Characterizations of free Meixner distributions

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Definition via Jacobi parameters.

$\beta, \gamma, b, c \in \mathbb{R}$, $1+\gamma, 1+c \geq 0$.

$$\{(\beta, b, b, \ldots), (1+\gamma, 1+c, 1+c, \ldots)\}.$$

Tridiagonal matrix

$$J = \begin{pmatrix} \beta & 1+\gamma & 0 & 0 & 0 \\ 1 & b & 1+c & 0 & 0 \\ 0 & 1 & b & 1+c & 0 \\ 0 & 0 & 1 & b & 1+c \\ 0 & 0 & 0 & 1 & b \end{pmatrix}.$$

Definition via Jacobi parameters.

Definition.

A free Meixner distribution with parameters β, γ, b, c is the measure with moments

$$\mu[x^n] = \langle e_0, J^n e_0 \rangle.$$

[Cohen-Trenholme '84, etc.]

$$J = \begin{pmatrix} \beta & 1+\gamma & 0 & 0 & 0 \\ 1 & b & 1+c & 0 & 0 \\ 0 & 1 & b & 1+c & 0 \\ 0 & 0 & 1 & b & 1+c \\ 0 & 0 & 0 & 1 & b \end{pmatrix}.$$

Examples.

Semicircular distribution $\{(0,0,0,\ldots),(t,t,t,\ldots)\}$.

Marchenko-Pastur distributions $\{(\beta,b,b,\ldots),(t,t,t,\ldots)\}$.

Kesten measures
$\left\{(0,0,0,\ldots),\left(\frac{1}{2n},\frac{1}{2n}\left(1-\frac{1}{2n}\right),\frac{1}{2n}\left(1-\frac{1}{2n}\right),\ldots\right)\right\}$.

Bernoulli distributions $\{(\beta,b,b,\ldots),(\gamma,0,0,\ldots)\}$.

Explicit formula.

Can always re-scale to have mean $\beta = 0$ and variance $1 + \gamma = 1$.

Then

$$\frac{1}{2\pi} \cdot \frac{\sqrt{4(1+c) - (x-b)^2}}{1 + bx + cx^2} \mathbf{1}_{[b-2\sqrt{1+c}, b+2\sqrt{1+c}]} \, dx$$

$+0, 1, 2$ atoms.

Why interesting?

Many limit theorems in non-commutative probability involve free Meixner distributions: [B,B,F,H,K,L,L,M,M,N,O,S,W,W,Y, . . .]

Free semigroups.

$$(\mu_{b,c})^{\boxplus t} = \begin{pmatrix} 0 & t & 0 & 0 & 0 \\ 1 & b & c+t & 0 & 0 \\ 0 & 1 & b & c+t & 0 \\ 0 & 0 & 1 & b & c+t \\ 0 & 0 & 0 & 1 & b \end{pmatrix}.$$

[Saitoh, Yoshida ’01] [A ’03] [Bożejko, Bryc ’06]

Boolean semigroups.

$$(\mu_{b,c})^{\uplus t} = \begin{pmatrix} 0 & t & 0 & 0 & 0 \\ 1 & b & c & 0 & 0 \\ 0 & 1 & b & c & 0 \\ 0 & 0 & 1 & b & c \\ 0 & 0 & 0 & 1 & b \end{pmatrix}.$$

[Bożejko, Wysoczański '01] [A '09]

$\mathbb{B}_t$-semigroups.

$$\mathbb{B}_t(\mu_{b,c,\beta,\gamma}) = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 \\ 1 & b & c+t & 0 & 0 \\ 0 & 1 & b & c+t & 0 \\ 0 & 0 & 1 & b & c+t \\ 0 & 0 & 0 & 1 & b \end{pmatrix}.$$

[Belinschi, Nica ’08], [A ’09]

Two-state free semigroups.

If

$$\nu(t) = \begin{pmatrix} \beta t & \gamma t & 0 & 0 & 0 \\ 1 & b+\beta t & c+\gamma t & 0 & 0 \\ 0 & 1 & b+\beta t & c+\gamma t & 0 \\ 0 & 0 & 1 & b+\beta t & c+\gamma t \\ 0 & 0 & 0 & 1 & b+\beta t \end{pmatrix},$$

$$\mu(t) = \begin{pmatrix} 0 & t & 0 & 0 & 0 \\ 1 & b+\beta t & c+\gamma t & 0 & 0 \\ 0 & 1 & b+\beta t & c+\gamma t & 0 \\ 0 & 0 & 1 & b+\beta t & c+\gamma t \\ 0 & 0 & 0 & 1 & b+\beta t \end{pmatrix}$$

then $(\mu(t), \nu(t))$ is a c-free semigroup.

Characterization: linear Jacobi parameters.

Recall:

$$(\mu_{b,c})^{\boxplus t} = \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & b & c+t & 0 \\ 0 & 1 & b & c+t \\ 0 & 0 & 1 & b \end{pmatrix}.$$

Proposition.

$$\mu^{\boxplus t} = \begin{pmatrix} \beta_0 + b_0 t & \gamma_1 + c_1 t & 0 & 0 \\ 1 & \beta_1 + b_1 t & \gamma_2 + c_2 t & 0 \\ 0 & 1 & \beta_2 + b_2 t & \gamma_3 + c_3 t \\ 0 & 0 & 1 & \beta_3 + b_3 t \end{pmatrix}.$$

if and only if μ is a free Meixner distribution.

A corollary of recent work of Młotkowski.

Characterization: orthogonal polynomials.

For a measure μ, denote by $\{P_n(x)\}$ its monic orthogonal polynomials, and

$$H(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n$$

their generating function.

Proposition. [A '03]

μ is a free Meixner distribution if and only if

the generating function of monic orthogonal polynomials of μ has the special form

$$H(x,z) = \frac{A(z)}{1 - xB(z)}.$$

Characterization: algebraic Riccati equations.

Denote

$$Df(z) = \frac{f(z) - f(0)}{z}.$$

Let $R(z)$ be the R-transform (free cumulant generating function) of μ.

Proposition. [A '07]

μ is a free Meixner distribution if and only if

$$D^2 R(z) = 1 + bDR(z) + c(DR(z))^2.$$

Characterization: algebraic Riccati equations.

Let $\eta(z)$ be the η-transform (Boolean cumulant generating function) of μ.

Proposition. [A '09]

μ is a free Meixner distribution if and only if

$$D^2\eta(z) = 1 + bD\eta(z) + (1 + c)(D\eta(z))^2.$$

Characterization: free Laha-Lukacs property.

Let $(\mathcal{A}, \mathrm{E})$ be a non-commutative probability space, $X, Y \in \mathcal{A}$.

Theorem. [Bożejko, Bryc '06]

Let X, Y be freely independent.

X, Y have free Meixner distributions if and only if

$$\mathrm{E}[X|X+Y]$$

is a linear function of $X+Y$ and

$$\mathrm{Var}[X|X+Y]$$

is a quadratic function of $X+Y$.

Reverse martingales.

$\{X_t\}$ a process with freely independent increments.

$\{P_n(x,t)\}$ polynomials, P_n of degree n.

P_n a martingale if for $s < t$,

$$\mathrm{E}[P_n(X_t,t)|X_s] = P_n(X_s,s).$$

P_n a reverse martingale if for $s < t$,

$$\mathrm{E}[P_n(X_s,s)|X_t] = \frac{f(s)}{f(t)} \cdot P_n(X_t,t).$$

Characterization: reverse martingales.

Proposition.

Each X_s has a free Meixner distribution if and only if

there is a family of polynomials P_n which are both martingales and reverse martingales.

Characterization: free Jacobi fields.

[Bożejko, Lytvynov '08+]

See the next talk.

The difference quotient.

$\partial =$ the difference quotient operation,

$$\partial f(x, y) = \frac{f(x) - f(y)}{x - y}.$$

For a measure ν, define the operator

$$L_\nu[f] = (I \otimes \nu)[\partial f].$$

Maps polynomials to polynomials, lowers degree by one.

Characterization: Orthogonality.

Proposition. [Very classical; Darboux?]

Let ν be a measure, $\{P_n\}$ orthogonal polynomials for it. Then

$$L_\nu[P_n] = Q_{n-1}$$

are always orthogonal with respect to some μ.

Fixed points.

Proposition. [A '08+]

Let ν be a measure,

$$L_\nu[A_n] = A_{n-1}.$$

$\{A_n\}$ are orthogonal with respect to some μ

if and only if μ has a free Meixner distribution.

Characterization: Bochner-Pearson.

Theorem. [A. 09+]

μ has a free Meixner distribution if and only if the operator

$$p(x)L_\mu^2 + q(x)L_\mu$$

has polynomial eigenfunctions for some polynomial p, q.

Free exponential families.

A free exponential family with variance function V is a family of **probability** measures

$$\left\{ \frac{V(m)}{V(m) + (m - m_0)(m - x)} \nu(dx) \right\}.$$

Proposition. [Bryc, Ismail '06+, Bryc '09]

Free Meixner distributions

generate free exponential families with quadratic variance functions: $V(m) = 1 + bm + cm^2$.

Free multiplicative properties.

Recall: four parameters b, c and β, γ.
If β is a root of

$$\gamma - bx + cx^2$$

then the S-transform of μ is a rational function

$$S(z) = \frac{cz + 1}{(2c\beta - b)z + \beta}.$$

Quadratic harnesses.

$\{X_t\} =$ stochastic process. No conditions on increments.

$\{X_t\}$ is a quadratic harness if for $s < t < u$,

$$\mathrm{E}[X_t | \leq s \text{ and } \geq u]$$

is a linear function of X_s, X_u and

$$\mathrm{Var}[X_t | \leq s \text{ and } \geq u]$$

is a quadratic function of X_s, X_u.

Investigated in detail by Bryc, Matysiak, Szabłowski, and Wesołowski.

Quadratic harnesses.

Theorem. [Bryc, Wesołowski '05]

A quadratic harness with the extra property

$$\mathrm{E}[X_t^2 - t | \leq s] = X_s^2 - s$$

is a q-Meixner process.

The distributions for $q = 0$ are precisely all the free Meixner distributions. In that case the process is a classical version of a process with freely independent increments.

Further results.

In the orthogonal polynomials characterization,

Boolean Meixner class $=$ free Meixner class $=$ two-state free Meixner class [A '08+, A '09].

In the Laha-Lukacs characterization,

Boolean Meixner class $=$ only Bernoulli distributions [A '09]

Two-state Laha-Lukacs class described in [Bożejko, Bryc '09].

Many limit theorems in non-commutative probability involve free Meixner distributions: [B,B,F,H,K,L,L,M,M,N,O,S,W,W,Y, $\ldots$]

Other appearances of classical Meixner distributions: representation theory, linearization coefficients.