Noncommutative analogs of symmetric polynomials

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Colloq. Math. **LXVI** (1993), 95-107.

The commutative case

- **Symmetric function (polynomial)** f of n variables $x_1, x_2, \ldots, x_n$ and with the coefficients from a commutative ring K with a unit:

$$f\left(x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}\right) = f\left(x_{\sigma(1)}^{j_1} x_{\sigma(2)}^{j_2} \ldots x_{\sigma(n)}^{j_n}\right)$$

  for every permutation σ of $\{1, 2, \ldots, n\}$ and integers (natural numbers for polynomials) j_s.

- **Elementary symmetric polynomials**

$$e_k = \sum_{i_s \neq i_t} x_{i_1} x_{i_2} \ldots x_{i_k} \ ,$$

  where $k \in \mathbb{N}_+$, form an algebraical basis of the algebra of symmetric polynomials.

- Another algebraical basis is given by power polynomials

$$p_k = \sum_{i=1}^{n} x_i^k,$$

  where $k \in \mathbb{N}_+$.

- The following **Newton's identities** hold:

$$(-1)^m m e_m + \sum_{k=0}^{m-1} (-1)^k e_k p_{m-k} = 0$$

  for $m \in \mathbb{N}_+$.

  For $m = n$ and $p_0 = n$ we obtain

$$\sum_{k=0}^{n} (-1)^k e_k p_{n-k} = 0$$

The noncommutative case

Problems with a definition of a symmetric function: is the polynomial

$$x_1^2 x_2 + x_2^2 x_1$$

symmetric for $n = 2$ or is it better to consider

$$x_1^2 x_2 + x_2^2 x_1 + x_1 x_2^2 + x_2 x_1^2 \ ?$$

What about $n > 2$ and the coefficients e. g. in

$$x_1 x_2 x_3 \text{ and in } x_1 x_2 x_1 \ ?$$

The first generalization

- The first **definition of a symmetric function**:

$$f\left(x_{i_1}^{j_1} x_{i_2}^{j_2} \ldots x_{i_k}^{j_k}\right) = f\left(x_{l_1}^{j_1} x_{l_2}^{j_2} \ldots x_{l_k}^{j_k}\right)$$

  for non–zero integers j_s, where $i_s \neq i_{s+1}, l_s \neq l_{s+1}$.

- Vector spaces spanned by such symmetric functions are algebras (we denote them by A in the case of rational functions and by B in the case of polynomials).

- We put **(analogs of elementary symmetric polynomials)**

$$E_k = \sum_{i_s \neq i_{s+1}} x_{i_1}^{\mathrm{sgn}(k)} x_{i_2}^{\mathrm{sgn}(k)} \ldots x_{i_{|k|}}^{\mathrm{sgn}(k)} ,$$

$$P_k = \sum_{i=1}^{n} x_i^k$$

  for non-zero integers k. We put $E_0 = P_0 = 1$.

- The following **analogs of Newton's identities** hold:

$$\sum_{k=0}^{|m|}(-1)^k E_{\mathrm{sgn}(m)k} P_{\mathrm{sgn}(m)(|m|-k)} = 0,$$

$$\sum_{k=0}^{|m|}(-1)^k P_{\mathrm{sgn}(m)k} E_{\mathrm{sgn}(m)(|m|-k)} = 0$$

for $m \in \mathbb{Z} \setminus \{0\}$.

- Each of the sets

$$\{E_k : k \in \mathbb{Z} \setminus \{0\}\},\ \{P_k : k \in \mathbb{Z} \setminus \{0\}\}$$

is an **algebraical basis** of A, each of the sets

$$\{E_k : k \in \mathbb{N}_+\},\ \{P_k : k \in \mathbb{N}_+\}$$

is an **algebraical basis** of B.

The second generalization

- The second **definition of a symmetric function**:

$$f\left(x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_k}^{j_k}\right) = f\left(x_{l_1}^{\varepsilon j_1} x_{l_2}^{\varepsilon j_2} \dots x_{l_k}^{\varepsilon j_k}\right)$$

for non–zero integers $j_s,\ \varepsilon \in \{-1, 1\}$, where $i_s \neq i_{s+1}, l_s \neq l_{s+1}$.

- Vector space spanned by such symmetric functions is an algebra and we denote it by C.

- We put **(the second version of analogs of elementary symmetric polynomials)**

$$E(\varepsilon_m) = \sum_{i_s \neq i_{s+1}} x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k} + \sum_{i_s \neq i_{s+1}} x_{i_1}^{-\varepsilon_1} x_{i_2}^{-\varepsilon_2} \dots x_{i_k}^{-\varepsilon_k}\ ,$$

for $k \in \mathbb{N}_+, (\varepsilon_m) \in \{-1, 1\}^k$.

Such functions form an algebraical basis of C.

The third generalization

- The third **definition of a symmetric function**:

$$f\left(x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_k}^{j_k}\right) = f\left(x_{l_1}^{\epsilon_1 j_1} x_{l_2}^{\epsilon_2 j_2} \dots x_{l_k}^{\epsilon_k j_k}\right)$$

  for non–zero integers $j_s, k \in \mathbb{N}_+, (\varepsilon_m) \in \{-1,1\}^k$, where $i_s \neq i_{s+1}, l_s \neq l_{s+1}$.

- Vector space spanned by such symmetric functions is an algebra and we denote it by D.

- We put **(the third version of analogs of elementary symmetric polynomials)**

$$E_{k,D} = \sum_{(\varepsilon_m)\in\{-1,1\}^k, i_s \neq i_{s+1}} x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k}$$

  for $k \in \mathbb{N}_+$.

  Such functions form an algebraical basis of D.

  Another algebraical basis is given by $P_{k,D} = \sum\limits_{m=0}^{n} x_m^k + \sum\limits_{m=0}^{n} x_m^{-k}$.

The methods

To calculate the coefficients in products of symmetric functions we use the function $L : \mathbb{N} \times \mathbb{Z}^4 \to K$, given by

$$L(a,b,c,d,e) = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } ab \neq 0 \\ (n-1)^{a-1} & \text{if } b = 0 \text{ and } ac \neq 0 \\ (n-2)(n-1)^{a-1} & \text{if } b = c = 0 \text{ and } a \neq 0 \neq de \\ (n-1)^{a} & \text{if } b = c = de = 0 \text{ and } a\left(d^2+e^2\right) \neq 0 \\ n(n-1)^(a-1) & \text{if } b = c = d = e = 0 \text{ and } a \neq 0. \end{cases}$$

We introduce useful injections $I_1 : (\mathbb{Z} \setminus \{0\})_\infty \to (\mathbb{Z} \setminus \{0\})_\infty$, $I_2 : (\{-1,1\}_\infty)_\infty \to (\{-1,1\}_\infty)_\infty$ by inductions.

An important role is played by the introduced three linear orderings on $(\mathbb{Z} \setminus \{0\})_\infty$ and a linear ordering on $(\{-1,1\}_\infty)_\infty$.