Interaction of a three-level atom with a single-mode field in a two photon resonant cavity

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1 Objective

- We extend Jaynes - Cummings model by adding a further atomic level to support a second resonance and cooperative effects in multi-atom systems.
- A successive passage of a three-level atom in the V configuration interacting with one quantized mode of electromagnetic field in a cavity is considered to study atomic inversion and entropy evolution of the state.

2 Basic Model

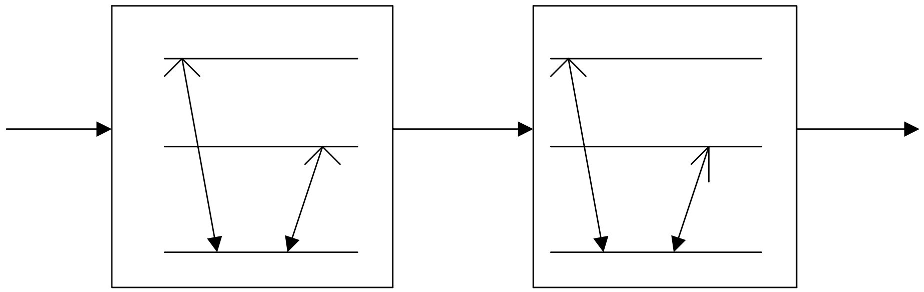


Figure 1: A successive passage of a three-level atom in the V configuration interacting with one mode of electromagnetic field.

3 Time Evolution of the State

- We consider a V-type three-level atomic system which consists of two allowed transitions

$$|e\rangle \leftrightarrow |g\rangle \text{and } |i\rangle \leftrightarrow |g\rangle$$

where $|e\rangle$, $|i\rangle$ and $|g\rangle$ are excited state, intermediate state and ground state respectively.

- Each interaction has same mode of the field.
- In the rotating-wave approximation, its Hamiltonian is described by

$$H = H_0 + H_1, \tag{1}$$

- where

$$H_0 = \omega_e|e\rangle\langle e| + \omega_i|i\rangle\langle i| + \omega_g|g\rangle\langle g| + \gamma a^\dagger a \ (\hbar = 1), \qquad (2)$$

- and

$$H_1 = g_1 a|e\rangle\langle g| + g_1 a^\dagger|g\rangle\langle e| + g_2 a|i\rangle\langle g| + g_2 a^\dagger|g\rangle\langle i|. \qquad (3)$$

- Here $a^\dagger$ and a are, respectively, the creation and annihilation operators for the field of frequency γ.
- $|m\rangle(m = e, i, g)$ are the eigenstates of the atom with eigenfrequencies $\omega_m(m = e, i, g)$, and $g_r(r = 1, 2)$ are the corresponding coupling constants.
- We assume the coupling constants g_1 and g_2 to be real throughout the work.

- In the interaction picture, the state vector of this atom-field coupling system at time t can be described by

$$|\psi^I(t)\rangle = \sum_{n,n}(C_{e,n,n}|e,n,n\rangle + C_{i,n,n}|i,n,n\rangle + C_{g,n,n}|g,n,n\rangle). \tag{4}$$

- To get the interaction Hamiltonian we have

$$
\begin{aligned}
V &= e^{jH_0t}H_1e^{-jH_0t} \\
&= H_1 + jt[H_0, H_1] + \frac{(jt)^2}{2!}[H_0, [H_0, H_1]] \\
&\quad + \frac{(jt)^3}{3!}[H_0, [H_0, [H_0, H_1]]] + \ldots \\
&= g_1 a|e\rangle\langle g|e^{jt(\omega_e-\omega_g-\gamma)} + g_1 a^\dagger|g\rangle\langle e|e^{-jt(\omega_e-\omega_g-\gamma)} + \\
&\quad g_2 a|i\rangle\langle g|e^{jt(\omega_i-\omega_g-\gamma)} + g_2 a^\dagger|g\rangle\langle i|e^{-jt(\omega_i-\omega_g-\gamma)}
\end{aligned}
\tag{5}
$$

- We assume here

$$\triangle \equiv \omega_e - \omega_g - \gamma = \omega_i - \omega_g - \gamma.$$

- Then

$$V = g_1 a|e\rangle\langle g|e^{j\triangle t} + g_1 a^\dagger|g\rangle\langle e|e^{-j\triangle t} + g_2 a|i\rangle\langle g|e^{j\triangle t} + g_2 a^\dagger|g\rangle\langle i|e^{-j\triangle t} \quad (6)$$

- Substituting equation(4) in the Schrödinger equation in the interaction picture

$$j\frac{d}{dt}|\psi^I(t)\rangle = V|\psi^I(t)\rangle, \quad (7)$$

- we get from (4), (6) and (7)

$$\dot{C}_{e,n-1,n} = -jg_1\sqrt{n}e^{j\triangle t}C_{g,n,n} \quad (8)$$

$$\dot{C}_{i,n,n-1} = -jg_2\sqrt{n}e^{j\triangle t}C_{g,n,n} \quad (9)$$

$$\dot{C}_{g,n,n} = -j(g_1\sqrt{n}C_{e,n-1,n} + g_2\sqrt{n}C_{i,n,n-1})e^{-j\triangle t} \quad (10)$$

- If the atom is initially in the state $|\psi_A(0)>$,

$$|\psi_A(0)>= \cos\frac{\varphi}{2}|e> + \sin\frac{\varphi}{2}e^{-i\psi}|i> \tag{11}$$

which means that the atom is in the coherent superposition state of its eigenkets $|e>$ and $|i>$,

- and the field is in the superposition of the photon number states at time $t = 0$[2]

$$|\psi_f(0)>= \sum_{n,n} F_{n,n}|n,n>, \tag{12}$$

where $\sum_{n,n}|F_{n,n}|^2 = 1$,

- then the state vector of the total system at $t = 0$ can be described as

$$|\psi(0)>= \sum_{n,n}[\cos\frac{\varphi}{2}F_{n-1,n}|e,n-1,n> + \sin\frac{\varphi}{2}e^{-j\psi}F_{n,n-1}|i,n,n-1> \quad (13)$$

- We now assume that

$$\varphi = 90^0 \text{ and } \psi = 0.$$

Also

$$F_{n,n} \approx F_{n-1,n} \approx F_{n,n-1}.$$

- Then the state vector (13) of the total system at $t = 0$ reduces to

$$|\psi(0) >= \sum_{n,n} \frac{1}{\sqrt{2}} F_{n,n}(|e,n,n> + |i,n,n>) \tag{14}$$

- On solving equations (8), (9) and (10) with the initial condition (14) we get

$$C_{g,n,n}(t) = -2B_1 j e^{-j\frac{\triangle t}{2}} \sin \beta t \tag{15}$$

- where

$$B_1 = \frac{\sqrt{n} F_{n,n}(g + g_2)}{2\sqrt{2}\beta} \tag{16}$$

and $\beta = \frac{1}{2}\sqrt{\Delta^2 + 4n(g_1^2 + g_2^2)}$.

- Also

$$C_{e,n-1,n}(t) = -g_1\sqrt{n}B_1[\frac{e^{j(\triangle/2+\beta)t}-1}{(\triangle/2+\beta)} - \frac{e^{j(\triangle/2-\beta)t}-1}{(\triangle/2-\beta)}] + \cos\frac{\varphi}{2}F_{n-1,n}$$
$$\approx -g_1\sqrt{n}B_1\{\frac{e^{j\frac{\triangle t}{2}}[j\Delta\sin\beta t - 2\beta\cos\beta t]}{\Delta^2/4-\beta^2} + \frac{2\beta}{\Delta^2/4-\beta^2}\}$$
$$+\frac{1}{\sqrt{2}}F_{n,n} \tag{17}$$

- and

$$C_{i,n,n-1}(t) = -g_2\sqrt{n}B_1[\frac{e^{j(\triangle/2+\beta)t}-1}{(\triangle/2+\beta)} - \frac{e^{j(\triangle/2-\beta)t}-1}{(\triangle/2-\beta)}]$$
$$+\sin\frac{\varphi}{2}e^{-j\psi}F_{n,n-1}$$
$$\approx -g_2\sqrt{n}B_1\{\frac{e^{j\frac{\triangle t}{2}}[j\Delta\sin\beta t - 2\beta\cos\beta t]}{\Delta^2/4-\beta^2}$$
$$+\frac{2\beta}{\Delta^2/4-\beta^2}\} + \frac{1}{\sqrt{2}}F_{n,n} \tag{18}$$

- Substituting the values of $C_{g,n,n}(t)$, $C_{e,n-1,n}(t)$ and $C_{i,n,n-1}(t)$ from (15), (17) and (18) respectively in equation (4) we can obtain the state vector of the system at time t in the **interaction picture**.
- If the radiation field is initially in the coherent state, namely,

$$F_n = e^{-\bar{n}/2} \frac{\alpha^n}{\sqrt{n}}, \tag{19}$$

where

$$\alpha = \sqrt{\bar{n}} e^{j\zeta} \tag{20}$$

- and $\bar{n}$ is the mean photon number of the field, and ζ is the phase angle of α.

- For $\bar{n} \gg 1$, then the photon number of the field can be well approximated by Gaussian distribution

$$F_n \approx (2\pi\bar{n})^{-1/4} e^{in\zeta}.e^{-\frac{n^2}{4\bar{n}}} \tag{21}$$

- Now

$$\sqrt{n} = \sqrt{\bar{n} + (n - \bar{n})} = \sqrt{\bar{n}}\sqrt{1 + \frac{n - \bar{n}}{\bar{n}}} \approx \sqrt{\bar{n}}. \tag{22}$$

- Again n large implies $\bar{n}$ is also large.

- Now after the interaction with the field if we detect the atom in the ground state $|g>$ after time t_1 then effectively atom absorbs no photon but projects the cavity field into the state

$$|\psi_1 >= \frac{1}{\eta} \sum_n C_{g,n,n}(t_1)|n > \tag{23}$$

- where

$$C_{g,n,n}(t_1) = -2B_1 j e^{-j\frac{\triangle t_1}{2}} \sin \beta t_1 \tag{24}$$

with B_1 given by (16).

- **The atom collapses in the ground state $|g\rangle$ with maximum fidelity[3] are shown numerically in the following table for different values of gt_1 and $\bar{n}$:**

$\bar{n}$	gt_1	Fidelity	$\bar{n}$	gt_1	Fidelity
01	2.17475	0.455027	12	0.111629	0.000066556
02	0.485741	0.907061	13	0.111267	6.35892×10^{-09}
03	0.346932	0.94599	14	0.1	2.68304×10^{-14}
04	0.267409	0.966372	15	0.1	5.9437×10^{-21}
05	0.64969	0.830403	16	0.1	6.04874×10^{-29}
06	0.54566	0.872593	17	0.1	6.04874×10^{-38}
07	0.156686	0.98797	18	0.1	3.32933×10^{-48}
08	0.137503	0.990682	19	0.1	1.35062×10^{-59}
09	0.122792	0.975713	20	0.1	4.2914×10^{-72}
10	0.344388	0.514773	21	0.1	1.12532×10^{-85}
11	0.112437	0.028343	22	0.1	2.54818×10^{-100}

(25)

- If we now consider the passage of a second identical atom through the cavity[6, 7], then the field inside the cavity becomes

$$|\psi_2(t)>=\sum_n (D_{e,n,n}|e,n,n>+D_{i,n,n}|i,n,n>+D_{g,n,n}|g,n,n>) \quad (26)$$

- As in equation (6) we get the interaction Hamiltonian V

$$V = g_1 a|e\rangle\langle g|e^{j\triangle t}+g_1 a^\dagger|g\rangle\langle e|e^{-j\triangle t}+g_2 a|i\rangle\langle g|e^{j\triangle t}+g_2 a^\dagger|g\rangle\langle i|e^{-j\triangle t} \quad (27)$$

- Substituting equation(26) in the Schrödinger equation in the interaction picture

$$j\frac{d}{dt}|\psi^I(t)\rangle = V|\psi^I(t)\rangle, \tag{28}$$

- we get from (26), (27) and (28)

$$\dot{D}_{e,n-1,n} = -jg_1\sqrt{n}e^{j\triangle t}D_{g,n,n} \tag{29}$$

$$\dot{D}_{i,n,n-1} = -jg_2\sqrt{n}e^{j\triangle t}D_{g,n,n} \tag{30}$$

$$\dot{D}_{g,n,n} = -j(g_1\sqrt{n}D_{e,n-1,n} + g_2\sqrt{n}D_{i,n,n-1})e^{-j\triangle t} \tag{31}$$

- If the atom is initially in the state $|\psi_A(0)>$,

$$|\psi_A(0)>= \cos\frac{\varphi}{2}|e> + \sin\frac{\varphi}{2}e^{-i\psi}|i> \tag{32}$$

- which means that the atom is in the coherent superposition state of its eigenkets $|e>$ and $|i>$,
- and the field is in the superposition of the photon number states at time $t = 0$

$$|\psi_f(0)>= \frac{1}{\eta}\sum_{n,n} C_{g,n,n}(t_1)|n, n>, \tag{33}$$

where $\sum_{n,n} |C_{g,n,n}|^2 = \eta^2$,

- then the state vector of the total system at $t = 0$ can be described as

$$
\begin{aligned}
|\psi(0)> \quad = \quad & \frac{1}{\eta}\sum_{n,n}[\cos\frac{\varphi}{2}C_{g,n-1,n}(t_1)|e,n-1,n> \\
& +\sin\frac{\varphi}{2}e^{-j\psi}C_{g,n,n-1}(t_1)|i,n,n-1>]
\end{aligned}
\tag{34}
$$

- We now assume that

$$\varphi = 90^0 \text{ and } \ \psi = 0.$$

- Also

$$F_{n,n} \approx F_{n-1,n} \approx F_{n,n-1}.$$

- Then the state vector (34) of the total system at $t = 0$ reduces to

$$|\psi(0)>=\sum_{n,n}\frac{1}{\eta\sqrt{2}}C_{g,n,n}(t_1)(|e,n,n>+|i,n,n>) \tag{35}$$

- On solving (29), (30) and (31) with the initial condition (35) we get

$$\begin{aligned} D_{g,n,n}(t) &= K_1\{e^{-j(\triangle/2+\beta)t} - e^{-j(\triangle/2-\beta)t}\} \\ &= -2K_1 j e^{-j\frac{\triangle t}{2}}\sin\beta t \end{aligned} \tag{36}$$

- with

$$K_1 = \frac{\sqrt{n}g_1 C_{g,n-1,n}(t_1) + \sqrt{n}g_2 C_{g,n,n-1}(t_1)}{2\sqrt{2}\beta\eta} \tag{37}$$

- From (36) we now have

$$
\begin{aligned}
& D_{g,n,n}(t) \\
= \quad & -2K_1 j e^{-j\frac{\triangle t}{2}} \sin\beta t \\
= \quad & -2\frac{\sqrt{n}g_1 C_{g,n-1,n}(t_1)+\sqrt{n}g_2 C_{g,n,n-1}(t_1)}{2\sqrt{2}\beta\eta} j e^{-j\frac{\triangle t}{2}} \sin\beta t \\
= \quad & -\left(\frac{\sqrt{n}g_1 C_{g,n-1,n}(t_1)}{\sqrt{2}\beta\eta} + \frac{\sqrt{n}g_2 C_{g,n,n-1}(t_1)}{\sqrt{2}\beta\eta}\right) j e^{-j\frac{\triangle t}{2}} \sin\beta t \\
= \quad & -\Big[\frac{\sqrt{n}g_1}{\sqrt{2}\beta\eta}\{-\frac{\sqrt{n}F_{n,n}(g_1+g_2)}{\sqrt{2}\beta} j e^{-j\frac{\Delta t_1}{2}} \sin\beta t_1\}+ \\
& +\frac{\sqrt{n}g_2}{\sqrt{2}\beta\eta}\{-\frac{\sqrt{n}F_{n,n}(g_1+g_2)}{\sqrt{2}\beta} j e^{-j\frac{\Delta t_1}{2}} \sin\beta t_1\}\Big] j e^{-j\frac{\Delta t}{2}} \sin\beta t \\
= \quad & -\Big[\frac{n g_1}{2\beta^2\eta} F_{n,n}(g_1+g_2)\sin\beta t_1 \\
& +\frac{n g_2}{2\beta^2\eta} F_{n,n}(g_1+g_2)\sin\beta t_1\Big] e^{-j\frac{\Delta t_1}{2}} e^{-j\frac{\Delta t}{2}} \sin\beta t_1 \sin\beta t \\
= \quad & -\Big[\frac{n F_{n,n}}{2\beta^2\eta}(g_1+g_2)^2\Big] e^{-j\frac{\Delta t_1}{2}} e^{-j\frac{\Delta t}{2}} \sin\beta t_1 \sin\beta t
\end{aligned}
\tag{38}
$$

- From (29) we have

$$\begin{aligned} \dot{D}_{e,n-1,n}(t) &= -jg_1\sqrt{n}e^{j\triangle t}D_{g,n,n} \\ &= -jg_1\sqrt{n}K_1\{e^{j(\triangle/2-\beta)t} - e^{j(\triangle/2+\beta)t}\} \end{aligned} \quad (39)$$

- Integrating with respect to t we get

$$\begin{aligned} & D_{e,n-1,n}(t) - D_{e,n-1,n}(0) \\ = \quad & g_1\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t-2\beta\cos\beta t]}{\triangle^2/4-\beta^2} + \tfrac{2\beta}{\triangle^2/4-\beta^2}\} \end{aligned} \quad (40)$$

- Hence

$$
\begin{aligned}
D_{e,n-1,n}(t) &= g_1\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t-2\beta\cos\beta t]}{\triangle^2/4-\beta^2}+\tfrac{2\beta}{\triangle^2/4-\beta^2}\} \\
&\quad +\tfrac{1}{\eta}cos\tfrac{\varphi}{2}C_{g,n-1,n}(t_1) \\
&\approx g_1\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t-2\beta\cos\beta t]}{\triangle^2/4-\beta^2}+\tfrac{2\beta}{\triangle^2/4-\beta^2}\} \\
&\quad +\tfrac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1)
\end{aligned}
\tag{41}
$$

- From (30) we have

$$
\begin{aligned}
\dot{D}_{i,n,n-1} &= -jg_2\sqrt{n}e^{j\triangle t}D_{g,n,n} \\
&= -jg_2\sqrt{n}K_1\{e^{j(\triangle/2-\beta)t}-e^{j(\triangle/2+\beta)t}\}
\end{aligned}
\tag{42}
$$

- Integrating with respect to t we get

$$\begin{aligned} & D_{i,n,n-1}(t) - D_{i,n,n-1}(0) \\ = \quad & g_2\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t - 2\beta\cos\beta t]}{\triangle^2/4-\beta^2} + \tfrac{2\beta}{\triangle^2/4-\beta^2}\} \end{aligned} \tag{43}$$

- Hence

$$\begin{aligned} D_{i,n,n-1}(t) \quad = \quad & g_2\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t - 2\beta\cos\beta t]}{\triangle^2/4-\beta^2} + \tfrac{2\beta}{\triangle^2/4-\beta^2}\} \\ & + \tfrac{1}{\eta}\sin\tfrac{\varphi}{2}e^{-j\psi}C_{g,n,n-1}(t_1) \\ \approx \quad & g_2\sqrt{n}K_1\{\tfrac{e^{j\triangle t/2}[j\triangle sin\beta t - 2\beta\cos\beta t]}{\triangle^2/4-\beta^2} + \tfrac{2\beta}{\triangle^2/4-\beta^2}\} \\ & + \tfrac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \end{aligned} \tag{44}$$

- Now, we take $\Delta = 0,\ g_1 = g_2 = g$ to get

$$\beta^2 = \frac{\Delta^2}{4} + (g_1^2 + g_2^2)n = 2g^2 n \tag{45}$$

- and

$$\begin{aligned} D_{g,n,n}(t) &= -[\tfrac{nF_{n,n}}{2\beta^2\eta}(g_1+g_2)^2]e^{-j\frac{\Delta t_1}{2}}e^{-j\frac{\Delta t}{2}}\sin\beta t_1 \sin\beta t \\ &= -[\tfrac{nF_{n,n}}{2.2g^2 n.\eta}4g^2]\sin\sqrt{2n}gt_1.\sin\sqrt{2n}gt \\ &= -\tfrac{F_{n,n}}{\eta}\sin\sqrt{2n}gt_1.\sin\sqrt{2n}gt \end{aligned} \tag{46}$$

- and

$$
\begin{aligned}
D_{e,n-1,n}(t) &= g_1\sqrt{n}K_1\{\frac{e^{j\triangle t/2}[j\triangle sin\beta t-2\beta\cos\beta t]}{\triangle^2/4-\beta^2}+\frac{2\beta}{\triangle^2/4-\beta^2}\} \\
&\quad +\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= g\sqrt{n}K_1\{\frac{[-2\beta\cos\beta t]}{-\beta^2}+\frac{2\beta}{-\beta^2}\}+\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= \sqrt{2}(\cos\sqrt{2n}gt-1)K_1+\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= \sqrt{2}(\cos\sqrt{2n}gt-1)\frac{1}{2\eta}C_{g,n,n}(t_1)+\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= \frac{C_{g,n,n}(t_1)}{\sqrt{2}\eta}\cos\sqrt{2n}gt.
\end{aligned}
\tag{47}
$$

- Also

$$
\begin{aligned}
D_{i,n,n-1}(t) &= g_2\sqrt{n}K_1\{\frac{e^{j\triangle t/2}[j\triangle sin\beta t-2\beta\cos\beta t]}{\triangle^2/4-\beta^2}+\frac{2\beta}{\triangle^2/4-\beta^2}\} \\
&\quad +\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= g\sqrt{n}K_1\{\frac{[-2\beta\cos\beta t]}{-\beta^2}+\frac{2\beta}{-\beta^2}\}+\frac{1}{\sqrt{2}\eta}C_{g,n,n}(t_1) \\
&= \frac{C_{g,n,n}(t_1)}{\sqrt{2}\eta}\cos\sqrt{2n}gt
\end{aligned} \tag{48}
$$

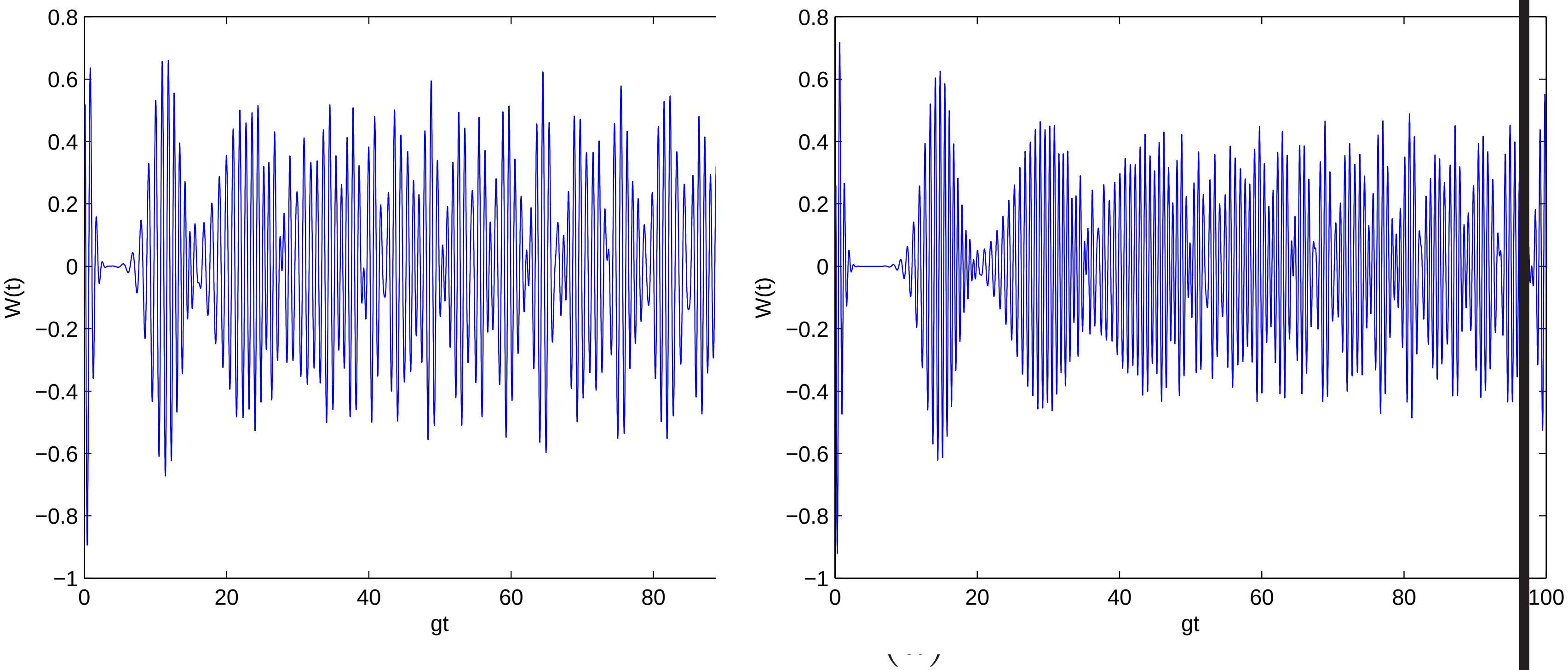
4 Population Inversion

- The inversion $W(t)$ for the three-level atom with ground state $|g>$ and two excited states $|i>$ and $|e>$ each coupled to $|g>$ is given by

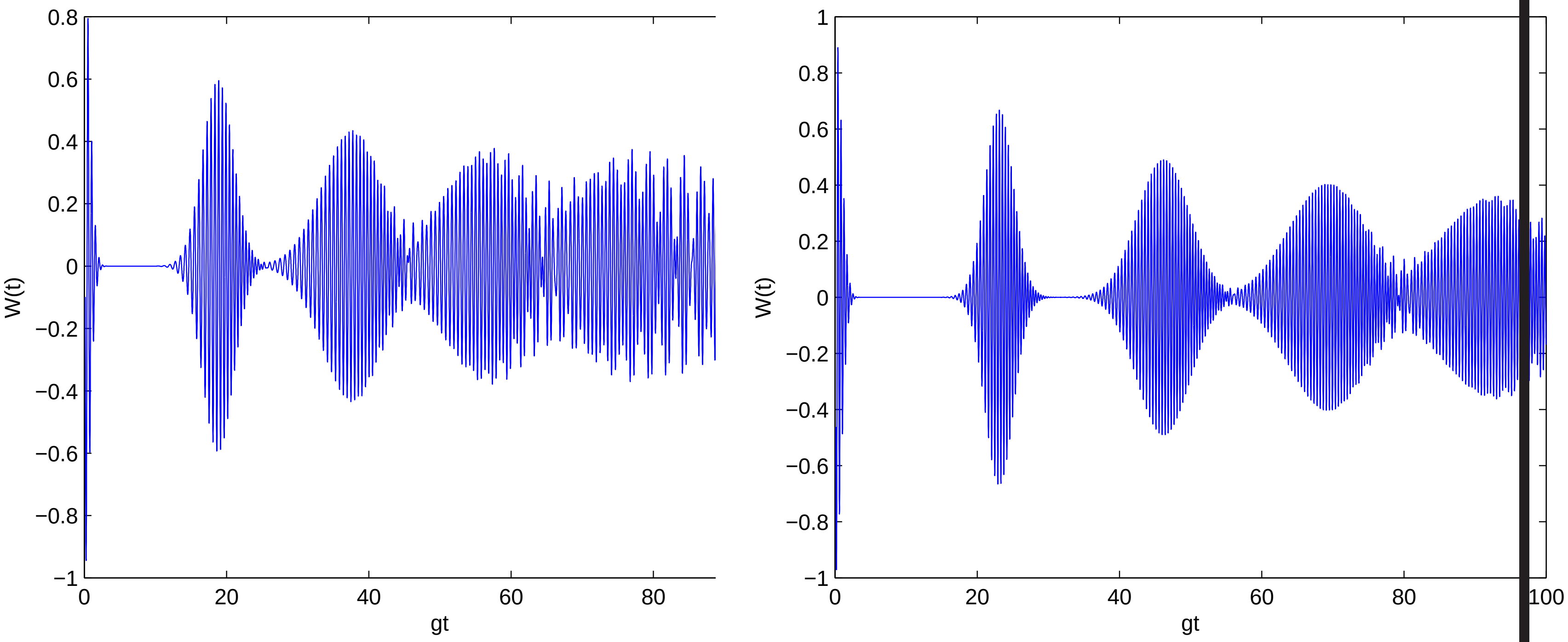
$$\begin{aligned} & W(t) \\ = \; & \sum_{n,n}[|D_{e,n-1,n}|^2 + |D_{i,n,n-1}|^2 - |D_{g,n,n}|^2] \\ = \; & \frac{e^{-|\alpha|^2}}{\sum_0^\infty |c'_{g,n+1}(t_1)|^2}\{\sum_0^\infty \frac{|\alpha|^{2n}}{n!} \sin^2(gt_1\sqrt{2n+2}) \cos^2(gt\sqrt{2n+4}) \\ & - \sum_0^\infty \frac{|\alpha|^{2n}}{n!} \sin^2(gt_1\sqrt{2n+2}) \sin^2(gt\sqrt{2n+4})\} \\ = \; & \frac{e^{-|\alpha|^2}}{\sum_0^\infty |c'_{g,n+1}(t_1)|^2} \sum_0^\infty \frac{|\alpha|^{2n}}{n!} \sin^2(gt_1\sqrt{2n+2}) \cos(2gt\sqrt{2n+4}). \end{aligned} \tag{49}$$

- On taking $\Delta = 0, g_1 = g_2 = g, \varphi = \frac{\pi}{2}, \psi = 0$ we find that the equations (47) and (48) are identical.
- This means that atom occupation probability of level $|e\rangle$ and $|i\rangle$ are identical.
- There are only one photon transition, $|e\rangle \leftrightarrow |g\rangle$ and $|i\rangle \leftrightarrow |g\rangle$, in the interaction of the field and the atom.
- In order to find the inversion property of the atom-field interaction different values of gt_1 and $\bar{n}$ are given to (49) to solve it numerically and the results are shown in the adjoining figures.

- Fig. 1(a), 1(b) indicate that the population inversion of two photons transition oscillate irregularly and atom collapses in between $3.34 \leq gt \leq 5.06$ for $\bar{n} = 2$ and $2.76 \leq gt \leq 7.22$ for $\bar{n} = 3$.
- Thus the initial coherent field with mean photon number $\bar{n} = 2$ or $\bar{n} = 3$ is not good for this model and is quite good for $\bar{n} = 4$ and $\bar{n} = 5$.
- If the initial mean photon number of the coherent field lies between $\bar{n} = 6$ and $\bar{n} = 9$, then this model is similar to that of two-level one photon JCM.



(b)



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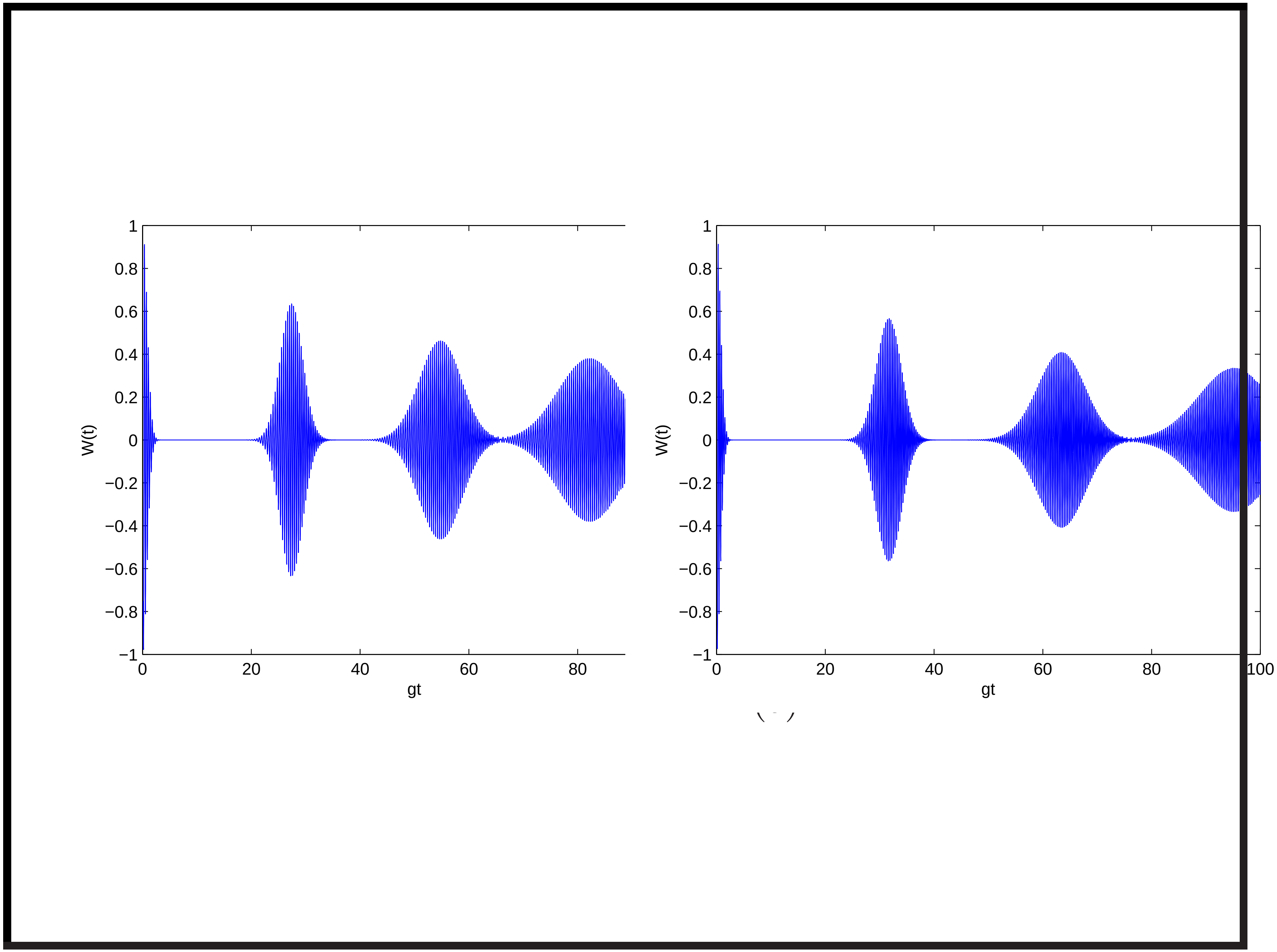

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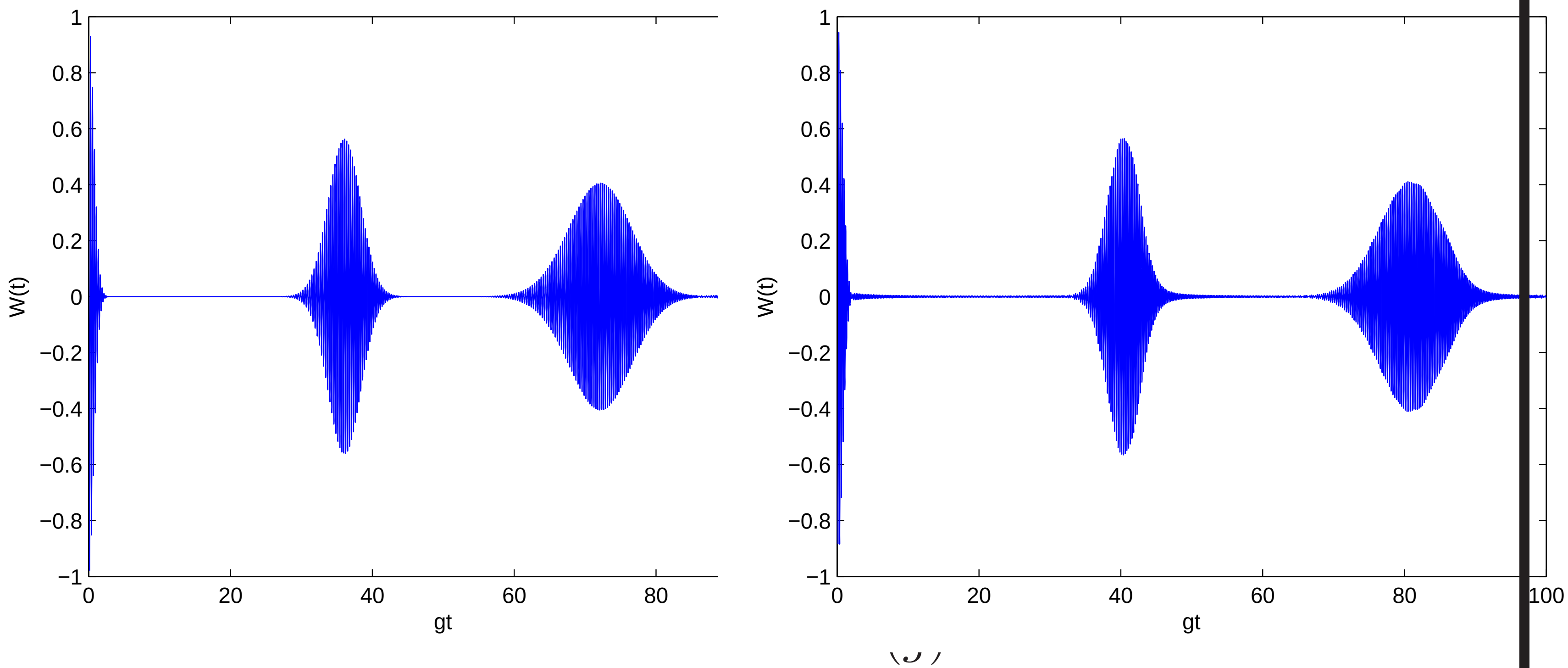


(h)

5 Evolution of the entropy of the field

- As we know, a measure of the uncertainty of the quantum-mechanical state is entropy and if ρ is a density operator for a given quantum system, $S(\rho)$ defined by

$$S(\rho) = -Tr(\rho ln \rho) \tag{50}$$

  is the von Neumann quantum mechanical entropy associated with ρ.

- $S = 0$ for a pure state ρ and $S \neq 0$ for a mixed state ρ.
- Thus entropy measures deviations from pure state behaviour.
- A non-zero S thus describes additional uncertainties in addition to those inherent quantum uncertainties which exist even for a pure state.

- Entropy is used to find out the disorder that evolve in a quantum system [10- 15].
- But the trace of $\rho ln \rho$ depends only on its eigenvalues and being governed by a unitary time evolution its eigenvalues remain constant.
- Thus the entropy defined in (50) is time independent which means that if it is prepared in a pure or mixed state then it remains in a pure or mixed state as we proceed with the time.
- But we are interested in the way a system interacts with its environment or how a sub-system interacts with the rest of the system.
- The density operator ρ of the whole system contains many information which gives a little information of particular sub-system's behaviour.

- We thus get a statistical operator which is called the reduced density operator and using this operator we define entropy for a sub-system to study interesting information about the dynamical behaviour of the subsystem.
- In our present study of atom-field system we denote by S the total entropy of the complete atom-field system.
- A remarkable theorem of Araki and Lieb [9] states that

$$|S_a - S_f| \leq S \leq |S_a + S_f|. \tag{51}$$

- As the atom-field system is initially a pure state we have $S = 0$ and we have an immediate consequence that if the total system is prepared in a pure state then the component systems have equal entropies throughout the entire evolution of the system.
- Thus the atom and the field entropies are identical.

- The time evolution of the light field (atom) entropy reflects the evolution property of the interdependence between the light field and the atom[4,5,8].
- Initially, since both light field and atom are in pure states and independent of each other then the whole system of the atom-field is zero and remains constant.
- Now, the entropy of the atom when treated as a separate system is defined through the corresponding reduced density operator by

$$S_A = -Tr_A(\rho_A ln \rho_A) \tag{52}$$

- where the reduced density operator ρ_A is

$$\rho_A = Tr_F(\rho). \tag{53}$$

- We now calculate

$$\rho_A = tr_F[|\psi_2(t) >< \psi_2(t)|] \tag{54}$$

- with

$$|\psi_2(t) >= \sum_n [D_{e,n,n}|e,n,n> + D_{i,n,n}|i,n,n> + D_{g,n,n}|g,n,n>] \tag{55}$$

Now

$$
\begin{aligned}
& |\psi_2(t)><\psi_2(t)| \\
= \quad & \sum_{m,n}[D_{e,n}|e,n> + D_{i,n}|i,n> + D_{g,n}|g,n>]. \\
& [\bar{D}_{e,m}<e,m| + \bar{D}_{i,m}<i,m| + \bar{D}_{g,m}<g,m|] \\
= \quad & \sum_{m,n}\{D_{e,n}\bar{D}_{e,m}|e,n><e,m| + D_{e,n}\bar{D}_{i,m}|e,n><i,m| \\
& + D_{e,n}\bar{D}_{g,m}|e,n><g,m| + D_{i,n}\bar{D}_{e,m}|i,n><e,m| \\
& + D_{i,n}\bar{D}_{i,m}|i,n><i,m| + D_{i,n}\bar{D}_{g,m}|i,n><g,m| \\
& + D_{g,n}\bar{D}_{e,m}|g,n><e,m| + D_{g,n}\bar{D}_{i,m}|g,n><i,m| \\
& + D_{g,n}\bar{D}_{g,m}|g,n><g,m|\}
\end{aligned} \tag{56}
$$

Hence

$$
\begin{aligned}
& \rho_A \\
= \quad & tr_F[|\psi_2(t)><\psi_2(t)|] \\
= \quad & (\textstyle\sum_n |D_{e,n}|^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\textstyle\sum_n D_{e,n}\bar{D}_{i,n}) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + ... \\
= \quad & \begin{bmatrix} (\sum_n |D_{e,n}|^2) & (\sum_n D_{e,n}\bar{D}_{i,n}) & (\sum_n D_{e,n}\bar{D}_{g,n}) \\ (\sum_n D_{i,n}\bar{D}_{e,n}) & (\sum_n |D_{i,n}|^2 & (\sum_n D_{i,n}\bar{D}_{g,n}) \\ (\sum_n D_{g,n}\bar{D}_{e,n}) & (\sum_n D_{g,n}\bar{D}_{i,n}) & (\sum_n |D_{g,n}|^2 \end{bmatrix} \\
= \quad & \begin{bmatrix} \rho_{ee} & \rho_{ei} & \rho_{eg} \\ \rho_{ie} & \rho_{ii} & \rho_{ig} \\ \rho_{ge} & \rho_{gi} & \rho_{gg} \end{bmatrix}
\end{aligned}
\tag{57}
$$

- and the atomic entropy is given by

$$S_F(t) = S_A(t) = -Tr[\rho_A(t) ln \rho_A(t)] = -\sum_{j=1}^{3} \lambda_j ln \lambda_j \qquad (58)$$

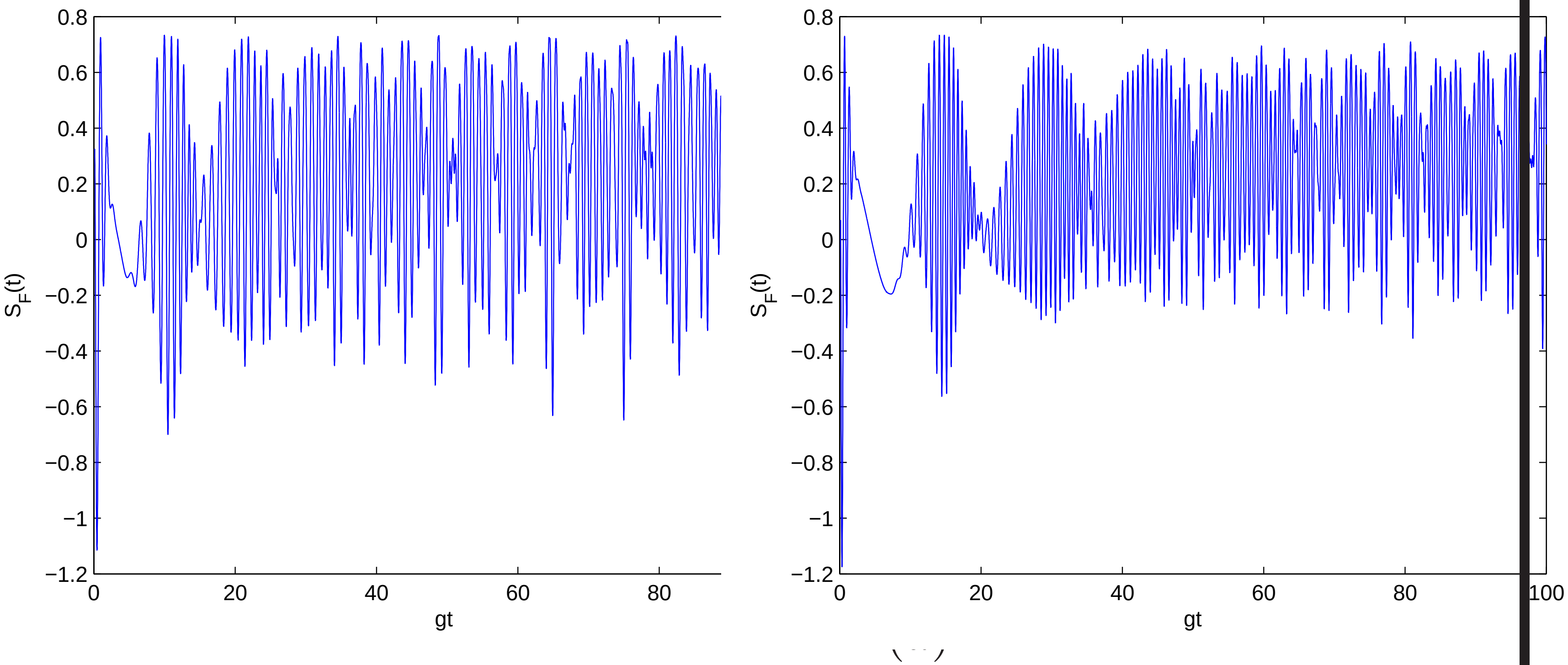
- where $\lambda_j' s (j = 1, 2, 3)$ are the eigenvalues of reduced density matrix of the atom whose characteristic equation is

$$\lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0 \qquad (59)$$

- with

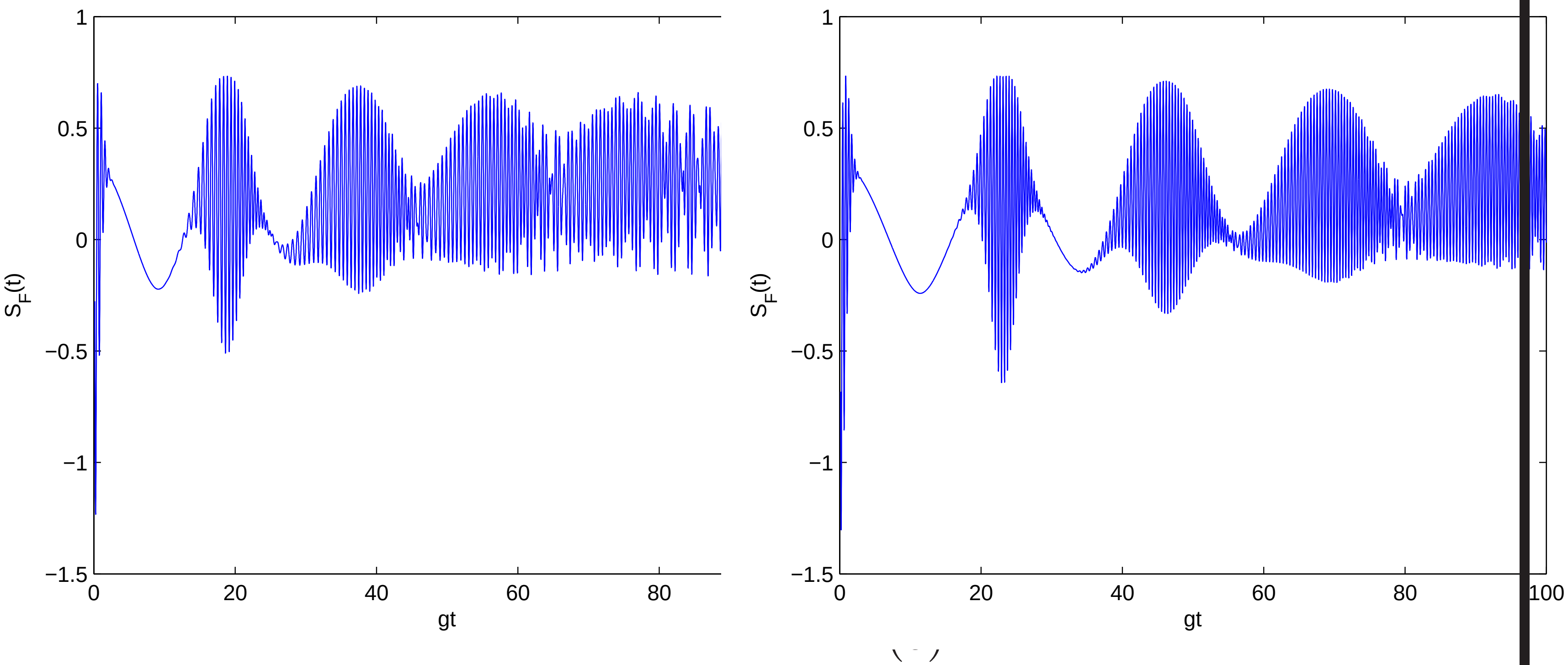
$$\begin{aligned}\alpha_0 &= -\rho_{ee}\rho_{ii}\rho_{gg} - \rho_{eg}\rho_{ig}\rho_{ge} - \rho_{eg}\rho_{gi}\rho_{ie} + \rho_{ee}\rho_{ig}\rho_{gi} + \rho_{ii}\rho_{ge}\rho_{eg} \\ &\quad +\rho_{gg}\rho_{ei}\rho_{ie} \\ \alpha_1 &= \rho_{ee}\rho_{ii} + \rho_{ii}\rho_{gg} + \rho_{gg}\rho_{ee} - \rho_{ei}\rho_{ie} - \rho_{ig}\rho_{gi} - \rho_{ge}\rho_{eg} \\ \alpha_2 &= -\rho_{ee} - \rho_{ii} - \rho_{gg}\end{aligned} \tag{60}$$

- In Fig. 2, on taking $\Delta = 0, g_1 = g_2 = g, \varphi = \frac{\pi}{2}, \psi = 0$ we have plotted the entropy for eight different values of mean photon numbers of the initial field.
- We observe from the figures 2(a) and 2(b) that the value of the field entropy oscillates and the coherent field does not keep its coherence but quietly decohere.
- In figures 2(c) and 2(d) the entropy oscillates from the initial value but after sometime it collapses for a short while and then revives for ever.
- From figures 2(e)- 2(h), entropy collapses and revives periodically and the system is similar to that of one photon transition of JCM.

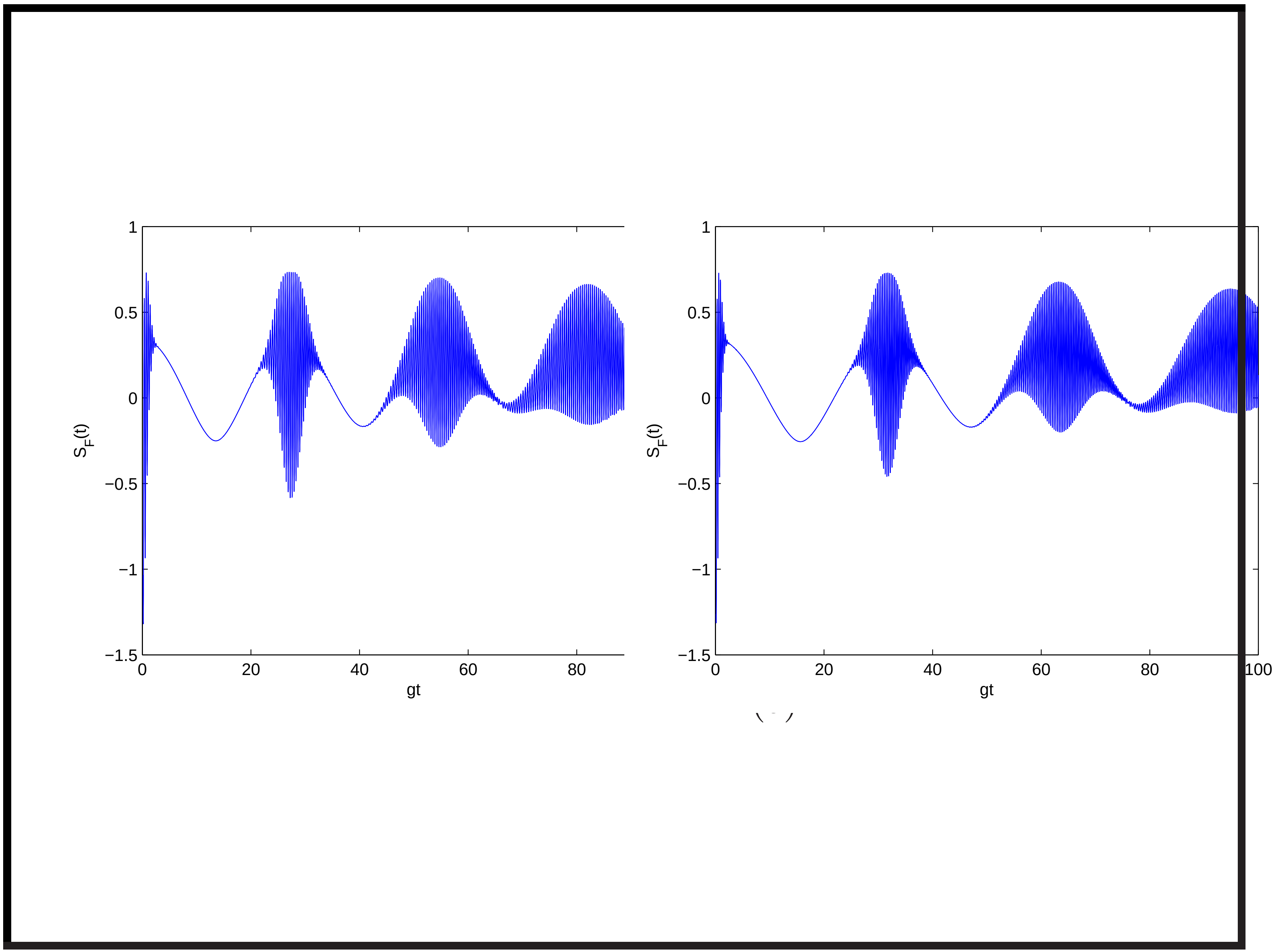


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
$S_F(t)$
0
20
40
60
80
gt
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
$S_F(t)$
0
20
40
60
80
100
gt

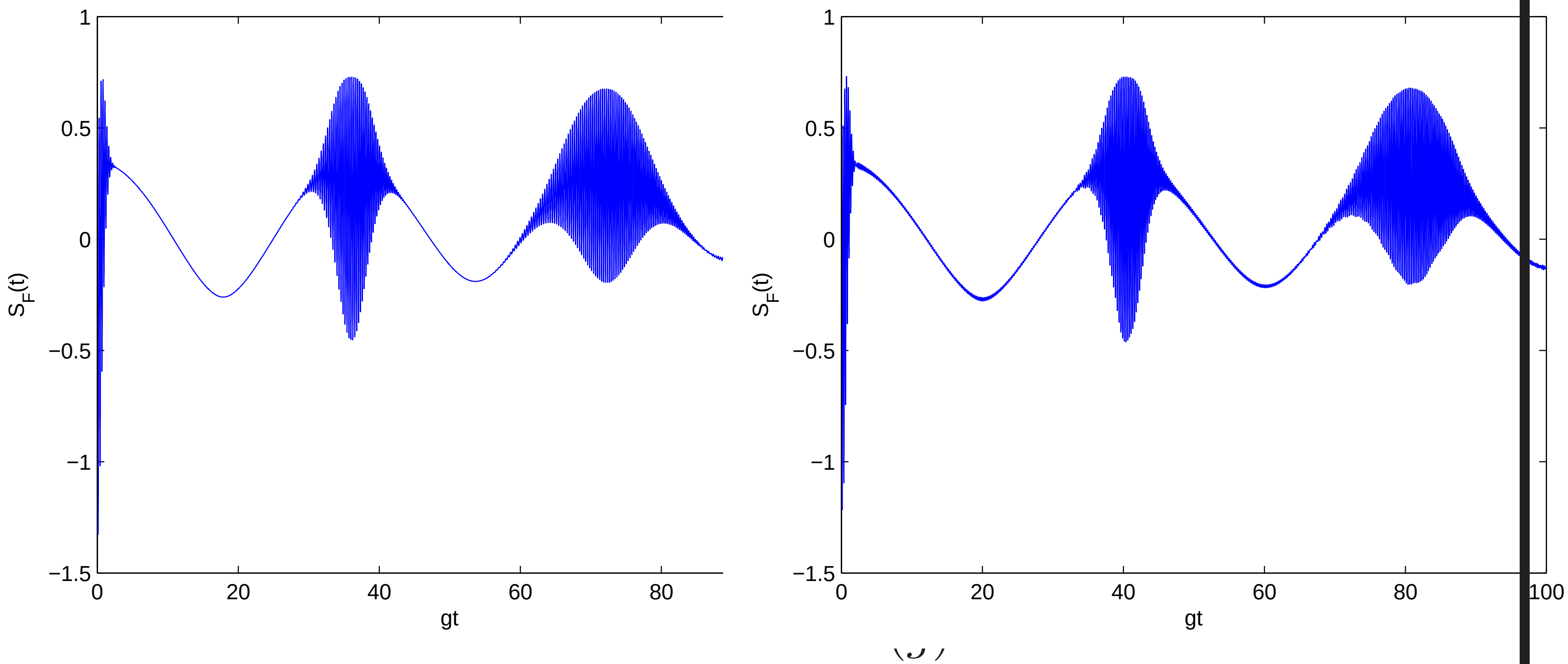
(b)



(d)



(f)



(h)

6 Conclusion

- We have thus shown the population inversion and the entropy evolution of the field interaction between the two atoms (passing one by one) and the single mode coherent field in a cavity.
- If we take the initial coherent field with mean photon number 8, then the atom collapses to the ground state with maximum fidelity .996082 after the laps of normalized time 0.137503.
- In the table (25) we have shown that the first atom collapses with maximum fidelity at different time in different mean photon number.
- In the picture of inversion and entropy evolution of the field we consider the coherent field with mean photon number between 2 and 9 and they are plotted for different values of normalized

time at which the first atom collapses to the ground state with maximum fidelity.

- If the mean photon number of the initial field is 1 then the fidelity is less than .5 and if the mean photon number is greater than 12 then the fidelity is nearly equal to 0 and we observe that atomic inversion and entropy of the field oscillate irregularly and the system will not collapse at all.
- If the fidelity is nearly equal to one then the system is similar to one-photon JCM.
- Thus the model is good when the fidelity is maximum and is bad when the fidelity is minimum.
- Thus the model is totally dependent on the initial field and the first atom when it collapses to the ground state with maximum fidelity.

THANKS