Superconformal field theory, Moonshine and operator algebras

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August 2009, Będlewo

Moonshine

Mysterious relations between finite simple groups and elliptic modular functions

Realize a given finite group as the automorphism group of some infinite dimensional structure in a natural way

Formulated first in terms of vertex operator algebra (VOA), which is an algebraic axiomatization of conformal field theory

A new approach based on von Neumann algebras
Also relations to superconformal field theory and noncommutative geometry
(“super” $\approx \mathbb{Z}_2$-grading $+\varepsilon$)

Classification of finite simple groups:

1. Cyclic groups of prime order
2. Alternative groups of degree 5 or higher
3. 16 series of Lie type groups (such as $PSL(n, \mathbb{F}_q)$)
4. 26 sporadic groups (since Mathieu, 1861)

Monster has the largest order among 26 sporadic finite simple groups. The order is
$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$
which is around 8×10^{53}.
Constructed by Griess as the automorphism group of a 196884-dimensional commutative nonassociative algebra.

The smallest dimension of Monster's non-trivial irreducible representation is 196883.

The following function, called j-function, has been classically studied.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

For $q = \exp(2\pi i\tau)$, $\mathrm{Im}\,\tau > 0$, we have modular invariance property, $j(\tau) = j\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and this is the only function satisfying this property and starting with q^{-1}, up to freedom of the constant term.

$$196884 = 196883 + 1 \quad \text{(McKay)}$$

Similar relations for other coefficients of the j-function and dimensions of irreducible representations of the Monster group.

Then Conway-Norton formulated the Moonshine conjecture roughly as follows, which has been now proved by Borcherds.

(1) We have a "natural" infinite dimensional graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with $\dim V_n < \infty$ having some algebraic structure whose automorphism group is the Monster group.

(2) For any element g in the Monster, the power series $\sum_{n=0}^{\infty} (\mathrm{Tr}\ g|_{V_n}) q^{n-1}$ is a special function called a Hauptmodul for some discrete subgroup of $SL(2, \mathbb{R})$. When g is the identity element, we obtain the j-function minus 744.

Construction of Frenkel-Lepowsky-Meurman of the Moonshine VOA (vertex operator algebra) for part (1): Leech lattice Λ: an exceptional lattice in dimension 24 $\Rightarrow$ lattice VOA V_Λ. We take a fixed point algebra under a natural action of $\mathbb{Z}/2\mathbb{Z}$, and then make an extension of order 2. This gives the Moonshine VOA $V^\natural$. The series $\sum_{n=0}^{\infty}(\dim V_n^\natural)q^{n-1}$ is indeed the j-function minus 744.

VOA is a family of vertex operators, which are basically operator-valued distributions on S^1. Such a family gives an algebraic axiomatization of conformal field theory, so it should "correspond" to a local conformal net.

We have already seen that conformal field theory can be axiomatized with a local conformal net as follows.
A family of von Neumann algebras $\{A(I)\}$, parametrized by intervals I on S^1, acting on the same Hilbert space H, with the following axioms.

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy (for the rotation symmetry)
5. Vacuum vector $\Omega \in H$

More functional analysis than VOA

Operator algebraic counterpart of the Moonshine VOA

We realize a Leech lattice local conformal net as an extension of the 48th tensor power of the Virasoro net with $c = 1/2$. Then an extension of a $\mathbb{Z}_2$-fixed point gives a local conformal net called the Moonshine net $A^\natural$. Theory of α-induction shows that this really "corresponds" to the Moonshine VOA. We then obtain the following properties. (K-Longo)

1. $c = 24$
2. Representation theory is trivial
3. The automorphism group is the Monster
4. Hauptmodul property (as in the Moonshine conjecture)

Baby Monster

This group has the second largest order among the 26 sporadic finite simple groups. The order is $2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$ which is around 4×10^{33}.

Höhn's construction of the shorter Moonshine super VOA can be translated into the operator algebraic framework and we have a local conformal net with $c = 23\frac{1}{2}$ whose automorphism group is the Baby Monster group.

We do not have "unified" constructions for the Monster and the Baby Monster, and Hauptmodul properties for elements in the Baby Monster group are not directly visible.

$N = 1$ Super Virasoro algebra:

Infinite dimensional super Lie algebra generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= (\frac{m}{2} - r)G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\end{aligned}
$$

Ramond algebra, if $r \in \mathbb{Z}$, and Neveu-Schwarz algebra, if $r \in \mathbb{Z} + 1/2$. Their unitary representations give the super Virasoro nets ($\mathbb{Z}_2$-grading and superconformal net).

First Conway group Co_1:

The order is $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$, which is around 4×10^{18}. This is one of the 26 sporadic finite simple groups, and also the automorphism group of the Leech lattice in dimension 24 divided by its center. The other Conway groups, Co_2 and Co_3, are subgroups of Co_1.

Duncan constructed a "super" analogue of the Moonshine VOA and showed that its automorphism group is Conway's group Co_1.

We now have its operator algebraic counterpart having $c = 12$, but in order to get Co_1, we have to consider the group of automorphisms fixing the super Virasoro subnet pointwise, rather than the entire automorphism group.

Rudvalis group:

Duncan considered the Rudvalis group, which is one of the six pariah groups not involved in the Monster group. (That is, it is not a quotient of a subgroup of the Monster group.) The order is $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, which is around 10^{11}.

He constructed two super VOA's with automorphic actions of the Rudvalis group with certain Moonshine type properties on two-variable power series arising from elements of the Rudvalis group.

We have an operator algebraic counterpart for one of the two, based on our general theory of superconformal nets. It has $c = 28$, and the group of automorphisms fixing a (strange) subnet pointwise gives the Rudvalis group.

Noncommutative geometry:

Slogan: Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.
In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the noncommutative smooth function algebra
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator, the Dirac operator, with $[D, a] \in B(H)$ for all $a \in \mathcal{A}$.

Construction of a family of spectral triples $(\mathcal{A}(I), H, D)$ parametrized by intervals $I \subset S^1$ from a representation of the $N = 1$ super Virasoro algebra.
One of the Ramond relations gives $G_0^2 = L_0 - c/24$.

$$\begin{array}{ccc} \text{The Laplacian} & \rightarrow & \text{square root} = \text{The Dirac operator} \\ \updownarrow & & \updownarrow \\ L_0 & \rightarrow & \text{square root} = G_0 \end{array}$$

Based on this analogy, we indeed show that the representation image of G_0 gives D in the Connes definition and we have a dense subalgebra $\mathcal{A}(I)$ of $A(I)$ appearing in some (generalized) superconformal net.
(Carpi-Hillier-K-Longo)