Time-independent properties of monotone convolution semigroups

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Motivation: Interplay among the notions of independence, Fock spaces, combinatorics of partitions, central limit theorems, orthogonal polynomials and **Lévy processes** (or martingales)

In classical probability theory, it is often the case that a property of a convolution semigroup μ_t is completely determined at an instant. Such a property is often stated as "if there exists some $t_0 > 0$ and μ_{t_0} satisfies $\cdots$, then μ_t satisfies $\cdots$ for all $t > 0$." We look for such properties in monotone case.

Definition 0.1. Let $\mathcal{A}$ be a $*$-algebra and let ϕ be a state.

(1) Let $\{\mathcal{A}_m\}_{m=1}^n$ be a sequence of $*$-subalgebras in $\mathcal{A}$. Then $\{\mathcal{A}_m\}_{m=1}^n$ is said to be *monotone independent* if the following condition holds.

$$\phi(a_1 a_2 \cdots a_n) = \phi(a_k)\phi(a_1 a_2 \cdots \check{a}_k \cdots a_n) \text{ if } a_m \in \mathcal{A}_{i_m}$$

$$\text{for all } 1 \leq m \leq n \text{ and } k \text{ satisfies } i_{k-1} < i_k > i_{k+1}.$$

If $k = 1$ (resp. $k = n$), the above inequality is understood to be $i_1 > i_2$ (resp. $i_{n-1} < i_n$).

(2) Let $\{b_i\}_{i=1}^n$ be a sequence of elements in $\mathcal{A}$. $\{b_i\}_{i=1}^n$ is said to be monotone independent if the $*$-algebras $\mathcal{A}_i$ generated by each b_i without unit form a monotone independent family.

(1) If $X \sim \mu$ and $Y \sim \nu$ are monotone independent in an algebraic probability space, we can define monotone convolution $\mu \triangleright \nu$ by the distribution of $X + Y$.

(2) Monotone convolution is characterized by the reciprocal of Cauchy (Stieltjes) transform:

$$H_{\mu \triangleright \nu} = H_\mu \circ H_\nu$$

where

$$H_\mu(z) := \left(\int_{\mathbb{R}} \frac{1}{z - x} d\mu(x) \right)^{-1}.$$

(3) μ is called monotone infinitely divisible if for any $n \in \mathbb{N}$ there exists μ_n such that $\mu = \mu_n \triangleright \cdots \triangleright \mu_n$ (n times).

Theorem 0.2. *(Muraki, Belinschi) There is a one-to-one correspondence among the following four objects:*

(1) *a monotone infinitely divisible distribution* μ*;*

(2) *a weakly continuous monotone convolution semigroup* $\{\mu_t\}$ *with* $\mu_0 = \delta_0, \mu_1 = \mu$*;*

(3) *a composition semigroup of reciprocal Cauchy transforms* $\{H_t\}$ $(H_t \circ H_s = H_{t+s})$ *with* $H_0 =$*id,* $H_1 = H_\mu$*, where* $H_t(z)$ *is a continuous function of* $t \geq 0$ *for any* $z \in \mathbb{C} \setminus \mathbb{R}$*;*

(4) *a vector field on the upper halfplane which has the form* $A(z) = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z} d\tau(x)$*, where* $\gamma \in \mathbb{R}$ *and* τ *is a positive finite measure. (This is the*

Lévy-Khintchine formula in monotone probability theory.)

The correspondence of (3) and (4) is obtained through the following ordinary differential equation (ODE):

$$\begin{aligned} &\frac{d}{dt}H_t(z) = A(H_t(z)), \\ &H_0(z) = z, \end{aligned} \tag{0.1}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$.

Main results

Theorem 0.3. *Let* $\{\mu_t\}_{t\geq 0}$ *be a weakly continuous* $\rhd$*-convolution semigroup with* $\mu_0 = \delta_0$*. Then the following statements are equivalent:*

(1) there exists $t_0 > 0$ *such that* $\mathrm{supp}\,\mu_{t_0} \subset [0,\infty)$*;*

(2) $\mathrm{supp}\,\mu_t \subset [0,\infty)$ *for all* $0 \leq t < \infty$*;*

(3) $\mathrm{supp}\,\tau \subset [0,\infty)$*,* $\tau(\{0\}) = 0$*,* $\int_0^\infty \frac{1}{x} d\tau(x) < \infty$ *and* $\gamma \geq \int_0^\infty \frac{1}{x} d\tau(x)$*.*

Remark 0.4. The same kind of theorem holds in the classical and Boolean cases, but does not hold in the free case.

Theorem 0.5. *Let* $\{\mu_t\}_{t\geq 0}$ *be a weakly continuous* $\triangleright$*-convolution semigroup with* $\mu_0 = \delta_0$*. Then the following statements are equivalent:*

(1) *there exists* $t_0 > 0$ *such that* $\operatorname{supp} \mu_{t_0}$ *is bounded below;*

(2) $\operatorname{supp} \mu_t$ *is bounded below for all* $0 \leq t < \infty$*;*

(3) $\operatorname{supp} \tau$ *is bounded below.*

Remark 0.6. The same kind of theorem holds in the free and Boolean cases. (1) and (2) are equivalent in the classical case, but (3) needs to be replaced by the following conditions: $\operatorname{supp} \tau \subset [0, \infty)$, $\tau(\{0\}) = 0$ and $\int_0^1 \frac{1}{x} d\tau(x) < \infty$.

Theorem 0.7. *Let* $\{\mu_t\}_{t\geq 0}$ *be a weakly continuous* $\rhd$*-convolution semigroup with* $\mu_0 = \delta_0$ *and let* $n \geq 1$ *be a natural number. Then the following statements are equivalent:*

(1) *there exists* $t_0 > 0$ *such that* $m_{2n}(\mu_{t_0}) < \infty$*;*

(2) $m_{2n}(\mu_t) < \infty$ *for all* $0 < t < \infty$*;*

(3) $m_{2n}(\tau) < \infty$*.*

Remark 0.8. The same kind of theorem holds in the classical, free and Boolean cases.

Bercovici-Pata bijection

Lévy-Khintchine formulae

$$\widehat{\mu}(u) = \exp\Big(i\gamma u + \int_{\mathbb{R}} \big(e^{ixu} - 1 - \frac{ixu}{1+x^2}\big)\frac{1+x^2}{x^2}\tau(dx)\Big),$$

$$A(z) = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z}\tau(dx),$$

where $\gamma \in \mathbb{R}$ and τ is a positive finite measure.

Let $ID(\triangleright)$ be the set of all monotone infinitely divisible distributions; let $ID(*)$ be the set of all infinitely divisible distributions.

We define a map $\Lambda_M : ID(*) \to ID(\triangleright)$ by sending the pair (γ, τ) in $\widehat{\mu}(u)$ to the pair (γ, τ) in $A(z)$. This map enjoys nice properties. Let D_λ be the dilation operator: $D_\lambda \mu(E) := \mu(\lambda^{-1} E)$ for Borel sets E and $\lambda > 0$.

Theorem 0.9. Λ_M *satisfies following properties.*

(1) Λ_M *is continuous;*

(2) $\Lambda_M(\delta_a) = \delta_a$ *for all* $a \in \mathbb{R}$*;*

(3) $D_\lambda \circ \Lambda_M = \Lambda_M \circ D_\lambda$ *for all* $\lambda > 0$.

(4) Λ_M *maps the Gaussian with mean 0 and variance* σ^2 *to the arcsine law with mean 0 and variance* σ^2*;*

(5) Λ_M *maps the Poisson distribution with parameter* λ *to the monotone Poisson distribution with parameter* λ*;*

(6) Λ_M *gives a one-to-one correspondence between the set* $\{\mu \in ID(*); \mathrm{supp}\, \mu \subset [0, \infty)\}$ *and the set* $\{\nu \in ID(\triangleright); \mathrm{supp}\, \nu \subset [0, \infty)\}$.

(7) *For all* $\alpha \in (0, 2)$, Λ_M *gives a one-to-one correspondence between strictly* α*-stable distributions and monotone strictly* α*-stable distributions.*

(8) *If* $\mathrm{supp}\,\tau$ *is compact, the symmetry of* $\mu \in ID(*)$ *is equivalent to the symmetry of* $\Lambda_M(\mu)$.

(9) *For each* $n \geq 1$, Λ_M *gives a one-to-one correspondence between the set* $\{\mu \in ID(*); \int_{\mathbb{R}} x^{2n}\mu(dx) < \infty\}$ *and the set* $\{\nu \in ID(\rhd); \int_{\mathbb{R}} x^{2n}\nu(dx) < \infty\}$.

Remark 0.10. Since monotone convolution is non-commutative, Λ_M does not preserve the structure of convolution: $\Lambda_M(\mu * \lambda) \neq \Lambda_M(\mu) \rhd \Lambda_M(\lambda)$ for some μ, λ.

Proof of the main results

$a(\mu) := \inf\{x \in \mathrm{supp}\mu\}$, $b(\mu) := \sup\{x \in \mathrm{supp}\mu\}$.

Proposition 0.11. *$\nu^{\rhd n}$ is supported on $[0, \infty)$, its n-th root ν is also supported on $[0, \infty)$.*

Proposition 0.12. *Let μ be a probability measure. We use the notation*

$$H_\mu(z) = z + b + \int_{\mathbb{R}} \frac{1 + xz}{x - z} \eta(dx), \tag{0.2}$$

Then $\mathrm{supp}\,\mu \subset [0, \infty)$ if and only if

(1) *$\mathrm{supp}\,\eta \subset [0, \infty)$;*

(2) *$H_\mu(-0) \leq 0$.*

Moreover, under the assumption (1), the condition (2) is *equivalent to the following condition:* (3) $\eta(\{0\}) = 0$; $\int_0^\infty \frac{1}{x} d\eta(x) < \infty$; $b + \int_0^\infty \frac{1}{x} d\eta(x) \leq 0$.

Proof. By the Stieltjes inversion formula, we have (1). Next we show (2). If $H_\mu(-0)$ were greater than 0, there would exists $u_0 < 0$ such that $H_\mu(u_0) = 0$. Then μ has an atom at $u_0 < 0$, which contradicts the assumption. Therefore, we have (2). $\square$

Lemma 0.13. *If there exists $t_0 > 0$ such that* $\operatorname{supp} \mu_{t_0}$ *is included in* $[0, \infty)$*, then* $A(z)$ *is analytic in* $\mathbb{C} \setminus [0, \infty)$.

Proof. We have $\operatorname{supp} \mu_{\frac{t_0}{n}} \subset [0, \infty)$ by Corollary 0.11. By definition we have

$$A(z) = \lim_{n \to \infty} \frac{H_{\frac{t_0}{n}}(z) - z}{\frac{t_0}{n}}.$$

We show that this limit is uniform on any compact set in $\mathbb{C} \setminus [0, \infty)$. Montel's theorem.

$$A_n(z) := \frac{H_{\frac{t_0}{n}}(z) - z}{\frac{t_0}{n}}$$

for $z \in \mathbb{C} \setminus [0, \infty)$. A_n is analytic in the domain. (1) We can show the uniform boundedness of A_n.
(2) By Montel's theorem there exists a subsequence of $\{A_n\}$ which converges uniformly on $\mathbb{C} \setminus [0, \infty)$ to some analytic function $\widetilde{A}(z)$. The limit $\widetilde{A}$ coincides with A on $\mathbb{C} \setminus \mathbb{R}$. Therefore, A has the analytic continuation to $\mathbb{C} \setminus [0, \infty)$. $\square$

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