PROBABILITY
AND
MATHEMATICAL STATISTICS
Published online 15.3.2024
doi:10.37190/0208-4147.00145
Online First version

POINT PROCESS OF CLUSTERS FOR A STATIONARY GAUSSIAN RANDOM FIELD ON A LATTICE*

BY

YINGYIN LU (CHENGDU) AND JINHUI GUO (CHENGDU)

Abstract. It is well established that the normalized exceedances resulting from a standard stationary Gaussian triangular array at high levels follow a Poisson process under the Berman condition. To model frequent cluster phenomena, we consider the asymptotic distribution of the point process of clusters for a Gaussian random field on a lattice. Our analysis demonstrates that the point process of clusters also converges to a Poisson process in distribution, provided that the correlations of the Gaussian random field meet certain conditions. Additionally, we provide a numerical example to illustrate our theoretical results.

2020 Mathematics Subject Classification: Primary 60G70; Secondary 60G15.

Key words and phrases: point process of clusters, Gaussian random field, lattice, asymptotic behavior.

1. INTRODUCTION

For a centered unit-variance stationary Gaussian sequence $\{X_{n,s},\, 0 \leqslant s \leqslant n\}$ with correlation $\rho_j = \operatorname{corr}(X_{n,s}, X_{n,s+j})$, under the Berman condition

$$\rho_j \log j \to 0 \quad \text{as } j \to \infty,$$

which requires the correlation to tend to zero as $j \to \infty$, [18] established that the exceedances point process $N_n(B)$ converges weakly to a Poisson process with intensity e^{-x}. The process $N_n(B)$ is defined by

$$N_n(B) = \sum_{s=0}^{n} I\{X_{n,s} > u_n(x),\, s/n \in B\}, \quad x \in \mathbb{R},$$

* The research is financially supported by the State Scholarship Fund (CSC No. 202208515005) from the China Scholarship Council.

where B is a Borel subset of $[0,1]$, and $u_n(x) = b_n + x/a_n$ is a high level. Here, a_n and b_n are defined by

$$a_n = \sqrt{2\log n}, \qquad b_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}. \tag{1.1}$$

This demonstrates that the number of exceedances of $u_n(x)$ by a Gaussian sequence $X_{n,s}$ in $\{0, 1, \ldots, n\}$ follows an asymptotic Poisson distribution. However, [15] proposed a more general stationary Gaussian triangular array with correlations that tend to 1 at a certain rate to better model the extremes of correlated Gaussian sequences. For such arrays, the exceedances of $u_n(x)$ may tend to occur in clusters, i.e., an exceedance is likely to have neighboring exceedances, resulting in clustering of exceedances. To define the point process of clusters, we can consider a sequence $s_n + \ell_n$ and events that occur within a distance of $s_n + \ell_n$ as belonging to the same cluster. The first systematic study of the point process of clusters for a stationary sequence $\{\xi_{n,s},\ 0 \leqslant s \leqslant n\}$ was reported by Leadbetter in 1983, where the point process of clusters was defined by

$$\widehat{N}_n(B) = \sum_{k=1}^{t_n} I\Big\{ \bigcup_{s\in Q_k} \xi_{n,s} > u_n(x),\ s/n \in B \Big\}, \qquad x \in \mathbb{R},$$

where $Q_k = \{(k-1)(s_n+\ell_n), (k-1)(s_n+\ell_n)+1, \ldots, k(s_n+\ell_n)-1\}$, and s_n, ℓ_n are positive integers. Here $t_n = [\frac{n+1}{s_n+\ell_n}]$, and $[\,\cdot\,]$ denotes the integer part. [17] showed that $\widehat{N}_n(B)$ can also converge in distribution to a Poisson process.

Using analogues of conditions of [15], which allow for strong local dependence among variables while keeping their asymptotic independence, [13] generalized the results of [15] and [14] to multivariate stationary Gaussian triangular arrays. The limiting distribution of the normalized maxima for Gaussian random vectors was derived, and [14] established the limit law for the bivariate stationary Gaussian triangular arrays. Furthermore, [11] and [19] extended the results to stationary random fields and it was shown that a two-dimensional Gaussian random field $\{X_{n,ij}, 0 \leqslant i, j \leqslant n\}$ also exhibits extremal clustering in the limit. The asymptotic distribution of the maximum of a Gaussian random field was established in which the spatial domain is rescaled. Recently, [12] established asymptotic behaviors of point processes of clusters for stationary bivariate Gaussian triangular arrays. However, to the best of our knowledge, there is not much research on the point process of a Gaussian random field on a lattice, because the theoretical development of multivariate extreme value theory is far behind its univariate counterpart. Moreover, there is no study on the asymptotics of point processes of clusters for two-dimensional Gaussian random fields.

Motivated by [11, 12], we consider asymptotic behaviors of the point process of clusters for a Gaussian random field $\{X_{n,ij}, 0 \leqslant i, j \leqslant n\}$ on a lattice. Similar to the definition of $\widehat{N}_n(B)$, the point process of clusters formed by $\{X_{n,ij}, 0 \leqslant i,j \leqslant n\}$ is defined by

$$N_{n^2} = \sum_{s=1}^{t_n} \sum_{t=1}^{t_n} I\Big\{ \bigcup_{(i,j)\in I_{st}} X_{n,ij} > u_{n^2}(x) \Big\}, \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_{n^2}(x) = x/a_{n^2} + b_{n^2}$, and

$$a_{n^2} = 2\sqrt{\log n}, \quad b_{n^2} = 2\sqrt{\log n} - \frac{\log\log n^2 + \log 4\pi}{4\sqrt{\log n}},$$

with

$$\begin{aligned} I_{st} = \{(i,j) : (s-1)(s_n+\ell_n) \leqslant i \leqslant s(s_n+\ell_n) - 1, \\ (t-1)(s_n+\ell_n) \leqslant j \leqslant t(s_n+\ell_n) - 1\} \end{aligned} \tag{1.3}$$

for $s, t = 1, \ldots, t_n$. As an immediate consequence of our results, one recovers the results of [11].

It is worth mentioning that the behavior of extremes of random fields has many applications. [7] considered the application of the behavior of extremes of random fields in brain mapping. [21] derived control methods for random fields and identified the clusters of galaxies in astronomy. When a smooth Gaussian random field is sampled on a discrete lattice, [23] focused on approximations of the exceedance probability of the maximum in a finite region and applied the results to image discrimination. [10] studied the multiple testing problem in the context of constructing confidence regions for level curves of Gaussian random fields. Furthermore, the theory of extremes of random fields has been applied to construct simultaneous confidence bands for electrical load curves (see e.g. [4]). For more advanced topics related to random fields and their applications, see [3, 9, 2, 22].

The paper is organized as follows. We present the main results in Section 2, and an illustrative numerical example in Section 3. Auxiliary results and the proofs are given in Sections 4 and 5, respectively.

2. MAIN RESULTS

By assuming conditions similar to those of [11], which generalized the cluster conditions for the correlations of [15] to Gaussian random fields on a two-dimensional lattice, we will show in Theorem 2.1 that the limiting point process of clusters of a Gaussian random field on a lattice is a Poisson process. Additionally, if the covariance function of the random field satisfies the condition that sample paths are almost surely continuous and have short-range dependence, Theorem 2.2 will show the result also holds.

THEOREM 2.1. *Let $\{X_{n,ij},\ 0 \leqslant i,j \leqslant n\}$ be a mean zero, variance one, stationary Gaussian random field on an $n \times n$ lattice. Denote the correlation between*

$X_{n,ij}$ and $X_{n,k\ell}$ by $\rho_{n,ij,k\ell} = \mathrm{E}(X_{n,ij}X_{n,k\ell})$. Suppose that

$$(1 - \rho_{n,ij,k\ell}) \log n \to \delta_{ij,k\ell} \in (0, \infty) \quad \text{as } n \to \infty \tag{2.1}$$

for all $i, j, k, \ell \in \mathbb{Z}$ such that $(i, j) \neq (k, \ell)$ and there exist positive integers s_n, ℓ_n such that

$$\frac{\ell_n}{s_n} \to 0, \quad \frac{s_n}{n} \to 0 \quad \text{as } n \to \infty, \tag{2.2}$$

$$\lim_{n\to\infty} \sup_{\sqrt{i^2+j^2} \geqslant \ell_n} |\rho_{n,ij,00}| \log n = 0, \tag{2.3}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{(i,j)\in\{0,1,\ldots,\ell_n\}^2 \setminus \{0,1,\ldots,m\}^2} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \frac{(\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}}{\sqrt{1-\rho_{n,ij,00}^2}} = 0. \tag{2.4}$$

Then the point process of clusters N_{n^2} converges to N in distribution as $n \to \infty$, where N is a Poisson process with intensity parameters $\vartheta \exp(-x)$ on $(0, 1]$ and

$$\vartheta = \mathrm{P}\left(\frac{E}{4} + \sqrt{\frac{1}{2}\delta_{ij,00}}W_{ij} \leqslant \delta_{ij,00},\ (i, j) \in K\right),$$

with $K = \{\mathbb{N} \times \{0\}\} \cup \{\mathbb{Z} \times \mathbb{N}\}$, E is a standard exponential random variable, and $\{W_{ij}\}$ is a mean zero, variance one, Gaussian random field independent of E with correlation

$$\mathrm{corr}(W_{ij}W_{k\ell}) = \frac{\delta_{ij,00} + \delta_{k\ell,00} - \delta_{ij,k\ell}}{2\sqrt{\delta_{ij,00}\delta_{k\ell,00}}}.$$

REMARK 2.1. The limiting distribution of the maximum of a stationary Gaussian random field $\{X_{n,ij},\ 0 \leqslant i, j \leqslant n\}$ on a lattice in [11] is a direct corollary of Theorem 2.1.

Since isotropic Gaussian random fields are common in practice [1], we consider the asymptotic behavior of the point process of clusters formed from an isotropic Gaussian random field $\{Y(\mathbf{s})\}$ in the following theorem.

THEOREM 2.2. *Let $\{Y(\mathbf{s})\}$ be an isotropic Gaussian random field on $\mathbb{R}^2$ whose correlation function $\rho(\mathbf{h}) = \rho(\|\mathbf{h}\|)$ is decreasing and such that*

(i) $\rho(\mathbf{h}) = 1 - \|\mathbf{h}\|^\beta(1 + o(1))$ *as* $\|\mathbf{h}\| \downarrow 0$, *for some* $0 < \beta \leqslant 2$,

(ii) $0 \leqslant \rho(\mathbf{h}) \leqslant D\exp(-\|\mathbf{h}\|^a)$ *for some constants D, $a > 0$ and all $\mathbf{h} \in \mathbb{R}^2$.*

Let $X_{n,ij} = Y((i,j)/(\log n)^{1/\beta})$, $i,j = 0,1,\ldots,n$. Then the assumptions of Theorem 2.1 *are satisfied by $X_{n,ij}$ and hence the conclusion of Theorem* 2.1 *holds with*

$$\delta_{ij,kl} = [(k-i)^2 + (l-j)^2]^{\beta/2}, \quad (i,j),(k,l) \in K.$$

REMARK 2.2. By [6], the Gaussian process $\{Y(s/(\log n)^{1/\beta})\}$, as given in Theorem 2.2, belongs to the domain of attraction of a max-stable process known as a Brown–Resnick process. See [16, 20, 8] for advanced works on this class of processes.

3. AN ILLUSTRATIVE EXAMPLE

In this section, we discuss how to compute the extremal index ϑ and illustrate our results with a numerical example. Consider a stationary Gaussian random field $\{X_{n,ij},\, 1 \leqslant i,j \leqslant n\}$ with mean zero and unit variance on an $n \times n$ lattice. Set

$$X_{n,21} = d_n X_{n,11} + \sqrt{1-d_n^2} Z_{n,21}, \ldots, X_{n,n1} = d_n X_{n,(n-1)\,1} + \sqrt{1-d_n^2} Z_{n,n\,1},$$
$$X_{n,22} = d_n X_{n,12} + \sqrt{1-d_n^2} Z_{n,22}, \ldots, X_{n,n2} = d_n X_{n,(n-1)\,2} + \sqrt{1-d_n^2} Z_{n,n\,2},$$
$$\vdots$$
$$X_{n,2n} = d_n X_{n,1n} + \sqrt{1-d_n^2} Z_{n,2n}, \ldots, X_{n,nn} = d_n X_{n,(n-1)\,n} + \sqrt{1-d_n^2} Z_{n,nn},$$

and

$$X_{n,12} = d_n X_{n,11} + \sqrt{1-d_n^2} Z_{n,12}, \ldots, X_{n,1n} = d_n X_{n,1(n-1)} + \sqrt{1-d_n^2} Z_{n,1n}$$

where $Z_{n,ij}$ are independent of $X_{n,ij}$ for $1 \leqslant i,j \leqslant n$ and $\{Z_{n,ij},\, 1 \leqslant i,j \leqslant n\}$ are independent and identically distributed standard normal random variables. Assume that

$$d_n = 1 - \frac{\zeta}{\log n} \quad \text{for some } \zeta \in (0,\infty). \tag{3.1}$$

By stationarity, we have the correlation

$$\rho_{n,ij,00} = d_n^{i+j} = \left(1 - \frac{\zeta}{\log n}\right)^{i+j}.$$

Thus, condition (2.1) in Theorem 2.1 holds with

$$\delta_{ij,00} = (i+j)\zeta.$$

Now let $\ell_n = (\log n)\log(\log n)^2$. Then we can show that

$$\sup_{\sqrt{i^2+j^2} \geqslant \ell_n} |\rho_{n,ij,00}| \log n \leqslant \exp\left(-\frac{\zeta \ell_n}{\log n} + \log\log n\right) \to 0 \quad \text{as } n \to \infty,$$

and hence condition (2.3) follows.

Next, we verify (2.4). For any $\epsilon \in (0,2)$, if $(i+j)\zeta/\log n > \epsilon$, then

$$\rho_{n,ij,00} \leqslant e^{-\epsilon}. \tag{3.2}$$

If $(i+j)\zeta/\log n \leqslant \epsilon$, then by Taylor expansion we obtain

$$\frac{(i+j)\zeta}{\log n}\left(1-\frac{\epsilon}{2}\right) \leqslant 1-\rho_{n,ij,00} \leqslant \frac{(i+j)\zeta}{\log n}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
&\sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \frac{(\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}}{\sqrt{1-\rho_{n,ij,00}^2}} \\
&= \sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \frac{(\log n)^{\frac{1-\rho_{n,ij,00}}{2(1+\rho_{n,ij,00})}}}{\sqrt{(1-\rho_{n,ij,00}^2)\log n}} \\
&\leqslant \Bigg\{\sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} n^{-2\frac{1-e^{-\epsilon}}{1+e^{-\epsilon}}} \frac{(\log n)^{1/2}}{\sqrt{(1-e^{-\epsilon})\log n}} \\
&\vee \sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} \exp\left[-\frac{(i+j)\zeta}{2}\left(1-\frac{\epsilon}{2}\right)\right] \frac{\exp\left(\frac{(i+j)\zeta\log\log n}{2\log n}\right)}{\sqrt{(i+j)\zeta(1-\epsilon/2)}}\Bigg\}.
\end{aligned}$$

Therefore, (2.4) is satisfied and the conclusion of Theorem 2.1 holds.

Now, we discuss how to compute the extremal index ϑ for this example. We can replace W_{ij} by $(i+j)^{-1/2}\sum_{t_1=1}^{i}\sum_{t_2=1}^{j}(Z_{t_1}+\varepsilon_{t_2})$ with ε_{t_2} independent of Z_{t_1}. Then

$$\vartheta = \mathrm{P}\left(\frac{E}{4}+\sqrt{\tfrac{1}{2}\zeta}\sum_{t_1=1}^{i}\sum_{t_2=1}^{j}(Z_{t_1}+\varepsilon_{t_2}) \leqslant (i+j)\zeta \text{ for all } i,j\geqslant 1\right), \tag{3.4}$$

where E denotes a standard exponential random variable independent of the Z_{t_1} and ε_{t_2}. Since Z_{t_1} and ε_{t_2} are independent and identically distributed standard normal random variables, a calculation based on (3.4) is straightforward. We computed the extremal indices ϑ for various sample sizes n and d, and recorded the results in Table 1. Noting that $d = d_n$, given by (3.1), it is clear that, for larger d, it takes a larger n for ϑ to approach 1.

To confirm our asymptotic results in Theorems 2.1 and 2.2, we choose a specific value of d, say $d = 0.9$. Then $\widetilde{\zeta} = (1-d)\log n$ and

$$\widetilde{\vartheta} = \mathrm{P}\left(\frac{E}{4}+\sqrt{\tfrac{1}{2}\widetilde{\zeta}}\sum_{t_1=1}^{i}\sum_{t_2=1}^{j}(Z_{t_1}+\varepsilon_{t_2}) \leqslant (i+j)\widetilde{\zeta} \text{ for all } i,j\geqslant 1\right).$$

TABLE 1. Extremal indices ϑ

	$d = 0.7$	$d = 0.75$	$d = 0.8$	$d = 0.85$	$d = 0.9$
$n = 20$	0.935	0.905	0.860	0.789	0.674
$n = 100$	0.979	0.965	0.939	0.890	0.795
$n = 1000$	0.995	0.990	0.979	0.954	0.890
$n = 10000$	0.998	0.997	0.993	0.979	0.939
$n = 50000$	0.999	0.998	0.996	0.988	0.959

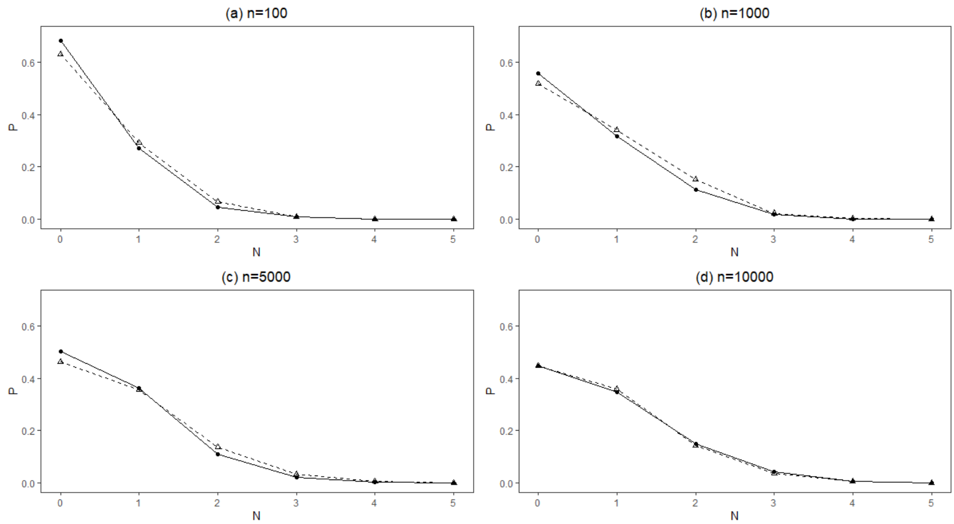

FIGURE 1. Solid lines are for the distribution of N_{n^2} based on the observed frequencies; dotted lines represent the distribution of the Poisson process N; n represents different lengths of sequences.

The distributions of the point process of clusters N_{n^2} and the Poisson process N for different sample sizes n when $x = 0$ are displayed in Figure 1. From this figure, we can see that the distribution of N_{n^2} approximates that of N better as n becomes larger, as guaranteed by our results in Theorems 2.1 and 2.2.

4. AUXILIARY RESULTS

In order to prove the theorems, we need some auxiliary lemmas. Define the indicator variable

$$\eta_{st} = \begin{cases} 1 & \text{when } \max\{X_{n,ij},\ (i,j) \in I^*_{st}\} > u_{n^2}(x), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} I^*_{st} = \{(i,j) : (s-1)(s_n + \ell_n) + \ell_n \leqslant i \leqslant s(s_n + \ell_n) - \ell_n, \\ (t-1)(s_n + \ell_n) + \ell_n \leqslant j \leqslant t(s_n + \ell_n) - \ell_n\}, \end{aligned}$$

and $I'_{st} = I_{st} \setminus I^*_{st}$, with I_{st} is given by (1.3). Here s_n and ℓ_n are positive integers and $s, t = 1, \ldots, t_n$, $t_n = \left[\frac{n+1}{r_n+\ell_n}\right]$.

To simplify the notation, we define $M_n = \max\{X_{n,ij},\, 0 \leqslant i,j \leqslant n\}$ and $M_{t_n} = \max\{X_{n,ij},\, 0 \leqslant i,j \leqslant t_n\}$. Additionally, let $M_{[i:j],[k:\ell]} = \max\{X_{n,st},\, i \leqslant s \leqslant j,\, k \leqslant t \leqslant \ell\}$ be the maximum of the set of random variables in horizontal positions i through j and vertical positions k through ℓ, and simplify this to $M_{[i:j],[k]} = \max\{X_{n,st},\, i \leqslant s \leqslant j\}$ when only random variables from a single row k are under consideration. Note that $M_{[i:j],[k:\ell]} = -\infty$ if $i > j$ or $k > \ell$. For simplicity, we use the same letter C to denote positive constants that may take different values at different places.

LEMMA 4.1. *For any random field $\{X_{n,ij},\, 0 \leqslant i,j \leqslant n\}$, we have*

$$\begin{aligned}\mathrm{P}(M_n > u_{n^2}(x)) &\\ = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathrm{P}\big(X_{n,ij} > u_{n^2}(x),\, M_{[i+1:n],[j]} \vee M_{[0:n],[j+1:n]} \leqslant u_{n^2}(x)\big).\end{aligned} \tag{4.1}$$

Proof. To calculate $\mathrm{P}(M_n > u_{n^2}(x))$, we first intersect the set $\{M_n > u_{n^2}(x)\}$ with the union events $\{X_{n,ij} > u_{n^2}(x)\} \cup \{X_{n,ij} \leqslant u_{n^2}(x)\}$ for $1 \leqslant i,j \leqslant n$, and then sum over the disjoint events and simplify. More precisely,

$$\begin{aligned}\mathrm{P}(M_n > u_{n^2}(x)) &= \mathrm{P}\big(M_n > u_{n^2}(x),\, \{X_{n,nn} > u_{n^2}(x)\} \cup \{X_{n,nn} \leqslant u_{n^2}(x)\}\big)\\ &= \mathrm{P}(X_{n,nn} > u_{n^2}(x))\\ &\quad + \mathrm{P}\big(M_{[0:n],[0:n-1]} \vee M_{[0:n-1],[n]} > u_{n^2}(x),\, X_{n,nn} \leqslant u_{n^2}(x)\big).\end{aligned}$$

Continuing this pattern by decrementing the horizontal position i by 1 each time until the first position of the row is reached, we have

$$\begin{aligned}&\mathrm{P}(M_n > u_{n^2}(x))\\ &= \mathrm{P}(X_{n,nn} > u_{n^2}(x)) + \mathrm{P}\big(X_{n,(n-1)n} > u_{n^2}(x),\, X_{n,nn} \leqslant u_{n^2}(x)\big)\\ &\quad + \mathrm{P}\big(M_{[0:n],[0:n-1]} \vee M_{[0:n-2],[n]} > u_{n^2}(x),\, X_{n,(n-1)n} \leqslant u_{n^2}(x),\\ &\qquad\qquad X_{n,nn} \leqslant u_{n^2}(x)\big)\\ &= \sum_{i=0}^{n} \mathrm{P}\big(X_{n,in} > u_{n^2}(x),\, M_{[i+1:n],[n]} \leqslant u_{n^2}(x)\big)\\ &\quad + \mathrm{P}\big(M_{[0:n],[0:n-1]} > u_{n^2}(x),\, M_{[0:n],[n]} \leqslant u_{n^2}(x)\big).\end{aligned}$$

Finally, we continue this pattern for the vertical position j until we reach $(i,j) = (0,0)$. Thus, we get the assertion. ■

LEMMA 4.2. *Under the assumptions of Theorem 2.1, we have*

$$\lim_{n\to\infty}\Big|\mathrm{P}(\eta_{st} = e_{st},\, s,t = 1,\ldots,t_n) - \prod_{s=1}^{t_n}\prod_{t=1}^{t_n} \mathrm{P}(\eta_{st} = e_{st})\Big| = 0, \tag{4.2}$$

where e_{st} are variables assuming only the values 0 and 1 for $s,t = 1,\ldots,t_n$.

Proof. As in the proof of [5, Lemma 8.1], we only need to prove that (4.2) holds when $e_{st} = 0$, $s, t = 1, \ldots, t_n$. By the Normal Comparison Lemma (see e.g. [18]), we have

$$
\begin{aligned}
&\Big|\mathrm{P}(\eta_{st} = 0,\ s, t = 1, \ldots, t_n) - \prod_{s=1}^{t_n} \prod_{t=1}^{t_n} \mathrm{P}(\eta_{st} = 0)\Big| \qquad (4.3)\\
&= \Big|\mathrm{P}\Big(\max_{(i,j)\in I_{st}^*} X_{n,ij} \leqslant u_{n^2}(x),\ s, t = 1, \ldots, t_n\Big) \\
&\qquad\qquad - \prod_{s=1}^{t_n} \prod_{t=1}^{t_n} \mathrm{P}\Big(\max_{(i,j)\in I_{st}^*} X_{n,ij} \leqslant u_{n^2}(x)\Big)\Big| \\
&\leqslant C \sum_{\substack{(i,j)\in I_{11}^*,\\ (k,\ell)\in I^*\setminus I_{11}^*}} |\rho_{n,ij,k\ell}| \exp\Big(-\frac{u_{n^2}^2(x)}{1 + |\rho_{n,ij,k\ell}|}\Big) \\
&\quad + C \sum_{\substack{(i,j)\in I_{12}^*,\\ (k,\ell)\in I^*\setminus I_{12}^*}} |\rho_{n,ij,k\ell}| \exp\Big(-\frac{u_{n^2}^2(x)}{1 + |\rho_{n,ij,k\ell}|}\Big) \\
&\quad + \cdots + C \sum_{\substack{(i,j)\in I_{t_nt_n}^*,\\ (k,\ell)\in I^*\setminus I_{t_nt_n}^*}} |\rho_{n,ij,k\ell}| \exp\Big(-\frac{u_{n^2}^2(x)}{1 + |\rho_{n,ij,k\ell}|}\Big) \\
&\leqslant C t_n^2 s_n^2 n^2 \rho_n^* \exp\Big(-\frac{u_{n^2}^2(x)}{1 + \rho_n^*}\Big),
\end{aligned}
$$

where

$$
I^* = \{I_{11}^*, I_{12}^*, \ldots, I_{t_nt_n}^*\}, \qquad \rho_n^* = \max\{|\rho_{n,ij,k\ell}|,\ \sqrt{(k-i)^2 + (\ell-j)^2} \geqslant \ell_n\}.
$$

From [18, (4.3.4)], we have

$$
\exp\Big(-\frac{u_n^2(x)}{2}\Big) \sim \frac{C u_n(x)}{n} \quad \text{and} \quad u_n(x) \sim \sqrt{2\log n}.
$$

So, $\exp(-u_{n^2}^2(x)) \sim C u_{n^2}^2(x)/n^4$ and

$$
\begin{aligned}
C t_n^2 s_n^2 n^2 |\rho_n^*| \exp\Big(-\frac{u_{n^2}^2(x)}{1 + |\rho_n^*|}\Big) &\leqslant C n^4 |\rho_n^*| \exp(-u_{n^2}^2) \exp\Big(\frac{u_{n^2}^2(x)|\rho_n^*|}{1 + |\rho_n^*|}\Big) \\
&\leqslant C|\rho_n^*| \log n \exp(u_{n^2}^2 |\rho_n^*|) \\
&\leqslant C|\rho_n^*| \log n \exp(4\log n\, |\rho_n^*|).
\end{aligned}
$$

From condition (2.3) of Theorem 2.1, we have

$$
\lim_{n\to\infty} |\rho_n^*| \log n = 0,
$$

and combining this with (4.3) yields (4.2). ■

LEMMA 4.3. *Under the assumptions of Theorem* 2.1*, for any bounded set* $K \subset \{\mathbb{N} \times \{0\}\} \cup \{\mathbb{Z} \times \mathbb{N}\}$*, we have*

$$\lim_{n\to\infty} \mathrm{P}\Big(\max_{(i,j)\in K} X_{n,ij} \leqslant u_{n^2}(x) \;\Big|\; X_{n,00} > u_{n^2}(x)\Big) = \mathrm{P}\bigg(\frac{E}{4} + \sqrt{\tfrac{1}{2}\delta_{ij,00}}W_{ij} \leqslant \delta_{ij,00},\ (i,j) \in K\bigg),$$

where E *is a standard exponential random variable and* W_{ij} *is a mean zero, variance one, Gaussian random field with correlation*

$$\operatorname{corr}(W_{ij}W_{k\ell}) = \frac{\delta_{ij,00} + \delta_{k\ell,00} - \delta_{ij,k\ell}}{2\sqrt{\delta_{ij,00}\delta_{k\ell,00}}}.$$

Proof. The proof can be found in [11, Lemma 5]. ■

LEMMA 4.4. *Under the assumptions of Theorem* 2.1*, let* r_n *be a positive integer that satisfies*

$$\frac{\ell_n}{r_n} \to 0, \qquad \frac{r_n}{s_n} \to 0, \qquad \frac{r_n}{n} \to 0 \quad \textit{as } n \to \infty. \tag{4.4}$$

Then

$$\lim_{n\to\infty} \mathrm{P}\big(M_{[1:r_n-\ell_n],[0]} \vee M_{[-r_n:r_n-\ell_n],[1:r_n-\ell_n]} \leqslant u_{n^2}(x) \;\big|\; X_{n,00} > u_{n^2}(x)\big) = \vartheta, \tag{4.5}$$

and

$$\lim_{n\to\infty} \mathrm{P}\big(M_{[1:s_n-\ell_n],[0]} \vee M_{[-s_n:s_n-\ell_n],[1:s_n-\ell_n]} \leqslant u_{n^2}(x) \;\big|\; X_{n,00} > u_{n^2}(x)\big) = \vartheta. \tag{4.6}$$

Proof. Define

$$\begin{aligned} A_n &= \{1,\ldots,r_n-\ell_n\} \times \{0\} \\ &\quad \cup \{-r_n, -r_n+1, \ldots, r_n-\ell_n\} \times \{1,\ldots,r_n-\ell_n\}, \\ G_{m,n} &= \{(i,j) \in A_n : |i|, |j| \leqslant m\}, \end{aligned}$$

and $H_{m,n} = A_n \setminus G_{m,n}$ for $m \in \mathbb{N}$. It follows from Lemma 4.3 that for (4.5) we only have to show

$$\lim_{m\to\infty}\lim_{n\to\infty} \mathrm{P}\Big(\bigcup_{(i,j)\in H_{m,n}} \{X_{n,ij} > u_{n^2}(x)\} \;\Big|\; X_{n,00} > u_{n^2}(x)\Big) = 0. \tag{4.7}$$

By similar arguments to those in the proof of Lemma 4.3, we have

$$
\begin{aligned}
&\mathrm{P}\Big(\bigcup_{(i,j)\in H_{m,n}} \{X_{n,ij} > u_{n^2}(x)\} \;\Big|\; X_{n,00} > u_{n^2}(x)\Big) \\
&\qquad \sim \int_0^\infty \mathrm{P}\Big(\bigcup_{(i,j)\in H_{m,n}} \{X_{n,ij} > u_{n^2}(x)\} \;\Big|\; X_{n,00} = u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\Big) \\
&\qquad\quad \times \exp\Big(-z - \frac{z^2}{2u_{n^2}^2(x)}\Big)\,\mathbf{d}z.
\end{aligned}
$$

Next, we assume that $\{Y_{n,ij},\ (i,j) \in H_{m,n}\}$ has the same distribution as $\{X_{n,ij},\ (i,j) \in H_{m,n} \mid X_{n,00} = u_{n^2}(x) + z/u_{n^2}(x)\}$. Then $(Y_{n,ij}, (i,j) \in H_{m,n}) \sim N(\mu, \mathbf{\Sigma})$, where

$$
\begin{aligned}
\mu &= \Big(\rho_{n,ij,00}\Big(u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\Big),\ (i,j) \in H_{m,n}\Big), \\
\mathbf{\Sigma} &= \big(\rho_{n,ij,k\ell} - \rho_{n,ij,00}\rho_{n,k\ell,00}\big)_{(i,j),(k,\ell)\in H_{m,n}}.
\end{aligned}
$$

Standardizing $Y_{n,ij}$ and letting

$$
Z_{n,ij} = \frac{Y_{n,ij} - \rho_{n,ij,00}\big(u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\big)}{\sqrt{1-\rho^2_{n,ij,00}}},
$$

we have

$$
\begin{aligned}
&\mathrm{P}\Big(\bigcup_{(i,j)\in H_{m,n}} \{X_{n,ij} > u_{n^2}(x)\} \;\Big|\; X_{n,00} > u_{n^2}(x)\Big) \\
&\quad \sim \int_0^\infty \exp\Big(-z - \frac{z^2}{2u_{n^2}^2(x)}\Big) \\
&\qquad \times \mathrm{P}\Big(\bigcup_{(i,j)\in H_{m,n}} \Big\{Z_{n,ij} > \frac{u_{n^2}(x) - \rho_{n,ij,00}\big(u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\big)}{\sqrt{1-\rho^2_{n,ij,00}}}\Big\}\Big)\,\mathbf{d}z.
\end{aligned} \tag{4.8}
$$

Note that, by similar arguments to those used in the proof of [11, Lemma 7], for large enough m and n, we have

$$
\theta_n = \frac{u_{n^2}(x) - u_{n^2}(x)\rho_{n,ij,00}}{\sqrt{1-\rho^2_{n,ij,00}}} - \frac{z\rho_{n,ij,00}}{u_{n^2}(x)\sqrt{1-\rho^2_{n,ij,00}}} > 0,
$$

and by Mill's inequality,

$$
\mathrm{P}(Z_{n,ij} > \theta_n) \leqslant \frac{1}{\sqrt{2\pi}\,\theta_n} \exp\Big(-\frac{1}{2}\theta_n^2\Big).
$$

From the definition of $u_{n^2}(x)$,

$$\theta_n^2 \geqslant \frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}(4\log n - \log(2\log n)) + C,$$

and so

$$\begin{aligned}
&\mathrm{P}\biggl(Z_{n,ij} > \frac{u_{n^2}(x) - \rho_{n,ij,00}\bigl(u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\bigr)}{\sqrt{1-\rho^2_{n,ij,00}}}\biggr) \\
&\qquad = \mathrm{P}(Z_{n,ij} > \theta_n) \leqslant C\frac{1}{\sqrt{1-\rho^2_{n,ij,00}}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{(4.9)}\qquad &\mathrm{P}\biggl(\bigcup_{(i,j)\in H_{m,n}}\Bigl\{Z_{n,ij} > \frac{u_{n^2}(x) - \rho_{n,ij,00}\bigl(u_{n^2}(x) + \frac{z}{u_{n^2}(x)}\bigr)}{\sqrt{1-\rho^2_{n,ij,00}}}\Bigr\}\biggr) \\
&\qquad \leqslant C\sum_{(i,j)\in H_{m,n}} \frac{1}{\sqrt{1-\rho^2_{n,ij,00}}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}.
\end{aligned}$$

It follows from (2.4) that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}\frac{(\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}}}{\sqrt{1-\rho^2_{n,ij,00}}} = 0,$$

and we have $\rho_{n,ij,00} \to 0$ when $i > \ell_n$ or $j > \ell_n$ by (2.3). Thus

$$\sup_{i>\ell_n \text{ or } j>\ell_n} \frac{1}{\sqrt{1-\rho^2_{n,ij,00}}} n^{2-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \to 1,$$

and by (4.4),

$$\begin{aligned}
\text{(4.10)}\qquad &\sum_{(i,j)\in\{0,1,\dots,r_n\}^2\setminus\{0,1,\dots,\ell_n\}^2} \frac{1}{\sqrt{1-\rho^2_{n,ij,00}}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \\
&\leqslant \frac{(r_n+1)^2 - (\ell_n+1)^2}{n^2} \frac{1}{\sqrt{1-\rho^2_{n,ij,00}}} n^{2-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \to 0
\end{aligned}$$

as $n \to \infty$. Combining this with (2.4) and (4.10), we have

$$
\begin{aligned}
(4.11) \qquad & \sum_{(i,j)\in\{0,1,\dots,r_n\}^2\setminus\{0,1,\dots,m\}^2} \frac{1}{\sqrt{1-\rho_{n,ij,00}^2}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \\
& \leqslant \sum_{(i,j)\in\{0,1,\dots,r_n\}^2\setminus\{0,1,\dots,\ell_n\}^2} + \sum_{(i,j)\in\{0,1,\dots,\ell_n\}^2\setminus\{0,1,\dots,m\}^2} \\
& \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
2 & \sum_{(i,j)\in\{0,1,\dots,r_n\}^2\setminus\{0,1,\dots,m\}^2} \frac{1}{\sqrt{1-\rho_{n,ij,00}^2}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} \\
& \geqslant \sum_{(i,j)\in H_{m,n}} \frac{1}{\sqrt{1-\rho_{n,ij,00}^2}} n^{-2\frac{1-\rho_{n,ij,00}}{1+\rho_{n,ij,00}}} (\log n)^{-\frac{\rho_{n,ij,00}}{1+\rho_{n,ij,00}}},
\end{aligned}
$$

combining (4.8), (4.9) and (4.11) we get (4.5). By similar arguments, (4.6) can be established. ■

5. PROOFS

5.1. Proof of Theorem 2.1. By the definition of N_{n^2} and η_{st} for $s,t = 1,,\dots,t_n$, we have

$$
\begin{aligned}
(5.1) \qquad & \left|\mathrm{E}(\omega^{N_{n^2}}) - \mathrm{E}\big(\omega^{\sum_{s=1}^{t_n}\sum_{t=1}^{t_n}\eta_{ij}}\big)\right| \\
& \leqslant \mathrm{P}\Big(\bigcup_{(i,j)\in I'_{st}} \{X_{n,ij} > u_{n^2}(x)\},\ s,t = 1,\dots,t_n\Big) \\
& \leqslant Ct_n^2 s_n \ell_n \,\mathrm{P}(X_{n,00} > u_{n^2}(x)) \\
& = \frac{Ct_n^2 s_n \ell_n}{n^2} n^2(1-\Phi(u_{n^2}(x))) \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

because $n^2(1-\Phi(u_{n^2}(x))) \to e^{-x}$ and by (2.2). By Lemmas 4.1 and 4.4, we have

$$
\begin{aligned}
& \mathrm{P}(\eta_{11} = 1) \\
& = \mathrm{P}(M_{[\ell_n:s_n],[\ell_n:s_n]} > u_{n^2}(x)) \\
& = \sum_{i=\ell_n}^{s_n}\sum_{j=\ell_n}^{s_n} \mathrm{P}\big(X_{n,ij} > u_{n^2}(x),\ M_{[i+1:s_n],[j]} \vee M_{[\ell_n:s_n],[j+1,s_n]} \leqslant u_{n^2}(x)\big) \\
& = \sum_{i=0}^{s_n-\ell_n}\sum_{j=0}^{s_n-\ell_n} \mathrm{P}\big(X_{n,00} > u_{n^2}(x), \\
& \qquad\qquad M_{[1:s_n-\ell_n-i],[0]} \vee M_{[-i:s_n-\ell_n-i],[1,s_n-\ell_n-j]} \leqslant u_{n^2}(x)\big) \\
& \geqslant (s_n-\ell_n)^2\, \mathrm{P}\big(X_{n,00} > u_{n^2}(x),\ M(Q_n) \leqslant u_{n^2}(x)\big),
\end{aligned}
$$

where $Q_n=\{1,\dots,s_n-\ell_n\}\times\{0\}\cup\{-s_n,-s_n+1,\dots,s_n-\ell_n\}\times\{1,\dots,s_n-\ell_n\}$, and

$$\begin{aligned}\mathrm{E}(\omega^{\eta_{11}}) &= \omega\,\mathrm{P}(\eta_{11}=1)+\mathrm{P}(\eta_{11}=0)=1-(1-\omega)\,\mathrm{P}(\eta_{11}=1)\\ &\leqslant 1-(1-\omega)(s_n-\ell_n)^2\,\mathrm{P}(X_{n,00}>u_{n^2}(x),\,M(Q_n)\leqslant u_{n^2}(x)).\end{aligned}\tag{5.2}$$

Then, by Lemmas 4.2 and 4.4, we have

$$\begin{aligned}&\mathrm{E}(\omega^{N_{n^2}})\\ &\quad\leqslant [1-(1-\omega)(s_n-\ell_n)^2\,\mathrm{P}(X_{n,00}>u_{n^2}(x),\,M(A_n)\leqslant u_{n^2}(x))]^{t_n^2}\\ &\quad\sim \exp(-(1-\omega)\vartheta e^{-x}) \quad \text{as } n\to\infty.\end{aligned}\tag{5.3}$$

Next, to develop a lower bound for $\mathrm{E}(\omega^{N_{n^2}})$, let $c_n=s_n+\ell_n-1$ and $p_n=r_n+\ell_n$ with $s_n=o(n)$, $\ell_n=o(r_n)$ and $r_n=o(s_n)$. Denote $A_n^*=\{p_n+1,p_n+2,\dots,c_n-p_n\}^2$ and $A_n'=\{0,1,\dots,c_n\}^2/A_n^*$. It follows from Lemmas 4.1 and 4.4 that, for large enough n, we have

$$\begin{aligned}&\mathrm{P}(M_{c_n}>u_{n^2}(x))\\ &=\mathrm{P}(M(A_n')>u_{n^2}(x))+\mathrm{P}\big(M(A_n')\leqslant u_{n^2}(x),M(A_n^*)>u_{n^2}(x)\big)\\ &\leqslant \frac{Cp_nc_n}{n^2}n^2(1-\Phi(u_{n^2}(x)))+\sum_{i=p_n+1}^{c_n-p_n}\sum_{j=p_n+1}^{c_n-p_n}\mathrm{P}\big(X_{n,ij}>u_{n^2}(x),\\ &\qquad M_{[i+1:c_n-p_n],[j]}\vee M_{[p_n+1:c_n-p_n],[j+1,c_n-p_n]}\vee M(A_n')\leqslant u_{n^2}(x)\big)\\ &\quad\leqslant (c_n-p_n)^2\,\mathrm{P}\big(X_{n,00}>u_{n^2}(x),M(A_n)\leqslant u_{n^2}(x)\big)+o(1),\end{aligned}\tag{5.4}$$

where $A_n=\{1,\dots,r_n-\ell_n\}\times\{0\}\cup\{-r_n,-r_n+1,\dots,r_n-\ell_n\}\times\{1,\dots,r_n-\ell_n\}$, and so

$$\begin{aligned}\mathrm{P}\big(M_{[\ell_n:s_n],[\ell_n:s_n]}>u_{n^2}(x)\big)&\leqslant \mathrm{P}\big(M_{c_n}>u_{n^2}(x)\big)\\ &\leqslant (c_n-p_n)^2\,\mathrm{P}\big(X_{n,00}>u_{n^2}(x),\,M(A_n)\leqslant u_{n^2}(x)\big)+o(1).\end{aligned}\tag{5.5}$$

Therefore, by similar arguments to those used in (5.3) and (5.4), we have

$$\begin{aligned}&\mathrm{E}(\omega^{N_{n^2}})\\ &\quad\geqslant \big[1-(1-\omega)(c_n-p_n)^2\,\mathrm{P}\big(X_{n,00}>u_{n^2}(x),\,M(A_n)\leqslant u_{n^2}(x)\big)\big]^{t_n^2}\\ &\quad\sim \exp(-(1-\omega)\vartheta e^{-x}) \quad \text{as } n\to\infty,\end{aligned}\tag{5.6}$$

and the conclusion of Theorem 2.1 follows by (5.3) and (5.6). ■

5.2. Proof of Theorem 2.2. Set $X_{n,ij} = Y((i,j)/(\log n)^{1/\beta})$, $i, j = 1, \dots n$. The distance $\|\mathbf{h}\|$ between any two random variables $X_{n,ij}$ and $X_{n,kl}$ will have the form

$$\|\mathbf{h}\| = h_{ij,kl} = \frac{1}{(\log n)^{1/\beta}} \sqrt{(k-i)^2 + (l-j)^2}.$$

Thus $\rho_{n,ij,kl} = \rho_n(h_{ij,kl})$ and it follows by condition (i) that

$$\lim_{n\to\infty} (1 - \rho_{n,ij,kl}) \log n = \delta_{ij,kl} = [(k-i)^2 + (l-j)^2]^{\beta/2} \in (0, \infty).$$

Hence, (2.1) in Theorem 2.1 holds. To prove Theorem 2.2, we also need to verify conditions (2.3) and (2.4) of Theorem 2.1 under conditions (i) and (ii). The proofs are similar to those of [11, Theorem 2.2] and the details are omitted here. ■

Acknowledgments. The authors would like to thank the Editor-in-Chief, the Associated Editor and the referee for a careful reading and comments which greatly improved the paper. They also wish to express their appreciation to Professor Wan Tang for fruitful discussions on this topic.

Yingyin Lu gratefully acknowledges the financial support by State Scholarship Fund (CSC No. 202208515005) from the China Scholarship Council.

REFERENCES

[1] P. Abbrahamsen, *A review of Gaussian random fields and correlation functionals*, Norwegian Computing Center, Oslo, 1997.

[2] P. Albin, E. Hashorva, L. Ji and C. Ling, *Extremes and limit theorems for difference of chi-type processes*, ESAIM Probab. Statist. 20 (2016), 349–366.

[3] M. Alodat, *An approximation to cluster size distribution of two Gaussian random fields conjunction with application to FMRI data*, J. Statist. Planning Inference 141 (2011), 2331–2347.

[4] J. Azaïs, S. Bercum, J. Fort, A. Lagnoux and P. Lé, *Simultaneous confidence bands in curve prediction applied to load curves*, J. Roy. Statist. Soc. Ser. C 59 (2010), 889–904.

[5] S. M. Berman, *Asymptotic independence of the numbers of high and low level crossings of stationary Gaussian processes*, Ann. Math. Statist. 42 (1971), 927–945.

[6] B. Brown and S. Resnick, *Extreme values of independent stochastic processes*, J. Appl. Probab. 14 (1997), 732–739.

[7] J. Cao and K. Worsley, *Applications of random fields in human brain mapping*, in: M. Moore (ed.), Spatial Statistics: Methodological Aspects and Applications, Springer, New York, 2001, 169–182.

[8] B. Das, S. Engelke and E. Hashorva, *Extremal behavior of squared Bessel processes attracted by the Brown–Resnick process*, Stochastic Process. Appl. 125 (2015), 780–796.

[9] K. Dębicki, E. Hashorva, L. Ji and C. Ling, *Extremes of order statistics of stationary processes*, TEST 24 (2015), 229–248.

[10] J. French, *Confidence regions for level curves and a limit theorem for the maxima of Gaussian random fields*, Ph.D. thesis, Colorado State Univ., Fort Collins, CO, 2009.

[11] J. P. French and R. A. Davis, *The asymptotic distribution of the maxima of a Gaussian random field on a lattice*, Extremes 16 (2013), 1–26.

[12] J. Guo and Y. Lu, *Joint behavior of point processes of clusters and partial sums for stationary bivariate Gaussian triangular arrays*, Ann. Inst. Statist. Math. 75 (2023), 17–37.

[13] E. Hashorva, L. Peng and Z. Weng, *Maxima of a triangular array of multivariate Gaussian sequence*, Statist. Probab. Lett. 103 (2015), 62–72.
[14] E. Hashorva and Z. Weng, *Limit laws for extremes of dependent stationary Gaussian arrays*, Statist. Probab. Lett. 83 (2013), 320–330.
[15] T. Hsing, J. Hüsler and R.-D. Reiss, *The extremes of a triangular array of normal random variables*, Ann. Appl. Probab. 6 (1996), 671–686.
[16] Z. Kabluchko, M. Schlather and L. De Haan, *Stationary max-stable fields associated to negative definite functions*, Ann. Probab. 37 (2009), 2042–2065.
[17] M. R. Leadbetter, *Extremes and local dependence in stationary sequences*, Z. Wahrsch. Verw. Gebiete 65 (1983), 291–306.
[18] M. R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and Related Properties of Stationary Sequences and Processes*, Springer, New York, 1983.
[19] C. Ling, *Extremes of stationary random fields on a lattice*, Extremes 22 (2019), 391–411.
[20] M. Oesting, Z. Kabluchko and M. Schlather, *Simulation of Brown–Resnick processes*, Extremes 15 (2012), 89–107.
[21] M. Pacifico, C. Genovese, I. Verdinelli and L. Wasserman, *False discovery control for random fields*, Journal of the American Statistical Association, 99 (2004), 1002–1014.
[22] L. Pereira, A. Martins and H. Ferreira, *Clustering of high values in random fields*, Extremes 20 (2017), 807–836.
[23] J. Taylor, K. Worsley and F. Gosselin, *Maxima of discretely sampled random fields, with an application to 'bubbles'*, Biometrika 94 (2007), 1–18.
[24] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference*, Springer, 2004.

Yingyin Lu
School of Science
Southwest Petroleum University
Chengdu, 610500, P.R. China
E-mail: luyingyin93@163.com

Jinhui Guo
School of Statistics
Southwestern University of Finance and Economics
Chengdu, 611130, P.R. China
E-mail: guojinhui94@163.com

Received 16.10.2023;
accepted 9.2.2024