On applications of Boolean cumulants in free probability

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joint work with:
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- $\varphi(I_{\mathcal{A}}) = 1$,
- $\varphi(aa^*) \geq 0 \ \forall a \in \mathcal{A}$.

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- $a_j\in\mathcal{A}_{i(j)}$ *and neighbours are from different subalgebras, i.e.:*

$$i(1)\neq i(2), i(2)\neq i(3),\ldots,i(k-1)\neq i(k)$$

Set partitions

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Partition π is called *noncrossing* if any ordered quadruple $p_1 < q_1 < p_2 < q_2$ cannot satisfy $p_1 \sim_\pi p_2$ and $q_1 \sim_\pi q_2$ unless p_1, p_2, q_1, q_2 are in the same block of π.

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Partition π is called *interval partition* if every block is an interval, i.e., $i \sim_\pi k$ and $i < j < k$ then i, j, k are in the same block of π.

Examples


noncrossing


interval

Cumulants

Free cumulants are defined recursively via

$$\varphi(X_1 \cdots X_n) = \sum_{\pi \in NC(n)} \kappa_\pi(X_1, \dots, X_n).$$

For example $\varphi(X) = \kappa_1(X)$ and $\kappa_2(X_1, X_2) = \varphi(X_1, X_2) - \varphi(X_1)\varphi(X_2)$.

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Boolean cumulants are defined via

$$\varphi(X_1 \cdots X_n) = \sum_{\pi \in Int(n)} \beta_\pi(X_1, \dots, X_n).$$

Freeness in terms of free cumulants

Theorem (Speicher)

Random variables $X_1, \ldots, X_n$ *are free if and only if for any* $k \geq 2$ *we have* $\kappa_k\left(X_{i_1}, \ldots, X_{i_k}\right) = 0$ *whenever at least two different indices appear.*

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$$NC(n; c) := \{\pi \in NC(n) \mid \ c \text{ is constant on every block of } \pi\}.$$

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2. For a fixed partition and a non outer block A we will denote by $\mathrm{Parent}_\pi(A)$ the block B such that A is nested inside B and if A is nested in a different block C then B is also nested in C.
3. We wil say that $\pi \in NC(n; c)$ has *vertical no-repeat* property (vnrp for short) when whenever $A \in \pi$ is not an outer block, one has
$$c(\,\mathrm{Parent}_\pi(A)\,) \neq c(A).$$

Freeness in terms of boolean cumulants cumulants

Theorem (Fevrier, Mastnak, Nica, Sz.)

Consider $\mathcal{A}_1, \ldots, \mathcal{A}_s \subset \mathcal{A}$ *unital subalgebras. Let* $n \geq 2$ *be an integer and let* $X_1, \ldots, X_n \in \mathcal{A}$ *be such that* $X_1 \in \mathcal{A}_{i(1)}, \ldots, X_n \in \mathcal{A}_{i(n)}$*. Define a colouring* $c(k) = i(k)$ *for* $k = 1, \ldots, n$*. Then* $\mathcal{A}_1, \ldots, \mathcal{A}_s$ *are free if and only if*

$$\beta_n(X_1, \ldots, X_n) = \sum_{\substack{\pi \in NC(n,c),\ \pi \ll 1_n \\ \pi \text{ has vnrp}}} \beta_\pi(X_1, \ldots X_n).$$

Distribution of an anti-commutator

The description above, is the main ingredient which allows to find combinatorial description of Boolean cumulants of $XY + YX$ as well as power series equation for Boolean cumulants generating function.

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$$\begin{bmatrix} f_{XX} & f_{X,X^*} \\ f_{X^*,X} & f_{X^*,X^*} \end{bmatrix}$$
$$= z\widetilde{\eta}_X\left(z\begin{bmatrix} f_{X^*X^*}(1-f_{XX^*})^{-1} & f_{X^*,X} + f_{X^*,X^*}(1-f_{X,X^*})^{-1}f_{X,X} \\ (1-f_{X,X^*})^{-1} & (1-f_{X,X^*})^{-1}f_{X,X} \end{bmatrix}\right).$$

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Then

$$\eta_{XY+YX}(z^2)=2\left(f_{X,X^*}(z)+\frac{f_{X,X}(z)f_{X^*,X^*}(z)}{1-f^2_{X^*,X}(z)}\right).$$

Examples

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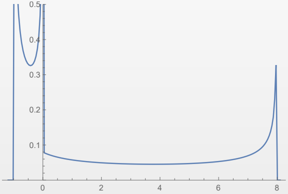
Solution2.pdf



(a) Plot of the density μ_{XY+YX}

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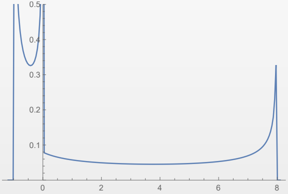

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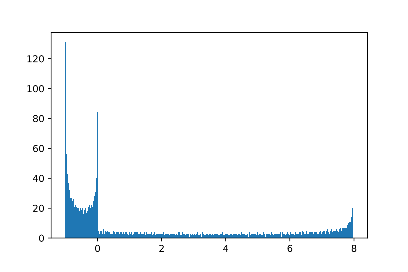

Moment-cumulant formula

Lemma (Lehner,Sz.)

Let $(X_1, \dots, X_n)$ *and* $(Y_1, \dots, Y_n)$ *be mutually free families in an ncps* $(\mathcal{A}, \varphi)$*, then*

$$\varphi\left(X_1 Y_1 \dots X_n Y_n\right) = \sum_{k=0}^{n-1} \sum_{0<i_1<\dots<i_k<n} \varphi\left(Y_{i_1} \dots Y_{i_k} Y_n\right) \cdot \prod_{j=0}^{k} \beta_{2(i_{j+1}-i_j)-1}(X_{i_j+1}, Y_{i_j+1}, \dots, X_{i_{j+1}}),$$

Subordination

Theorem (Biane)

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$$\mathbb{E}_\varphi\left[zX^{1/2}YX^{1/2}/(1 - zX^{1/2}YX^{1/2})^{-1} \mid X\right] = F(z)X(1 - F(z)X)^{-1},$$

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We have of course $G_{X+Y}(z) = G_X(\omega(z))$.

Subordination functions and Boolean cumulants

Proposition

(i) *If X, Y are bounded free random variables then the additive subordination function ω has the expansion*

$$\omega(z) = z - \sum_{n=0}^{\infty} \beta_{2n+1}(Y, (z-X)^{-1}, \ldots, (z-X)^{-1}, Y)$$

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2. *If X and Y are positive bounded free random variables then the multiplicative subordination function F has the expansion*

$$F(z) = \sum_{n=0}^{\infty} \beta_{2n+1}(Y, X, \dots, X, Y) z^{n+1}$$

in some neighbourhood of 0*.*

Distribution of $X + f(X)Yf^*(X)$

Theorem (Lehner, Sz.)

Let X, Y be free and bounded, then

$$\mathbb{E}_\varphi \left((z - X - f(X)Yf^*(X))^{-1} | X \right) = (z - X - \omega(z)|f(X)|^2)^{-1},$$

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Remark

To find the distribution of $X + f(X)Yf^*(X)$ it suffices to determine $\omega(z)$ and calculate the integral

$$G_{X+f(X)Yf^*(X)}(z) = \int_{\mathbb{R}} \frac{1}{z - x - \omega(z)|f(x)|^2} d\mu_X(x).$$

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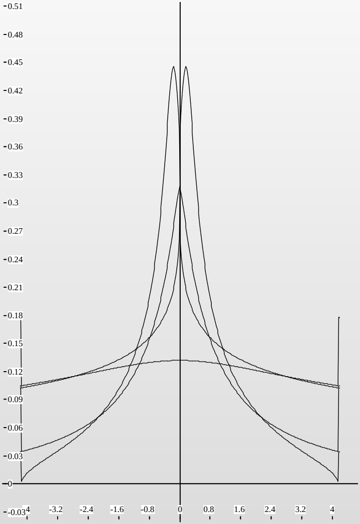

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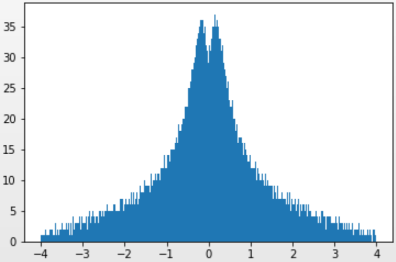

(a) Histogram of the spectrum of a 4000×4000 random matrix model for $X + XYX$

Figure: The spectral density of $X + XYX$

Further applications

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where

$$B_1(z) = \frac{\eta_U(\omega_2(z)) - \eta_U(1)}{\omega_2(z)-1}\varphi\left((1-U)^{-1}\right),$$

$$B_2(z) = \frac{\omega_2(z)(\eta_U(\omega_2(z)) - \eta_U(1) - (\omega_2(z)-1)\eta_U'(1))}{(\omega_2(z)-1)^2}\varphi^2\left((1-U)^{-1}\right).$$

and $M_{UV}(z) = M_V(\omega_1(z))$ *and* $M_{UV}(z) = M_U(\omega_2(z))$.

Thank you for your attention!