \Phi_{\sigma_{k-1}}) = 0. \tag{2.25}$$

Therefore, there are only two possibilities: either $(\alpha_t,\beta_t) \subset (\tau_k,\sigma_k)$ or $(\alpha_t,\beta_t) \subset (\sigma_{k-1},\tau_k)$ for some $k \geqslant 1$. We only consider the second case as the first case is analogous. It is enough to show that K is a constant on $[a_t,b_t]$ for any $[a_t,b_t] \subset (\alpha_t,\beta_t)$. Recall that when $t \in [\sigma_{k-1},\tau_k)$, we have $K_t = \inf_{s\in[\sigma_{k-1},t]} \Phi_s$. Set

$$\rho = \inf\{s \in [a_t,b_t] : K_s < K_{a_t}\}.$$

Suppose that $\rho < \infty$. The right-continuity of K yields $K_s = K_{a_t}$ for any $s \in [a_t,\rho)$ and either $K_\rho = \Phi_\rho < K_{a_t}$ or $K_\rho = \Phi_\rho = K_{a_t}$. In either case,

$$l_\rho = L(\rho, X_\rho) = L(\rho, S_\rho + \Phi_\rho) = 0,$$

which contradicts $\rho \in A$. Hence, $\rho = \infty$ and K is a constant on $[a_t,b_t]$.

To complete the proof, it suffices to show that for any $\alpha_t \in A$ with $t \in J$, K is continuous at a_t. Recalling (2.25), there exists some $k \geqslant 1$ such that either $\alpha_t \subset (\tau_k,\sigma_k)$ or $\alpha_t \subset (\sigma_{k-1},\tau_k)$. By the definition of α_t, we may find a sequence $\{\gamma_n\}_{n=1}^\infty \subset (0,\alpha_t) \cap A^c$ such that $\gamma_n \uparrow \alpha_t$.

We first consider the case that $l_{\gamma_n} = 0$ or equivalently $K_{\gamma_n} = \Phi_{\gamma_n}$ for infinitely many values of n. Applying (2.24), we have $\gamma_n \in [\sigma_{k-1},\tau_k)$ for some $k \geqslant 1$. There exists some k^* independent of n such that for n large enough, $\gamma_n \in [\sigma_{k^*-1},\tau_{k^*})$ and $\alpha_t \in (\sigma_{k^*-1},\tau_{k^*})$. Therefore,

$$\Phi_{\gamma_n} = K_{\gamma_n} = \inf_{s\in[\sigma_{k^*-1},\gamma_n]} \Phi_s.$$

Letting $n \to \infty$ implies that

$$\Phi_{\alpha_t-} = K_{\alpha_t-} = \inf_{s\in[\sigma_{k^*-1},\alpha_t)} \Phi_s.$$

Since $\alpha_t \in [\sigma_{k^*-1},\tau_{k^*})$, we have $K_{\alpha_t} = \inf_{s\in[\sigma_{k^*-1},\alpha_t]} \Phi_s$, which yields $K_{\alpha_t} \leqslant K_{\alpha_t-}$. Suppose that $K_{\alpha_t} < K_{\alpha_t-}$, which implies $K_{\alpha_t} = \Phi_{\alpha_t}$. This leads to

$$l_{\alpha_t} = L(\alpha_t, X_{\alpha_t}) = L(\alpha_t, S_{\alpha_t} + \Phi_{\alpha_t}) = 0,$$

which contradicts $\alpha_t \in A$. Therefore, $K_{\alpha_t} = K_{\alpha_t-}$, that is, K is continuous at α_t.

For the case when $l_{\gamma_n} = 0$ does not hold for infinitely many values of n, $r_{\gamma_n} = 0$ must hold for infinitely many values of n. By a similar analysis, we can also show that K is continuous at α_t. The proof is complete. ■

Proof of Theorem 2.1. The uniqueness of solution is a direct consequence of Proposition 2.1. Let K be defined as in (2.4) and set $X_t = S_t + K_t$ and

$$K_t^r = \int_0^t \mathbf{1}_{\{R(s,X_s)=0\}}\, d|K|_s, \qquad K_t^l = \int_0^t \mathbf{1}_{\{L(s,X_s)=0\}}\, d|K|_s.$$

Clearly, K^r, K^l are nondecreasing functions. By Theorem 2.3, we have $L(t, X_t) \leqslant 0 \leqslant R(t, X_t)$, $K_t = K_t^r - K_t^l$ for any $t \geqslant 0$, and

$$\int_0^\infty \mathbf{1}_{\{L(s,X_s)<0\}}\, dK_s^l = 0, \qquad \int_0^\infty \mathbf{1}_{\{R(s,X_s)>0\}}\, dK_s^r = 0.$$

That is, (X, K) solves $\mathbb{SP}_L^R(S)$. ∎

REMARK 2.10. Let (X, K) solve $\mathbb{SP}_L^R(S)$. For any $t > 0$, it is easy to check that

$$K_t - K_{t-} = X_t - X_{t-} - (S_t - S_{t-}).$$

By the proof of Theorem 2.1, K^r, K^l do not increase simultaneously. If $K_t - K_{t-} > 0$, we have

$$K_t^r - K_{t-}^r = K_t - K_{t-} = X_t - X_{t-} - (S_t - S_{t-}) \quad \text{and} \quad K_t^l - K_{t-}^l = 0.$$

Similarly, if $K_t - K_{t-} < 0$, we have

$$K_t^l - K_{t-}^l = -(K_t - K_{t-}) = -(X_t - X_{t-} - (S_t - S_{t-})) \quad \text{and} \quad K_t^r - K_{t-}^r = 0.$$

Therefore,

$$\begin{aligned} K_t^r - K_{t-}^r &= (X_t - X_{t-} - (S_t - S_{t-}))^+, \\ K_t^l - K_{t-}^l &= (X_t - X_{t-} - (S_t - S_{t-}))^-. \end{aligned}$$

REMARK 2.11. Suppose that $L \equiv -\infty$. Then the Skorokhod problem with two nonlinear reflecting boundaries turns into the Skorokhod problem with one constraint. More precisely, given $S \in D[0, \infty)$, we need to find $(X, K) \in D[0, \infty) \times I[0, \infty)$ such that

(i) $X_t = S_t + K_t$;

(ii) $R(t, X_t) \geqslant 0$;

(iii) $K_{0-} = 0$ and K is a nondecreasing function satisfying

$$\int_0^\infty \mathbf{1}_{\{R(s,X_s)>0\}}\, dK_s = 0.$$

For simplicity, we then write that (X,K) solves $\mathbb{SP}^R(S)$.

Since $L \equiv -\infty$, Φ may be interpreted as $+\infty$. Recalling (2.4), we have

$$K_t = \sup_{s\in[0,t]} \Psi_s^+.$$

In particular, if $R(t,x) = x$, the Skorokhod problem with nonlinear constraint degenerates to the classical Skorokhod problem. In this case, we have $\Psi_t = -S_t$. Consequently, $K_t = \sup_{s\in[0,T]} S_t^-$, which coincides with the result in [11].

REMARK 2.12. Suppose that (X,K) solves $\mathbb{SP}_L^R(S)$ and K admits the decomposition $K = K^r - K^l$. Then (X,K^r) may be interpreted as the solution to the Skorokhod problem with nonlinear constraint $\mathbb{SP}^R(S-K^l)$. For any $t \geqslant 0$, let Ψ_t^r be the solution

$$R(t, S_t - K_t^l + \Psi_t^r) = 0.$$

Then $K_t^r = \sup_{s\in[0,t]}(\Psi_s^r)^+$. On the other hand, $(-X, K^l)$ may be regarded as the solution to $\mathbb{SP}^{\tilde{L}}(-S-K^r)$, where $\tilde{L}(t,x) := -L(t,-x)$. For any $t \geqslant 0$, let Φ_t^l be the solution to

$$\tilde{L}(t, -S_t - K_t^r + \Phi_t^l) = -L(t, S_t + K_t^l - \Phi_t^l) = 0.$$

Then $K_t^l = \sup_{s\in[0,t]}(\Phi_s^l)^+$.

REMARK 2.13. It is worth pointing out that (iv) in Assumption 2.1 is necessary for K to be of bounded variation. The readers may refer to [12, Example 2.1] for a counterexample.

3. PROPERTIES OF SOLUTIONS TO SKOROKHOD PROBLEMS WITH TWO NONLINEAR REFLECTING BOUNDARIES

3.1. Nonanticipatory properties. In this subsection, suppose (X,K) is the solution to a Skorokhod problem with two nonlinear reflecting boundaries. We investigate if the pair of shifted functions is still the solution to some other Skorokhod problem. For this purpose, for any fixed $d \geqslant 0$, we define two operators $T_d, H_d : D[0,\infty) \to D[0,\infty)$ as follows:

$$(T_d(\psi))_t = \psi_{d+t} - \psi_d, \qquad (H_d(\psi))_t = \psi_{d+t}, \qquad t \geqslant 0. \tag{3.1}$$

Moreover, we define two functions $L^d, R^d : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$L^d(t,x) = L(t+d,x), \qquad R^d(t,x) = R(t+d,x).$$

THEOREM 3.1. *Under Assumption* 2.1*, for a given* $S \in D[0,\infty)$*, if* (X,K) *solves* $\mathbb{SP}_L^R(S)$*, then* $(H_d(X), T_d(K))$ *solves* $\mathbb{SP}_{L^d}^{R^d}(T_d(S)+X_d)$.

Proof. Clearly, if L, R satisfy Assumption 2.1, so do L^d, R^d. For any $t \geqslant 0$, it is easy to check that

$$\begin{aligned}(H_d(X))_t = X_{d+t} = S_{d+t} + K_{d+t} &= (T_d(S))_t + (T_d(K))_t + (S_d + K_d) \\ &= (T_d(S))_t + X_d + (T_d(K))_t.\end{aligned}$$

Moreover, we have

$$L^d(t, (H_d(X))_t) = L(t+d, X_{t+d}) \leqslant 0 \leqslant R(t+d, X_{t+d}) = R^d(t, (H_d(X))_t)$$

and

$$\begin{aligned}\int_0^\infty \mathbf{1}_{\{L^d(s,(H_d(X))_s)<0\}}\, d(T_d(K^l))_s &= \int_d^\infty \mathbf{1}_{\{L(s,X_s)<0\}}\, dK_s^l = 0, \\ \int_0^\infty \mathbf{1}_{\{R^d(s,(H_d(X))_s)>0\}}\, d(T_d(K^r))_s &= \int_d^\infty \mathbf{1}_{\{R(s,X_s)>0\}}\, dK_s^r = 0.\end{aligned}$$

The proof is complete. ∎

REMARK 3.1. Theorem 3.1 is an extension of [12, Theorem 3.1] to the nonlinear reflecting case.

3.2. Comparison properties. In this subsection, we present some comparison properties of Skorokhod problems with two nonlinear reflecting boundaries. In the proofs, the following inequalities are frequently used:

$$(a+b)^\pm \leqslant a^\pm + b^\pm, \quad (a-b)^\pm \geqslant a^\pm - b^\pm, \quad \text{for any } a, b \in \mathbb{R}.$$

Before investigating the comparison property for the doubly reflected problem, we first establish the comparison property for the singly reflected case, which may be of independent interest.

PROPOSITION 3.1. *Let Assumption* 2.1(i)–(iii) *hold for* R. *Given* $c_0^i \in \mathbb{R}$ *and* $S^i \in D[0,\infty)$ *for* $i = 1, 2$ *with* $S_0^1 = S_0^2 = 0$, *suppose that there exists a nonnegative* $\nu \in I[0,\infty)$ *such that* $S^2 \leqslant S^1 \leqslant S^2 + \nu$. *Let* (X^i, K^i) *solve* $\mathbb{SP}^R(c_0^i + S^i)$ *for* $i = 1, 2$. *Then*

(1) $K_t^1 - (c_0^2 - c_0^1)^+ \leqslant K_t^2 \leqslant K_t^1 + \nu_t + (c_0^1 - c_0^2)^+$;

(2) $X_t^2 - \nu_t - (c_0^2 - c_0^1)^+ \leqslant X_t^1 \leqslant X_t^2 + \nu_t + (c_0^1 - c_0^2)^+$.

Proof. Let Ψ_s^i be such that $R(s, c_0^i + S_s^i + \Psi_s^i) = 0$ for $i = 1, 2$ and $s \geqslant 0$. Recalling Remark 2.11, we have $K_t^i = \sup_{s\in[0,t]}(\Psi_s^i)^+$. Noting that $S^2 \leqslant S^1$ and

$$R(s, c_0^1 + S_s^1 + \Psi_s^1) = 0 = R(s, c_0^2 + S_s^2 + \Psi_s^2),$$

we have

$$\Psi_s^2 \geqslant \Psi_s^1 + c_0^1 - c_0^2. \tag{3.2}$$

Since $S^1 \leqslant S^2 + \nu$, it follows that

$$R(s, c_0^2 + S_s^2 + \Psi_s^2) = 0 \leqslant R(s, c_0^2 + S_s^2 + \nu_s + \Psi_s^1 + c_0^1 - c_0^2).$$

Consequently,

$$\Psi_s^2 \leqslant \Psi_s^1 + \nu_s + c_0^1 - c_0^2. \tag{3.3}$$

Recalling that $\nu \in I[0, \infty)$ is nonnegative, by (3.3) we obtain

$$\begin{aligned}
K_t^2 &= \sup_{s\in[0,t]} (\Psi_s^2)^+ \leqslant \sup_{s\in[0,t]} (\Psi_s^1 + \nu_s + c_0^1 - c_0^2)^+ \\
&\leqslant \sup_{s\in[0,t]} (\Psi_s^1 + \nu_t + c_0^1 - c_0^2)^+ \\
&\leqslant \sup_{s\in[0,t]} (\Psi_s^1)^+ + \nu_t + (c_0^1 - c_0^2)^+ \\
&= K_t^1 + \nu_t + (c_0^1 - c_0^2)^+.
\end{aligned}$$

Applying (3.2) yields

$$\begin{aligned}
K_t^1 &= \sup_{s\in[0,t]} (\Psi_s^1)^+ \leqslant \sup_{s\in[0,t]} (\Psi_s^2 + c_0^2 - c_0^1)^+ \\
&\leqslant \sup_{s\in[0,t]} (\Psi_s^2)^+ + (c_0^2 - c_0^1)^+ = K_t^2 + (c_0^2 - c_0^1)^+.
\end{aligned}$$

We have obtained property (1).

Based on (1), together with the facts that $K^2 = X^2 - c_0^2 - S^2$ and $S^1 \geqslant S^2$, it is easy to check that

$$\begin{aligned}
X^1 &= c_0^1 + S^1 + K^1 \geqslant c_0^1 + S^1 + K^2 - \nu - (c_0^1 - c_0^2)^+ \\
&= c_0^1 - c_0^2 + S^1 - S^2 + X^2 - \nu - (c_0^1 - c_0^2)^+ \\
&\geqslant X^2 - \nu - (c_0^2 - c_0^1)^+.
\end{aligned}$$

On the other hand, since $S^1 \leqslant S^2 + \nu$, we obtain

$$\begin{aligned}
X^1 &= c_0^1 + S^1 + K^1 \leqslant c_0^1 + S^1 + K^2 + (c_0^2 - c_0^1)^+ \\
&= c_0^1 - c_0^2 + S^1 - S^2 + X^2 + (c_0^2 - c_0^1)^+ \\
&\leqslant X^2 + \nu + (c_0^1 - c_0^2)^+.
\end{aligned}$$

The proof is complete. ∎

Now we establish the comparison property for the double nonlinear reflected problem.

PROPOSITION 3.2. *Let Assumption* 2.1 *hold. Given* $c_0^i \in \mathbb{R}$ *and* $S^i \in D[0,\infty)$ *for* $i = 1,2$ *with* $S_0^1 = S_0^2 = 0$*, suppose that there exists a nonnegative* $\nu \in I[0,\infty)$ *such that* $S^2 \leqslant S^1 \leqslant S^2 + \nu$*. Let* (X^i, K^i) *solve* $\mathbb{SP}_L^R(c_0^i + S^i)$ *for* $i = 1,2$*. Then*

(1) $K_t^1 - (c_0^2 - c_0^1)^+ \leqslant K_t^2 \leqslant K_t^1 + \nu_t + (c_0^1 - c_0^2)^+$;

(2) $X_t^2 - \nu_t - (c_0^2 - c_0^1)^+ \leqslant X_t^1 \leqslant X_t^2 + \nu_t + (c_0^1 - c_0^2)^+$.

Proof. The analysis in the proof of Lemma 3.1 implies that

$$\begin{aligned}\Psi_s^1 + c_0^1 - c_0^2 \leqslant \Psi_s^2 \leqslant \Psi_s^1 + \nu_s + c_0^1 - c_0^2,\\ \Phi_s^1 + c_0^1 - c_0^2 \leqslant \Phi_s^2 \leqslant \Phi_s^1 + \nu_s + c_0^1 - c_0^2.\end{aligned}$$

By Theorem 2.1, we have

$$K_t^i = \min\Big((-(\Phi_0^i)^-) \vee \sup_{r\in[0,t]} \Psi_r^i, \inf_{s\in[0,t]}\Big[\Phi_s^i \vee \sup_{r\in[s,t]} \Psi_r^i\Big]\Big).$$

For any $0 \leqslant s \leqslant t$, it is easy to check that

$$\begin{aligned}\sup_{r\in[s,t]} \Psi_r^1 + c_0^1 - c_0^2 \leqslant \sup_{r\in[s,t]} \Psi_r^2 &\leqslant \sup_{r\in[s,t]} (\Psi_r^1 + \nu_r + c_0^1 - c_0^2)\\ &\leqslant \sup_{r\in[s,t]} \Psi_r^1 + \nu_t + c_0^1 - c_0^2,\\ \Phi_s^1 + c_0^1 - c_0^2 \leqslant \Phi_s^2 \leqslant \Phi_s^1 + \nu_s &+ c_0^1 - c_0^2 \leqslant \Phi_s^1 + \nu_t + c_0^1 - c_0^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}-(\Phi_0^2)^- &\leqslant -(\Phi_0^1 + \nu_t + c_0^1 - c_0^2)^- \leqslant -(\Phi_0^1)^- + \nu_t + (c_0^1 - c_0^2)^+,\\ -(\Phi_0^2)^- &\geqslant -(\Phi_0^1 + c_0^1 - c_0^2)^- \geqslant -(\Phi_0^1)^- - (c_0^1 - c_0^2)^-.\end{aligned}$$

All the above inequalities indicate that

$$K_t^1 - (c_0^1 - c_0^2)^- \leqslant K_t^2 \leqslant K_t^1 + \nu_t + (c_0^1 - c_0^2)^+.$$

Consequently,

$$\begin{aligned}X_t^2 - X_t^1 &= S_t^2 - S_t^1 + c_0^2 - c_0^1 + K_t^2 - K_t^1\\ &\leqslant c_0^2 - c_0^1 + \nu_t + (c_0^1 - c_0^2)^+ = \nu_t + (c_0^1 - c_0^2)^-\end{aligned}$$

and

$$\begin{aligned}X_t^2 - X_t^1 &= S_t^2 - S_t^1 + c_0^2 - c_0^1 + K_t^2 - K_t^1\\ &\geqslant -\nu_t + c_0^2 - c_0^1 - (c_0^1 - c_0^2)^- = -\nu_t - (c_0^2 - c_0^1)^-.\end{aligned}$$

The proof is complete. ■

REMARK 3.2. Propositions 3.1 and 3.2 extend [9, Lemma 4.1 and Corollary 4.2] to the nonlinear reflecting case. Proposition 3.2 is also an extension of [1, Proposition 3.4]. However, in Proposition 3.2, we do not need to assume that $S^1 = S^2 + \nu$, while this condition appears in [9, Corollary 4.2] and [1, Proposition 3.4]. Moreover, suppose (X^i, K^i) solves the Skorokhod problem on $[0, a]$ for $c_0^i + S^i$, $i = 1, 2$, with fixed $a > 0$ (i.e., (X^i, K^i) solves $\mathbb{SP}_L^R(c_0^i + S^i)$ with $L(t, x) = x - a$ and $R(t, x) = x$). Corollary 4.2 in [9] shows that

$$K_t^1 - 2(c_0^2 - c_0^1)^+ \leqslant K_t^2 \leqslant K_t^1 + 2\nu_t + 2(c_0^1 - c_0^2)^+.$$

Compared with this result, our estimate in Proposition 3.2 is more accurate. Actually, Remark 4.3 in [9] finally provides the strengthened inequality

$$K_t^1 - (c_0^2 - c_0^1)^+ \leqslant K_t^2 \leqslant K_t^1 + \nu_t + (c_0^1 - c_0^2)^+.$$

It is worth pointing out that the proof needs the nonanticipatory properties. Therefore, our proof is simpler.

Proposition 3.2 only provides the comparison between the net constraining terms K^1 and K^2. A natural question is whether we could compare the individual constraining terms. The answer is affirmative; the following theorem generalizes [9, Theorem 1.7] and [1, Proposition 3.5].

THEOREM 3.2. *Let Assumption* 2.1 *hold. Given* $c_0^i \in \mathbb{R}$ *and* $S^i \in D[0, \infty)$ *for* $i = 1, 2$ *with* $S_0^1 = S_0^2 = 0$*, suppose that there exists* $\nu \in I[0, \infty)$ *such that* $S^1 = S^2 + \nu$*. Let* (X^i, K^i) *solve* $\mathbb{SP}_L^R(c_0^i + S^i)$ *with decomposition* $K^i = K^{i,r} - K^{i,l}$ *for* $i = 1, 2$*. Then*

(1) $K_t^{1,r} - (c_0^2 - c_0^1)^+ \leqslant K_t^{2,r} \leqslant K_t^{1,r} + \nu_t + (c_0^1 - c_0^2)^+$*;*

(2) $K_t^{2,l} - (c_0^2 - c_0^1)^+ \leqslant K_t^{1,l} \leqslant K_t^{2,l} + \nu_t + (c_0^1 - c_0^2)^+$.

Proof. Define

$$\alpha = \inf\{t > 0 : K_t^{1,r} + \nu_t + (c_0^1 - c_0^2)^+ < K_t^{2,r} \text{ or } K_t^{1,l} + (c_0^2 - c_0^1)^+ < K_t^{2,l}\}.$$

We claim that $\alpha = \infty$. Then, for any $t \geqslant 0$, we have

$$K_t^{1,r} + \nu_t + (c_0^1 - c_0^2)^+ \geqslant K_t^{2,r} \quad \text{and} \quad K_t^{1,l} + (c_0^2 - c_0^1)^+ \geqslant K_t^{2,l},$$

which are the second inequality in (1) and the first inequality in (2). We will argue by way of contradiction. Suppose that $\alpha < \infty$. The proof will be in two steps.

STEP 1. We claim that

$$K_\alpha^{2,r} \leqslant K_\alpha^{1,r} + \nu_\alpha + (c_0^1 - c_0^2)^+, \tag{3.4}$$

$$K_\alpha^{2,l} \leqslant K_\alpha^{1,l} + (c_0^2 - c_0^1)^+. \tag{3.5}$$

First, by the definition of α, for any $s \in [0, \alpha)$ we have

$$K_s^{2,r} \leqslant K_s^{1,r} + \nu_s + (c_0^1 - c_0^2)^+, \tag{3.6}$$

$$K_s^{2,l} \leqslant K_s^{1,l} + (c_0^2 - c_0^1)^+. \tag{3.7}$$

Noting that $\nu, K^{1,r}, K^{1,l}$ are nondecreasing, if $K^{2,r}, K^{2,l}$ are continuous at α, (3.4) and (3.5) hold true by using (3.6) and (3.7), respectively.

Now, suppose that $K_\alpha^{2,r} > K_{\alpha-}^{2,r}$. Then $K_\alpha^{2,l} = K_{\alpha-}^{2,l}$ and

$$R(\alpha, X_\alpha^2) = R(\alpha, c_0^2 + S_\alpha^2 + K_\alpha^{2,r} - K_\alpha^{2,l}) = 0. \tag{3.8}$$

On the other hand, since (X^1, K^1) solves $\mathbb{SP}_L^R(c_0^1 + S^1)$, we have

$$R(\alpha, c_0^1 + S_\alpha^1 + K_\alpha^{1,r} - K_\alpha^{1,l}) \geqslant 0.$$

Combining the above inequality and (3.8) implies that

$$\begin{aligned} K_\alpha^{2,r} &\leqslant c_0^1 - c_0^2 + S_\alpha^1 - S_\alpha^2 + K_\alpha^{2,l} - K_\alpha^{1,l} + K_\alpha^{1,r} \\ &\leqslant c_0^1 - c_0^2 + \nu_\alpha + K_{\alpha-}^{2,l} - K_{\alpha-}^{1,l} + K_\alpha^{1,r}, \end{aligned} \tag{3.9}$$

where we have used the facts that $K_\alpha^{2,l} = K_{\alpha-}^{2,l}$, $S^1 = S^2 + \nu$ and that $K^{1,l}$ is nondecreasing. Letting $s \uparrow \alpha$ in (3.7) yields

$$K_{\alpha-}^{2,l} - K_{\alpha-}^{1,l} \leqslant (c_0^2 - c_0^1)^+.$$

Plugging this inequality into (3.9), we obtain

$$\begin{aligned} K_\alpha^{2,r} &\leqslant c_0^1 - c_0^2 + \nu_\alpha + (c_0^2 - c_0^1)^+ + K_\alpha^{1,r} \\ &= K_\alpha^{1,r} + \nu_\alpha + (c_0^1 - c_0^2)^+, \end{aligned}$$

which is indeed (3.4). The proof of (3.5) under the assumption that $K_\alpha^{2,l} > K_{\alpha-}^{2,l}$ is similar and therefore omitted.

STEP 2. By the definition of α and recalling (3.4) and (3.5), there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ converging to 0 decreasingly such that for any $n \in \mathbb{N}$, one of the following two cases holds:

$$K_{\alpha+s_n}^{2,r} > K_{\alpha+s_n}^{1,r} + \nu_{\alpha+s_n} + (c_0^1 - c_0^2)^+, \tag{3.10}$$

or

$$K_{\alpha+s_n}^{2,l} > K_{\alpha+s_n}^{1,l} + (c_0^2 - c_0^1)^+. \tag{3.11}$$

We claim that in fact neither (3.10) nor (3.11) can hold.

Suppose that (3.10) holds. Letting $n \to \infty$, the right-continuity of $K^{2,r}, K^{1,r}$ and ν implies that

$$K^{2,r}_\alpha \geqslant K^{1,r}_\alpha + \nu_\alpha + (c^1_0 - c^2_0)^+.$$

The above inequality together with (3.4) yields

$$K^{2,r}_\alpha = K^{1,r}_\alpha + \nu_\alpha + (c^1_0 - c^2_0)^+. \tag{3.12}$$

We first claim that $R(\alpha, X^1_\alpha) = R(\alpha, X^2_\alpha) = 0$. In fact, since $K^{1,r} + \nu$ is non-decreasing, it follows from (3.10) and (3.12) that $K^{2,r}_{\alpha+s_n} > K^{2,r}_\alpha$ for any $n \in \mathbb{N}$. Since $s_n \downarrow 0$, we have $R(\alpha, X^2_\alpha) = 0$. Applying (3.5) and (3.12), it is easy to check that

$$\begin{aligned}
0 &\leqslant R(\alpha, X^1_\alpha) = R(\alpha, c^1_0 + S^1_\alpha + K^{1,r}_\alpha - K^{1,l}_\alpha) \\
&\leqslant R\big(\alpha, c^1_0 + S^2_\alpha + \nu_\alpha + K^{2,r}_\alpha - \nu_\alpha - (c^1_0 - c^2_0)^+ - K^{2,l}_\alpha + (c^2_0 - c^1_0)^+\big) \\
&= R(\alpha, X^2_\alpha) = 0.
\end{aligned}$$

Hence, the claim holds true, which implies that $L(\alpha, X^1_\alpha) = L(\alpha, X^2_\alpha) < 0$ and $X^1_\alpha = X^2_\alpha$. Since X^i, $i = 1, 2$, are right-continuous, there exists some $\varepsilon > 0$ such that for any $s \in [0, \varepsilon]$, $L(\alpha + \varepsilon, X^1_{\alpha+\varepsilon}) < 0$ and $L(\alpha + \varepsilon, X^2_{\alpha+\varepsilon}) < 0$. Thus, for any $s \in [0, \varepsilon]$ we have $K^{i,l}_{\alpha+s} = K^{i,l}_\alpha$ for $i = 1, 2$. Therefore, $(X^i_{\alpha+s}, K^{i,r}_{\alpha+s} - K^{i,r}_\alpha)_{s\in[0,\varepsilon]}$ can be seen as the solution to the Skorokhod problem with one constraint, $\mathbb{SP}^{R^\alpha}(S^{i,\alpha})$, on the time interval $[0, \varepsilon]$, where

$$R^\alpha(t, x) := R(\alpha + t, x), \quad S^{i,\alpha}_t := X^i_\alpha + S^i_{\alpha+t} - S^i_\alpha.$$

Applying Proposition 3.1(1) and noting that $X^1_\alpha = X^2_\alpha$, for any $s \in [0, \varepsilon]$ we have

$$K^{2,r}_{\alpha+s} - K^{2,r}_\alpha \leqslant K^{1,r}_{\alpha+s} - K^{1,r}_\alpha + \nu_{\alpha+s} - \nu_\alpha.$$

Plugging (3.12) into this inequality implies that

$$K^{2,r}_{\alpha+s} \leqslant K^{1,r}_{\alpha+s} + \nu_{\alpha+s} + (c^1_0 - c^2_0)^+,$$

which contradicts (3.10). Thus, (3.10) cannot hold.

It remains to show (3.11) does not hold. By the above analysis, together with (3.6) and (3.4), there exists some $\delta > 0$ such that for any $s \in [0, \alpha + \delta]$,

$$K^{2,r}_s \leqslant K^{1,r}_s + \nu_s + (c^1_0 - c^2_0)^+. \tag{3.13}$$

Recalling Remark 2.12, $(-X^i, K^{i,l})$ may be interpreted as the solution to the Skorokhod problem with one constraint $\mathbb{SP}^{\tilde{L}}(-c^i_0 - S^i - K^{i,r})$, $i = 1, 2$, where $\tilde{L}(t, x) = -L(t, -x)$. For any $s \geqslant 0$, let $\Phi^{i,l}_s$ be the solution to

$$\tilde{L}(s, -c^i_0 - S^i_s - K^{i,r}_s + \Phi^{i,l}_s) = 0.$$

Then, for any $t \in [0, \alpha+\delta]$, we have $c_0^1+S_t^1+K_t^{1,r}-\Phi_t^{1,l} = c_0^2+S_t^2+K_t^{2,r}-\Phi_t^{2,l}$ and

$$\begin{aligned}
K_t^{2,l} &= \sup_{s\in[0,t]} (\Phi_s^{2,l})^+ = \sup_{s\in[0,t]} (c_0^2 + S_s^2 + K_s^{2,r} - c_0^1 - S_s^1 - K_s^{1,r} + \Phi_s^{1,l})^+ \\
&\leqslant \sup_{s\in[0,t]} (c_0^2 - c_0^1 - \nu_s + \nu_s + (c_0^1 - c_0^2)^+ + \Phi_s^{1,l})^+ \\
&\leqslant \sup_{s\in[0,t]} (\Phi_s^{1,l})^+ + (c_0^2 - c_0^1)^+ = K_t^{1,l} + (c_0^2 - c_0^1)^+,
\end{aligned}$$

where we have used (3.13) and the fact that $S^2 = S^1 + \nu$. However, the above inequality contradicts (3.11).

All the above analysis indicates that neither (3.10) nor (3.11) holds, which means that $\alpha = \infty$. That is, the second inequality in (1) and the first inequality in (2) are satisfied.

Now, set

$$\beta = \inf\{t > 0 : K_t^{2,l} + \nu_t + (c_0^1 - c_0^2)^+ < K_t^{1,l} \text{ or } K_t^{2,r} + (c_0^2 - c_0^1)^+ < K_t^{1,r}\}.$$

By a similar analysis, we may show that $\beta = \infty$. This implies that the first inequality in (1) and the second inequality in (2) are satisfied. The proof is complete. ■

REMARK 3.3. The solution to the Skorokhod problem $\mathbb{SP}_L^R(S)$ is a pair of functions (X, K), where K has the decomposition $K = K^r - K^l$. In fact, K^r can be regarded as the force aiming to push the solution upwards, while K^l represents the force aiming to pull the solution downwards. Proposition 2.1 only provides the uniqueness for the overall function K. Applying the comparison properties of Theorem 3.2, we could also obtain the uniqueness for the individual constraining functions K^r, K^l, i.e., the decomposition of K is unique.

All the above results in this subsection give the comparison properties of solutions to Skorokhod problems with respect to the input function S. In the following, we provide the monotonicity property of the individual constraining functions with respect to the nonlinear reflecting boundaries L, R.

LEMMA 3.1. *Suppose (L^i, R^i) satisfy Assumption 2.1 for $i = 1, 2$ with $R^1 \equiv R^2$ and $L^1 \leqslant L^2$. For any given $S \in D[0,\infty)$, let (X^i, K^i) be the solution to the Skorokhod problem $\mathbb{SP}_{L^i}^{R^i}(S)$ with $K^i = K^{i,r} - K^{i,l}$. Then, for any $t \geqslant 0$, we have*

$$K_t^{2,r} \geqslant K_t^{1,r} \quad \textit{and} \quad K_t^{2,l} \geqslant K_t^{1,l}.$$

Proof. For any $t \geqslant 0$, let Φ_t^i, Ψ_t^i, $i = 1, 2$, be the solutions to

$$L^i(t, S_t + \Phi_t^i) = 0, \qquad R^i(t, S_t + \Psi_t^i) = 0.$$

It is easy to check that $\Psi_t^1 = \Psi_t^2$ and $\Phi_t^1 \geqslant \Phi_t^2$. By Theorem 2.1, we have $X_t^i = S_t + K_t^i$, $i = 1, 2$, where

$$K_t^i := \min\Big((-(\Phi_0^i)^-) \vee \sup_{r\in[0,t]} \Psi_r^i, \inf_{s\in[0,t]} \Big[\Phi_s^i \vee \sup_{r\in[s,t]} \Psi_r^i\Big]\Big).$$

Thus, we obtain $K_t^1 \geqslant K_t^2$ and

$$X_t^1 \geqslant X_t^2. \tag{3.14}$$

Note that $K_t^{i,l} = -K_t^i + K_t^{i,r}$. It suffices to prove that for any $t \geqslant 0$,

$$K_t^{2,r} \geqslant K_t^{1,r}. \tag{3.15}$$

At time 0, if $R^1(0, S_0) = R^2(0, S_0) \geqslant 0$, then $K_0^{1,r} = K_0^{2,r} = 0$. If $R^1(0, S_0) = R^2(0, S_0) < 0$, then $K_0^{1,r} = K_0^{2,r} = \Psi_0^1 > 0$. Hence, (3.15) holds at the initial time.

Now, set

$$t^* = \inf\{s \geqslant 0 : K_s^{2,r} < K_s^{1,r}\}.$$

We claim that $t^* = \infty$, which will complete the proof.

Towards a contradiction, suppose that $t^* < \infty$. Then

$$K_{t^*-}^{2,r} \geqslant K_{t^*-}^{1,r}, \tag{3.16}$$

and for any $\varepsilon_0 > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that

$$K_{t^*+\varepsilon}^{2,r} < K_{t^*+\varepsilon}^{1,r}. \tag{3.17}$$

We first show that $K_{t^*}^{2,r} \geqslant K_{t^*}^{1,r}$. If $K_{t^*}^{1,r} - K_{t^*-}^{1,r} = 0$, it is clear that $K_{t^*}^{2,r} - K_{t^*-}^{2,r} \geqslant K_{t^*}^{1,r} - K_{t^*-}^{1,r}$ since $K^{2,r}$ is nondecreasing. If $K_{t^*}^{1,r} - K_{t^*-}^{1,r} > 0$, we have $R^1(t^*, X_{t^*}^1) = 0$. The facts that $R^1 \equiv R^2$ and that (X^2, K^2) solves $\mathbb{SP}_{L^2}^{R^2}(S)$ imply that

$$R^2(t^*, X_{t^*}^1) = 0 \leqslant R^2(t^*, X_{t^*}^2),$$

which together with (3.14) indicates that $X_{t^*}^2 = X_{t^*}^1$. Recalling Remark 2.10, we obtain

$$\begin{aligned} K_{t^*}^{2,r} - K_{t^*-}^{2,r} &= (X_{t^*}^2 - X_{t^*-}^2 - (S_{t^*} - S_{t^*-}))^+ \\ &\geqslant (X_{t^*}^1 - X_{t^*-}^1 - (S_{t^*} - S_{t^*-}))^+ \\ &= K_{t^*}^{1,r} - K_{t^*-}^{1,r}, \end{aligned}$$

where we have used (3.14) in the second inequality. Therefore, the inequality $K_{t^*}^{2,r} - K_{t^*-}^{2,r} \geqslant K_{t^*}^{1,r} - K_{t^*-}^{1,r}$ always holds true. Combining it with (3.16) yields $K_{t^*}^{2,r} \geqslant K_{t^*}^{1,r}$. Now, we consider the following two cases.

CASE 1: $K_{t^*}^{2,r} > K_{t^*}^{1,r}$. Due to the right-continuity of $K^{1,r}$ and $K^{2,r}$, for $\varepsilon > 0$ small enough we have $K_{t^*+\varepsilon}^{2,r} > K_{t^*+\varepsilon}^{1,r}$, which contradicts (3.17).

CASE 2: $K_{t^*}^{2,r} = K_{t^*}^{1,r}$. Noting that $K^{i,r}$ are nondecreasing, by (3.17), for any $\varepsilon_0 > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that $K_{t^*}^{1,r} < K_{t^*+\varepsilon}^{1,r}$. According to Definition 2.1, this implies that $R^1(t^*, X_{t^*}^1) = 0$. Recalling (3.14) and the fact that $R^1 \equiv R^2$, it follows that

$$0 = R^2(t^*, X_{t^*}^1) \geqslant R^2(t^*, X_{t^*}^2) \geqslant 0,$$

which indicates that $X_{t^*}^1 = X_{t^*}^2$ and $R^1(t^*, X_{t^*}^1) = R^2(t^*, X_{t^*}^2) = 0$. Consequently,

$$L^1(t^*, X_{t^*}^1) < 0 \quad \text{and} \quad L^2(t^*, X_{t^*}^2) < 0.$$

Due to the right-continuity of X^i, there exists some $\delta > 0$ small enough such that for any $t \in [t^*, t^* + \delta]$,

$$L^1(t, X_t^1) < 0, \qquad L^2(t, X_t^2) < 0.$$

Therefore, for any $t \in [t^*, t^* + \delta]$, we have $K_t^{i,l} = K_{t^*}^{i,l}$ and thus $K_t^i - K_{t^*}^i = K_t^{i,r} - K_{t^*}^{i,r}$, $i = 1, 2$. Then, similar to Theorem 3.1, we deduce that on the time interval $[0, \delta]$,

$$(H_{t^*}(X^i), T_{t^*}(K^{i,r})) \text{ solves } \mathbb{SP}^{R^{i,t^*}}(X_{t^*}^i + T_{t^*}(S)), \qquad i = 1, 2,$$

where $R^{i,t^*}(t, x) = R^i(t + t^*, x)$ and T_{t^*}, H_{t^*} are defined in (3.1). For any $s \in [0, \delta]$, let Ψ_s^{i,t^*}, $i = 1, 2$, be the solution to

$$R^{i,t^*}(s, X_{t^*}^i + (T_{t^*}(S))_s + \Psi_s^{i,t^*}) = 0.$$

Since $R^1 \equiv R^2$ and $X_{t^*}^1 = X_{t^*}^2$, it follows that $\Psi_s^{1,t^*} = \Psi_s^{2,t^*}$. By Remark 2.11,

$$\begin{aligned} K_{t^*+\delta}^{1,r} - K_{t^*}^{1,r} &= (T_{t^*}(K^{1,r}))_\delta \\ &= \sup_{s \in [0,\delta]} (\Psi_s^{1,t^*})^+ = \sup_{s \in [0,\delta]} (\Psi_s^{2,t^*})^+ \\ &= (T_{t^*}(K^{2,r}))_\delta = K_{t^*+\delta}^{2,r} - K_{t^*}^{2,r}. \end{aligned}$$

Since we are considering the case $K_{t^*}^{2,r} = K_{t^*}^{1,r}$, we deduce that $K_{t^*+\delta}^{2,r} = K_{t^*+\delta}^{1,r}$, which contradicts (3.17).

Therefore, all the above analysis indicates that $t^* = \infty$, and the desired result holds true. ∎

REMARK 3.4. Suppose $l, \tilde{l}, r, \tilde{r} \in D[0, \infty)$ with $l \equiv \tilde{l}$, $r \leqslant \tilde{r}$ and $\inf_{t \geqslant 0}(r_t - l_t) > 0$. Let $R^1(t, x) = x - \tilde{l}_t$, $R^2(t, x) = x - l_t$, $L^1(t, x) = x - \tilde{r}_t$ and $L^2(t, x) = x - r_t$. Clearly, R^i, L^i, $i = 1, 2$, satisfy the assumptions in Lemma 3.1. In this case, our result reduces to [1, Lemma 3.1].

By a similar analysis to the proof of Lemma 3.1, we obtain the following lemma.

LEMMA 3.2. *Suppose* (L^i, R^i) *satisfy Assumption* 2.1 *for* $i = 1, 2$ *with* $R^1 \geqslant R^2$ *and* $L^1 \equiv L^2$. *For any given* $S \in D[0, \infty)$, *let* (X^i, K^i) *solve the Skorokhod problem* $\mathbb{SP}_{L^i}^{R^i}(S)$ *with* $K^i = K^{i,r} - K^{i,l}$. *Then, for any* $t \geqslant 0$,

$$K_t^{2,r} \geqslant K_t^{1,r} \quad \text{and} \quad K_t^{2,l} \geqslant K_t^{1,l}.$$

PROPOSITION 3.3. *Suppose* (L^i, R^i) *satisfy Assumption* 2.1 *for* $i = 1, 2$ *with* $R^1 \geqslant R^2$ *and* $L^1 \leqslant L^2$. *For any given* $S \in D[0, \infty)$, *let* (X^i, K^i) *solve the Skorokhod problem* $\mathbb{SP}_{L^i}^{R^i}(S)$ *with* $K^i = K^{i,r} - K^{i,l}$. *Then, for any* $t \geqslant 0$,

$$K_t^{2,r} \geqslant K_t^{1,r} \quad \text{and} \quad K_t^{2,l} \geqslant K_t^{1,l}.$$

Proof. Let (X^*, K^*) solve $\mathbb{SP}_{L^1}^{R^2}(S)$ with $K^* = K^{*,r} - K^{*,l}$. By Lemma 3.1,

$$K_t^{2,r} \geqslant K_t^{*,r} \quad \text{and} \quad K_t^{2,l} \geqslant K_t^{*,l}.$$

By Lemma 3.2,

$$K_t^{*,r} \geqslant K_t^{1,r} \quad \text{and} \quad K_t^{*,l} \geqslant K_t^{1,l}.$$

The above inequalities yield the desired result. ∎

REMARK 3.5. Proposition 3.3 is an extension of [1, Proposition 3.3] to the case of two nonlinear reflecting boundaries.

3.3. Continuity properties. In this subsection, we discuss the continuity properties of Skorokhod problems with two nonlinear reflecting boundaries under the uniform metric and J_1 metric d_0. The proofs are based on the representation for K obtained in (2.4). Therefore, we first establish some estimates for Φ and Ψ.

LEMMA 3.3. *Suppose that* (L^i, R^i) *satisfy Assumption* 2.1 *for* $i = 1, 2$. *Given* $S^i \in D[0, \infty)$, $i = 1, 2$, *for any* $t \geqslant 0$, *let* Φ^i, Ψ^i, $i = 1, 2$, *be the solutions to*

$$L^i(t, S_t^i + \Phi_t^i) = 0, \qquad R^i(t, S_t^i + \Psi_t^i) = 0.$$

Then

$$|\Phi_t^1 - \Phi_t^2| \leqslant \frac{C}{c}|S_t - S_t'| + \frac{1}{c} \sup_{x \in \mathbb{R}} |L^1(t, x) - L^2(t, x)|,$$
$$|\Psi_t^1 - \Psi_t^2| \leqslant \frac{C}{c}|S_t - S_t'| + \frac{1}{c} \sup_{x \in \mathbb{R}} |R^1(t, x) - R^2(t, x)|.$$

Proof. It suffices to prove the first inequality. A simple calculation yields

$$\begin{aligned}
&c|\Phi_t^1-\Phi_t^2|\\
\leqslant&|L^1(t,S_t^1+\Phi_t^1)-L^1(t,S_t^1+\Phi_t^2)|=|L^2(t,S_t^2+\Phi_t^2)-L^1(t,S_t^1+\Phi_t^2)|\\
\leqslant&\ |L^2(t,S_t^2+\Phi_t^2)-L^2(t,S_t^1+\Phi_t^2)|+|L^2(t,S_t^1+\Phi_t^2)-L^1(t,S_t^1+\Phi_t^2)|\\
\leqslant&\ C|S_t^1-S_t^2|+\sup_{x\in\mathbb{R}}|L^1(t,x)-L^2(t,x)|.
\end{aligned}$$

The proof is complete. ∎

PROPOSITION 3.4. *Suppose that* (L^i, R^i) *satisfy Assumption* 2.1 *for* $i=1,2$. *Given* $S^i\in D[0,\infty)$, *let* (X^i,K^i) *solve the nonlinear Skorokhod problem* $\mathbb{SP}_{L^i}^{R^i}(S^i)$. *Then*

$$\sup_{t\in[0,T]}|K_t^1-K_t^2|\leqslant\frac{C}{c}\sup_{t\in[0,T]}|S_t^1-S_t^2|+\frac{1}{c}(\bar{L}_T\vee\bar{R}_T),$$

where

$$\begin{aligned}
\bar{L}_T&:=\sup_{(t,x)\in[0,T]\times\mathbb{R}}|L^1(t,x)-L^2(t,x)|,\\
\bar{R}_T&:=\sup_{(t,x)\in[0,T]\times\mathbb{R}}|R^1(t,x)-R^2(t,x)|.
\end{aligned}$$

Proof. Note that for any $x_i,y_i\in\mathbb{R}$, $i=1,2$, the following inequalities hold:

$$\begin{gathered}
|x_1\wedge x_2-y_1\wedge y_2|\leqslant|x_1-y_1|\vee|x_2-y_2|,\\
|x_1^+-x_2^+|\leqslant|x_1-x_2|.
\end{gathered}$$

It is easy to check that

$$\begin{aligned}
&\left|(-\Phi_0^1)^+\wedge\inf_{r\in[0,t]}(-\Psi_r^1)-(-\Phi_0^2)^+\wedge\inf_{r\in[0,t]}(-\Psi_r^2)\right|\\
&\qquad\leqslant|(-\Phi_0^1)^+-(-\Phi_0^2)^+|\vee\left|\inf_{r\in[0,t]}(-\Psi_r^1)-\inf_{r\in[0,t]}(-\Psi_r^2)\right|\\
&\qquad\leqslant|\Phi_0^1-\Phi_0^2|\vee\sup_{r\in[0,t]}|\Psi_r^1-\Psi_r^2|
\end{aligned}$$

and

$$
\begin{aligned}
&\Big|\sup_{s\in[0,t]}\Big[(-\Phi^1_s)\wedge\inf_{r\in[s,t]}(-\Psi^1_r)\Big]-\sup_{s\in[0,t]}\Big[(-\Phi^2_s)\wedge\inf_{r\in[s,t]}(-\Psi^2_r)\Big]\Big|\\
&\qquad\qquad\leqslant \sup_{s\in[0,t]}\Big|(-\Phi^1_s)\wedge\inf_{r\in[s,t]}(-\Psi^1_r)-(-\Phi^2_s)\wedge\inf_{r\in[s,t]}(-\Psi^2_r)\Big|\\
&\qquad\qquad\leqslant \sup_{s\in[0,t]}\Big[|\Phi^1_s-\Phi^2_s|\vee|\inf_{r\in[s,t]}(-\Psi^1_r)-\inf_{r\in[s,t]}(-\Psi^2_r)|\Big]\\
&\qquad\qquad\leqslant \sup_{s\in[0,t]}\Big[|\Phi^1_s-\Phi^2_s|\vee\sup_{r\in[s,t]}|\Psi^1_r-\Psi^2_r|\Big]\\
&\qquad\qquad\leqslant \sup_{s\in[0,t]}|\Phi^1_s-\Phi^2_s|\vee\sup_{s\in[0,t]}|\Psi^1_s-\Psi^2_s|.
\end{aligned}
$$

Recalling the construction of K^i in (2.4) and applying Lemma 3.3, we obtain the desired result. ∎

REMARK 3.6. Proposition 3.4 is a generalization of [9, Corollary 1.6] and [12, Proposition 4.1]. In fact, for any given $\alpha^i,\beta^i\in D[0,\infty)$ with $\inf_t(\beta^i_t-\alpha^i_t)>0$, $i=1,2$, let $L^i(t,x)=x-\beta^i_t$ and $R^i(t,x)=x-\alpha^i_t$. The result in Proposition 3.4 coincides with [12, (4.2)].

For any fixed $T>0$, let $\mathcal{M}_T$ be the collection of strictly increasing continuous functions λ of $[0,T]$ onto itself. The J_1 *metric* $d_{0,T}$ on $D[0,T]$ is defined by

$$
d_{0,T}(f,g)=\inf_{\lambda\in\mathcal{M}_T}\Big(\sup_{t\in[0,T]}|\lambda(t)-t|\vee\sup_{t\in[0,T]}|f_t-g_{\lambda(t)}|\Big).
$$

PROPOSITION 3.5. *Suppose that Assumption* 2.1 *holds. Given* $S,S'\in D[0,\infty)$*, let* (X,K)*,* (X',K') *solve the nonlinear Skorokhod problems* $\mathbb{SP}^R_L(S)$ *and* $\mathbb{SP}^R_L(S')$*, respectively. Then, for any* $T>0$*,*

$$
d_{0,T}(K,K')\leqslant\frac{1}{c}(\hat L_T\vee\hat R_T)+\frac{C}{c}d_{0,T}(S,S'),
$$

where

$$
\begin{aligned}
\hat L_T &:= \sup_{(t,s,x)\in[0,T]\times[0,T]\times\mathbb{R}}|L(t,x)-L(s,x)|,\\
\hat R_T &:= \sup_{(t,s,x)\in[0,T]\times[0,T]\times\mathbb{R}}|R(t,x)-R(s,x)|.
\end{aligned}
$$

Proof. Without loss of generality, we assume that $S\neq S'$. By the definition of $d_{0,T}$, for any $\delta>0$ there exists some $\lambda\in\mathcal{M}_T$ such that

$$
\begin{aligned}
\sup_{t\in[0,T]}|\lambda(t)-t| &\leqslant d_{0,T}(S,S')+\delta[1\wedge d_{0,T}(S,S')],\\
\sup_{t\in[0,T]}|S'_t-S_{\lambda(t)}| &\leqslant d_{0,T}(S,S')+\delta[1\wedge d_{0,T}(S,S')].
\end{aligned}
$$

Given $\lambda \in \mathcal{M}_T$, for any $t \in [0,T]$, applying the definition of K in (2.4), it is easy to check that

$$K_{\lambda(t)} = -\max\Big((-\Phi_0)^+ \wedge \inf_{r\in[0,t]}(-\Psi_{\lambda(r)}),\ \sup_{s\in[0,t]}\Big[(-\Phi_{\lambda(s)}) \wedge \inf_{r\in[s,t]}(-\Psi_{\lambda(r)})\Big]\Big).$$

That is, $(X\circ\lambda, K\circ\lambda)$ solves $\mathbb{SP}_{L\circ\lambda}^{R\circ\lambda}(S\circ\lambda)$ on $[0,T]$, where $(f\circ\lambda)_t = f_{\lambda(t)}$ for $f = X, K, S$ and $(g\circ\lambda)(t,x) = g(\lambda(t),x)$ for $g = L, R$. By Proposition 3.4, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} |K'_t - K_{\lambda(t)}| \\
&\quad\leqslant \frac{1}{c}\Big[\sup_{(t,x)\in[0,T]\times\mathbb{R}} |L(t,x) - L(\lambda(t),x)| \vee \sup_{(t,x)\in[0,T]\times\mathbb{R}} |R(t,x) - R(\lambda(t),x)|\Big] \\
&\qquad + \frac{C}{c}\sup_{t\in[0,T]} |S'_t - S_{\lambda(t)}| \\
&\quad\leqslant \frac{1}{c}(\hat{L}_T \vee \hat{R}_T) + \frac{C}{c}\big(d_{0,T}(S,S') + \delta[1 \wedge d_{0,T}(S,S')]\big).
\end{aligned}$$

Since the above inequality holds for any $\delta > 0$, we obtain the desired result. ■

REMARK 3.7. Proposition 3.5 is a generalization of [9, Corollary 1.6] and [12, Proposition 4.2]. In contrast to the results in [9] for the Skorokhod map on a time-independent interval $[0,a]$, where a is a positive constant, the solution to the Skorokhod problem with two nonlinear reflecting boundaries is not continuous under the J_1 metric. The terms $\hat{L}_T$, $\hat{R}_T$ may be regarded as the "oscillations" of L and R, which cannot be omitted (see [12, Example 4.1]).

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