Subfactors and representation theory for von Neumann algebras

Yasuyuki Kawahigashi

University of Tokyo

August 2009, Będlewo

A von Neumann algebra M

1. $*$-subalgebra of $B(H)$ containing I
2. closed in the strong operator topology

If the center of M is $\mathbb{C}I$, it is called a factor.

Set $\mathrm{tr}(A) = \frac{1}{2}\mathrm{Tr}(A)$ for $A \in M_2(\mathbb{C})$.
We have $\bigotimes_{n=1}^{\infty} \mathrm{tr}$ on $\bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$
$\longrightarrow$ GNS construction and strong operator closure gives a type II_1 factor M with tr. (Noncommutative version of $\prod_{n=1}^{\infty}\{0,1\} \cong [0,1]$ as measure spaces.)
For projections $p, q \in M$, we have $u \in M$ with $uu^* = p$, $u^*u = q$ if and only if $\mathrm{tr}(p) = \mathrm{tr}(q)$.

Use

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{a}{1+\lambda} + \frac{\lambda d}{1+\lambda}$$

on $M_2(\mathbb{C})$ instead of tr. Then the GNS construction for $\bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ and strong operator closure gives a type III factor. ($0 < \lambda < 1$.)

We now have $u \in N$ with $uu^* = p$, $u^*u = q$ for any nonzero projections $p, q \in N$.

In a type II$_1$ factor, the functional tr measures the "size" of projections, but in a type III factor, all nonzero projections are of "the same size".

Representation theory for a factor M

We consider only separable Hilbert spaces and von Neumann algebras acting on them.

Study $*$-homomorphisms $\pi : M \to B(K)$, where K is some Hilbert space.

If M is a type II_1 factor, such a representation is completely classified by the coupling constant, $\dim_M H \in (0, \infty]$.
(Note $\dim_M L^2(M)p = \mathrm{tr}(p)$ for a projection $p \in M$.)
If M is a type III factor, all representations are unitarily equivalent.
In both cases, a representation is never irreducible.
More useful representation theory?

Bimodules: Let M, N be factors. Consider a Hilbert space H with mutually commuting left and right actions of M and N. We write ${}_M H_N$.

For ${}_M H_N$, let P be the commutant of the right N-action. P is automatically a factor, so we get a subfactor $M \subset P$.

Suppose we have a type II_1 subfactor $N \subset M$. Then we get a bimodule ${}_N L^2(M)_M$. (Note that the commutant of the right M-action gives the left M-action.)

Considering a bimodule and a subfactor are essentially the same.

We have the Jones index $[M : N] = \dim_N L^2(M)$. (We can define the Jones index also for type III subfactors.)

Direct sums and irreducible decompositions for bimodules (if both left and right dimensions are finite).
Relative tensor product of bimodules: ${}_N H \otimes_M K_P$

Compare the above with the case of group representations.
Consider a type II_1 subfactor $N \subset M$ with finite index. Start with ${}_N L^2(M)_M$. Make tensor powers ${}_N L^2(M) \otimes_M L^2(M) \otimes_N L^2(M) \otimes_M \cdots$ and decompose them into irreducible ones.
If we have only finitely many irreducible bimodules in this way, these bimodules and homomorphisms between them completely recover hyperfinite type II_1 subfactors. (Popa)

Principal graph of s subfactor

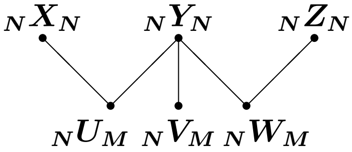

Figure: An example of a principal graph

We have $_N Y \otimes_N L^2(M)_M \cong {}_N U_M \oplus {}_N V_M \oplus {}_N W_M$.

We also have the dual principal graph for M-N and M-M bimodules.

If these graphs are finite, the Jones index is given by the Perron-Frobenius eigenvalues of their incidence matrices.

Classification of subfactors with index less than 4

Possible values of the Jones index are as follows. (Jones)

$$\{4\cos^2(\pi/n) \mid n = 3, 4, 5, \dots\} \cup [4, \infty]$$

Jones index $< 4 \Rightarrow$ the principal graphs must be one of A_n, D_n, E_6, E_7, E_8. (Jones)

Only A_n, D_{2n}, E_6, E_8 are possible. Unique type II_1 hyperfinite subfactor for each of A_n, D_{2n} and two such for each of E_6, E_8. (Ocneanu)

Principal graph contains only partial information on representation theory, but still is often enough to (almost) recover the entire representation theory.

Suppose we have a type III factor M and an M-M bimodule H. The Tomita-Takesaki theory gives a standard bimodule ${}_ML^2(M)_M$, where the left and right actions of M give commutants of each other.

If we have an endomorphism ρ of M, which is necessarily injective, we can replace the right action of ${}_ML^2(M)_M$ with the one given through ρ, that is, $x \cdot \xi \cdot y = x\xi\rho(y)$. We write H_ρ for this. It turns out that an arbitrary M-M bimodules are isomorphic to one in this form and $H_\rho \otimes_M H_\sigma$ is isomorphic to $H_{\rho\sigma}$.

That is, the relative tensor product is simply given by composition of endomorphisms.

Conformal Field Theory:

In quantum mechanics, an observable is given by a (possibly unbounded) self-adjoint operator.

In quantum field theory, we consider a certain spacetime together with its symmetry group, and study observables on spacetime regions and von Neumann algebras they generate.

We now start with $1+1$-dimensional Minkowski space, and decompose it as the product of $\{t=x\}$ and $\{t=-x\}$ and compactify these into two copies of S^1.

We regard each S^1 now as our new spacetime and use its orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$ as the spacetime symmetry group.

Operator algebraic axioms of a conformal field theory:
We have a family of von Neumann algebras $\{A(I)\}$, parametrized by intervals I on S^1, acting on the same Hilbert space H, with the following axioms. (An interval is an open, connected, nonempty and nondense subset of S^1.)

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. **(locality)**
3. $\mathrm{Diff}(S^1)$**-covariance (conformal covariance)**
4. **Positive energy (for the rotation symmetry)**
5. **Vacuum vector $\Omega \in H$**

Such a family $\{A(I)\}$ is called a local conformal net.

Consequences of the axioms:

Each $A(I)$ is a type III factor (unless $H = \mathbb{C}$)

[Additivity] $I \subset \bigcup_i I_i \Rightarrow A(I) \subset \bigvee_i A(I_i)$

[Haag duality] Let I' be the interior of the complement of an interval I. We have $A(I') = A(I)'$.

Diffeomorphism covariance involves a projective unitary representation of $\mathrm{Diff}(S^1)$ on H. $\mathrm{Diff}(S^1)$ is an infinite dimensional Lie group, and the central extension of its infinite dimensional Lie algebra is the famous Virasoro algebra. We automatically have a unitary representation of the Virasoro algebra on H.

The Virasoro algebra is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a single central element c, the central charge, with the following relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

An irreducible unitary representation of the Virasoro algebra maps c to a real number, also called the central charge, in $\{1 - 6/m(m+1) \mid m = 3, 4, 5, \dots\} \cup [1, \infty)$.
(Friedan-Qiu-Shenker) [Formal similarity to the Jones index]
A local conformal net gives a (possibly reducible) unitary representation of the Virasoro algebra and it gives a central charge value, which is also denoted by c.

Examples of local conformal nets:

It is difficult to construct even a single example.

Two basic methods are as follows.

Kac-Moody/Virasoro algebras: produce examples through positive energy representations (Wassermann and others)

Even lattices ($\subset$ Euclidean spaces): produce examples (Dong-Xu, K-Longo)

New onstructions from some known examples

1. $\{A(I)\}$, $\{B(I)\}$ given $\Rightarrow$ Tensor product
2. A group action of G on $\{A(I)\}$ $\Rightarrow$ Fixed point algebras $\{A(I)^G\}$
3. $\{A(I)\}$, $\{B(I)\}$ given, with $A(I) \subset B(I)$ $\Rightarrow$ $\{A(I)' \cap B(I)\}$ (the relative commutant)

Representation theory of a local conformal net

All $\mathcal{A}(I)$'s act on H, but we also consider their representation on another Hilbert space, that is, a family $\{\pi_I\}$ of representations $\pi_I : \mathcal{A}(I) \to B(K)$.
Consider such a representation $\{\pi_I\}$ and fix an interval $I \subset S^1$. We may and do assume the representation $\pi_{I'}$ is the identity map on $B(H)$ (by uniqueness of representation of $A(I')$).
For $x \in A(I)$, $y \in A(I')$, we have $xy = yx$, so we have $\pi_I(x)y = y\pi_I(x)$, which implies $\pi_I(x) \in A(I)$ by the Haag duality. So π_I gives an endomorphism λ of $A(I)$. This single endomorphism is the only nontrivial part of $\{\pi_I\}$.

A representation of a local conformal net is reduced to an endomorphism of a factor $A(I)$. Such endomorphisms can be composed and this composition gives a right notion of a tensor product of two representations of a local conformal net.

It is trivial to make a tensor product of representations of a group or a Hopf algebra, but we do not have a general notion of a tensor product of representations of an algebra.

Then $\lambda(\mathcal{A}(I)) \subset \mathcal{A}(I)$ is a subfactor and we have its Jones index. Its square root is the statistical dimension of the representation (Longo).
Representation theory of a local conformal net $\longrightarrow$ subfactor theory

Braiding

For group representations π and σ, $\pi \otimes \sigma$ and $\sigma \otimes \pi$ are trivially unitarily equivalent.

Now take endomorphisms λ, μ of $A(I)$ arising from representations of a local conformal net. There seems no reason to expect $\lambda\mu = \mu\lambda$, but it turns out that we have a unitary $u \in A(I)$ with $\mathrm{Ad}(u) \cdot \lambda\mu = \mu\lambda$. Such a choice of u can be made systematically for all λ's, and is called a braiding. (Fredenhagen-Rehren-Schroer, Longo)

Representation theory of quantum groups produces similar braidings and they lead to knot invariants such as the Jones polynomial and its many variations.

In representation theory of a quantum group, it sometimes happens that we have only finitely many irreducible representations. Such finiteness is often called rationality.

K-Longo-Müger gave an operator algebraic characterization of such rationality for a local conformal net $\{A(I)\}$ as follows and it is called complete rationality.

Split the circle into four intervals I_1, I_2, I_3, I_4. Finiteness of the Jones index for a subfactor

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$$

together with a minor extra condition called gives complete rationality. (Similar to represenations of a finite group)

α-induction:

$H \subset G$ and a representation of $H \longrightarrow$ induced representation of G

$\{A(I) \subset B(I)\}$ and a representation of $\{A(I)\} \longrightarrow$ induced representation of $\{B(I)\}$

Assume complete rationality. The induction arises from $\pm$-braiding structure of representations of $\{A(I)\}$: α-induction, α_λ^+ and α_λ^-

The matrix $(\dim(\mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)))$ has a special property called modular invariance, where Hom means the space of intertwiners.

Böckenhauer-Evans-K based on [Ocneanu, Longo-Rehren, Xu]

Modular invariants:

Fix a completely rational net $\{A(I)\} \Rightarrow$ A finite dimensional unitary representation π of the modular group $SL(2,\mathbb{Z})$ on H_π with basis given by irreducible representations of $\{A(I)\}$.

A matrix Z acting on H_π is called a modular invariant if the following are satisfied.

1. $Z_{\lambda\mu} \in \{0, 1, 2, 3, \dots\}$.
2. $Z \in \pi(SL(2,\mathbb{Z}))'$.
3. $Z_{00} = 1$, where 0 denotes the trivial representation.

Only finitely many such Z for a given $\{A(I)\}$.
Cappelli-Itzykson-Zuber and Gannon have classified such Z for many $\{A(I)\}$.

If $c < 1$, all possible $\{A(I)\}$ have been classified by K-Longo as extensions of the Virasoro nets, which arise from unitary representations of the Virasoro algebra.

1. Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$
2. Extensions of the Virasoro nets with index 2
3. Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$

They are labeled with pairs of Dynkin diagrams A-D_{2n}-$E_{6,8}$, since two subfactors naturally appear through α-induction

The classification list contains a new example, which does not arise from any other known constructions.