A theory of induction for tensor C^*–categories

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joint work with John E. Roberts
arXiv:0907.2459

12th Workshop: Noncommutative Harmonic Analysis
16-22.08.2009, Bedlewo

Tensor C^*–categories

Abstract tensor C^*–categories have been introduced by Doplicher and Roberts in the 70's as a basic mathematical structure underlying AQFT.

Motivated by 4-dim AQFT, attention was first dedicated to categories with conjugation and permutation symmetry, leading to an abstract duality theory for compact groups (Doplicher–Roberts, 89).

A connection between Jones' theory of subfactors and AQFT was established by Longo in 89 for type *III* factors.

Longo and Roberts (97), motivated by AQFT and Jones theory of subfactors, but also by quantum groups with C^*–structures (Wenzl, Woronowicz), studied a unifying concept, that of a tensor C^*–category with conjugation.

I shall briefly review the above developments and then discuss a more recent connection with ergodic theory of compact quantum groups, leading to the construction of a concrete realization of an abstract tensor C^*–category. The main tool is a theory of induction (P–Roberts)

Tensor C^*–category with conjugation (Longo–Roberts, 97).

A category $\mathcal{T}$ such that

- arrow spaces (ρ, σ) are abstract Banach spaces with involutions

$$(\rho, \sigma) \xrightarrow{*} (\sigma, \rho)$$

  satisfying the C^*–property

$$\|T^* T\| = \|T\|^2, \quad T \in (\rho, \sigma),$$

- with (associative) tensor products:

$$\rho, \sigma \to \rho\sigma$$

$$S \in (\rho, \rho'), T \in (\sigma, \sigma') \to S \otimes T \in (\rho\rho', \sigma\sigma'),$$

  with a 'trivial' unit object ι. We shall assume ι irreducible:

$$(\iota, \iota) = \mathbb{C}.$$

Conjugation is defined by:

- For every object ρ there is another one $\overline{\rho}$ and two arrows $R \in (\iota, \overline{\rho}\rho)$, $\overline{R} \in (\iota, \rho\overline{\rho})$ s.t.

$$\overline{R}^* \otimes 1_\rho \circ 1_\rho \otimes R = 1_\rho \qquad R^* \otimes 1_{\overline{\rho}} \circ 1_{\overline{\rho}} \otimes \overline{R} = 1_{\overline{\rho}}$$

Conjugation implies

- existence of an intrinsic dimension function

$$\rho \to d(\rho) := \inf \|R\| \|\overline{R}\|,$$

- $\dim(\rho, \sigma) < \infty, \quad$ for all ρ, σ

Simplest examples: G a compact group, $\mathrm{Rep}(G)$ category of representations of G,

Objects: finite dim unitary reps π of G

Arrows: intertwining operators (π, π')

Conjugation: $\pi \to \overline{\pi}$,

Special features:

- $\pi \otimes \pi' \simeq \pi' \otimes \pi$ ($\Rightarrow$ $\mathrm{Rep}(G)$ has permutation symmetry)
- $d(\pi) = \dim(\pi) \in \mathbb{N}$

A more general class of examples: G a compact quantum group of Woronowicz (90's) $\mathrm{Rep}(G)$ is a tensor C^*–category with conjugation. No permutation symmetry or integer dimension in general. I shall come back to cqg later.

These are examples of embedable categories

$$\mathrm{Rep}(G) \to \text{Hilbert spaces}$$

The embedding problem

The categories from AQFT do not come with an embedding into Hilbert spaces. Objects are endomorphisms of the local observable algebra. Tensor product is composition of endomorphisms. In 4-dim AQFT there is permutation symmetry, implying:

$$d(\rho) \in \mathbb{N} \quad \text{for any object } \rho$$

An abstract tensor C^*–category with conjugation and permutation symmetry is isomorphic to Rep(G), with G a unique compact group (Doplicher–Roberts, 89)

Deligne (90) proved an analogous duality result for algebraic groups

Permutation symmetry is essential for uniqueness: Izumi and Kosaki (02) found two different finite groups whose representation categories are isomorphic as (non-symmetric) tensor C^*–categories.

In low dim QFT permutation symmetry is replaced by a braided symmetry (unitary representations of the braid groups $\mathbb{B}_n$'s in the category)

Now $\rho \to d(\rho)$ is no longer integer-valued.

Original problem: Which noncommutative group-like structure replaces the compact group G appearing in DR duality in low dim AQFT?

The influence of the theory of subfactors of V. Jones led to extend the problem to categories with no symmetries:

Can we construct a concrete realization of an abstract tensor C^*–category with conjugation?

Subfactors

An important advance to the original problem was initiated in the early 80's by V. Jones with his work on II_1–subfactors.

A factor is a von Neumann algebra M with trivial center. It is of type II_1 if it has a finite and continuous dimension function (Murray–von Neumann) on projections: $E \in M \to d(E) \in [0, 1]$.

Jones considered an inclusion

$$N \subset M$$

of factors of type II_1 and associated an invariant, the index $[M : N]$. He showed the celebrated result

$$[M : N] \in \{4\cos^2\frac{\pi}{n}, n = 3, 4, \dots\} \cup [4, +\infty).$$

(Jones, 83)

For each value $4\cos^2 \pi/m$ Jones constructed a subfactor, the Jones subfactor.

Jones constructed a tower of II_1 factors

$$N \subset M \subset M_1 \subset M_2 \dots$$

The higher relative commutants, $M_r' \cap M_s$, give rise to a lattice of finite dimensional algebras. This is called the standard invariant of $N \subset M$

The standard invariant is a complete invariant for II_1 inclusions, under amenability assumptions (Popa, 94)

Ocneanu's category

It is an alternative description of the standard invariant. One starts with two bimodules over N and M:

$$X := {}_M M_N, \quad \overline{X} := {}_N M_M.$$

We can form Connes tensor products of X and $\overline{X}$ when right and left algebras match, e.g.

$$\overline{X}_M \otimes_M {}_M X = M \quad \text{as} \quad N - N \quad \text{bimodule}$$

$$X_N \otimes_N {}_N \overline{X} = M_1 \quad \text{as} \quad M - M \quad \text{bimodule}$$

The arrows of the category are the bimodule mappings.

Jones index and conjugation in a tensor C^*–category are related:

For II_1 subfactors, Ocneanu's category has conjugation. Conjugate equations are solved by Pimsner–Popa module bases

Longo clarified the connection between algebraic QFT and infinite subfactors

For an endomorphism ρ of a type III factor M

$$d(\rho)^2 = [M : \rho(M)]$$

(Longo 89)

thus

$$d(\rho) \in \{2\cos\frac{\pi}{n}, n = 3, 4, \dots\} \cup [2, +\infty).$$

Quantum groups

Wenzl: Drinfeld's algebraic quantum groups $\Rightarrow$ Tensor C^*–categories with conjugation and unitary braidings $\Rightarrow$ Subfactors

with emphasis at roots of 1 case, as these provide desired unitary reps of the braid groups.

In particular, Wenzl constructed subfactors from quantum $SL(d)$ at roots of 1. The special case of quantum $SL(2)$ gives the Jones subfactor.

Several classification results of tensor categories with prescribed fusion rules are known, e.g.:

A tensor category with conjugation and Grothendieck semiring isomorphic to that of $SL(d)$ *is equivalent to a suitable twist of* $Rep(SL_q(d))$ *for q not a root of* 1.
There is a generalization for q root of 1 *(Kazhdan–Wenzl, 93)*

Compact quantum groups

Woronowicz adopted a noncommutative topological approach to quantum groups, leading to the notion of a compact quantum group $G = (\mathcal{A}, \Delta)$, with $\mathcal{A}$ a unital C^*–algebra, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ the coproduct.

He associated $\text{Rep}(G)$, the category of unitary f.d. representations. It is a tensor C^*–categories with conjugation embedded into the category of Hilbert spaces.

The conjugate representation $\overline{u}$ of u is defined by an antilinear invertible j, defining the quantum dimension:

$$q - \dim(u) := \text{Tr}(j^*j)^{1/2}\text{Tr}((j^*j)^{-1})^{1/2} \geq \dim(H_u)$$

It is not an integer in general.

It never takes the values $2\cos\pi/m$ of the discrete part of the Jones spectrum.

Like in the classical case, compact quantum groups admit a spectral description, provided by a Tannaka–Krein duality result.

The embedded tensor C^*–categories with conjugation are precisely the representation categories of the compact quantum groups (Woronowicz, 88)

TK duality is useful to construct quantum groups, e.g. Woronowicz constructed $S_\mu U(d)$ in this way.

An abstract characterization of rep categories of certain compact quantum groups is known

e.g., $A_o(F)$, $A_u(F)$ (Banica, Yamagami), $S_\mu U(d)$ (P).

However, for general compact quantum groups it is an open problem.

Turning to abstract categories, there are plenty of examples of non-embedable ones from low dim QFT and subfactors:

An object ρ of a tensor C^*–category with conjugation and unitary braided symmetry is amenable. As such, it generates a non-embedable category if $d(\rho) \notin \mathbb{N}$ (Longo–Roberts, 97)

Similarly, a consequence of Popa's work:

If $N \subset M$ is amenable in the sense of Popa, the tensor category generated by ${}_N M_N$ is not embedable if $[M : N] \notin \mathbb{N}$

Hence, the predual of a tensor C^*–category with conjugation is not a compact quantum group in general.

Several constructions yield subfactors from wide classes of quantum groups. However, not all subfactors are known to arise from quantum groups (e.g. Asaeda–Haagerup subfactor). Subfactors are more general structures.

If we can't get the Hilbert spaces, we look for a natural embedding

$$\mathcal{T} \to \text{Hilbert } C^*\text{–bimodules}$$

A Hilbert C^*–bimodule X is a generalization of a Hilbert space, where the role of $\mathbb{C}$ is replaced by a C^*–algebra $\mathcal{C}$.

Hence X is a $\mathcal{C}$–bimodule with $\mathcal{C}$–valued inner product

$$x, x' \in X \to < x, x' > \in \mathcal{C}$$

$$< x, x'c > = < x, x' > c \qquad < x, cx' > = < c^* x, x' >$$

$$< x, x' >^* = < x', x >$$

$$< x, x > \geq 0$$

Why bimodules?

- Although a tensor category is generically non-embedable, it often contains a subcategory with 'few arrows' equivalent to the rep category of a quantum group G (e.g. Rep of quantum $SU(2)$ in Ocneanu's category). We may start with

$$\mathrm{Rep}(G) \subset \mathcal{T}$$

  However, G is not unique

- Subgroups $K \leq G$ give rise to prime examples with inclusion

$$\mathrm{Rep}(G) \subset \mathrm{Rep}(K)$$

  given by restriction functor:

$$v \to v \restriction_K$$

Classical induction

Let K be a closed subgroup of a compact group G, and $v \in \mathrm{Rep}(K)$. Mackey's induced representation:

$$\mathrm{Ind}(v) = \text{right translation}$$

on L^2–completion of the space of continuous functions

$$X_v := \{f : G \to H_v, f(kg) = v(k)f(g)\}$$

Frobenius reciprocity theorem

$$(u \restriction_K, v) \simeq (u, \mathrm{Ind}(v))$$

Now X_v is a bimodule over $C(K\backslash G)$ and it is in fact finite projective by Swan's theorem.

We may regard $\mathrm{Ind}(v)$ as a representation on X_v, and it is in fact convenient, as G–bimodules, $X_v \otimes_{C(K\backslash G)} X_{v'} = X_{v\otimes v'}$

Ind : *Rep*(K) $\to$ *Bimod*(G) *is a faithful tensor functor with full image*

Strategy

Now given Rep(G) $\subset \mathcal{T}$ replacing Rep(G) $\subset$ Rep(K) we look for a Frobenius isomorphism

$$\text{Ind} : \mathcal{T} \to \text{Bimod}(G)$$

Need to construct:

- a noncommutative C^*–algebra $\mathcal{C}$, replacing $C(K\backslash G)$
- induced G–bimodules Ind(ρ) over $\mathcal{C}$ induced from objects ρ of $\mathcal{T}$

Now Frobenius reciprocity plays the role of the main principle leading to the construction of the induced bimodules.

If G is a compact group,

$$K \to K\backslash G$$

is a bijective correspondence between conjugacy classes of closed subgroups K and transitive G–actions on compact spaces.

In the noncommutative situation transitive G–actions correspond to ergodic actions of compact quantum groups $G = (\mathcal{A}, \Delta)$ on unital C^*–algebras $\mathcal{C}$:

$$\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$$

$$\mathcal{C}^\alpha = \mathbb{C}.$$

Now there are more ergodic actions than subgroups.

For compact quantum groups, Podles found many non-quotient quantum 2–spheres.

Morever, quantum quotient spaces $K\backslash G$ are definitely too restrictive to treat non-embedable tensor C^*–categories $\mathcal{T}$.
Hence we can not hope to find a quantum subgroup.

Hence we need to construct an ergodic G–action and then develop a theory of induction from it.

In measurable ergodic theory there is an analogous situation.

Mackey introduced the idea of the virtual subgroup in the 60's to unify group theory and measurable ergodic theory.

He regarded a non-transitive ergodic measure G–space (X, ω) as a generalized subgroup, or virtual subgroup.

As such, one can talk about induced representations from the virtual subgroup to G.

Our main point is that there exists a large class of noncommutative virtual subgroups (ergodic C^*–algebras over compact quantum groups) arising from tensor C^*–categories with conjugation.

The class includes quantum quotients $K\backslash G$. However, it is much richer:

e.g., any proper II_1 subfactor $N \subset M$ is a source. The involved quantum groups are the quantum versions $A_o(F)$ of $SU(2)$ and $A_u(F)$ of Van Daele and Wang.

Bimodule representations of quantum groups

These are the noncommutative analogue of G–equivariant Hermitian bundles over compact spaces.

Let $G = (\mathcal{A}, \Delta)$ be a compact quantum group acting on a unital C^*–algebra $\mathcal{C}$:

$$\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}.$$

A bimodule representation of G is a $\mathbb{C}$–linear map

$$v : X_v \to X_v \otimes \mathcal{A},$$

with X_v a Hilbert $\mathcal{C}$–bimodule, such that

$$v(xc) = v(x)\alpha(c), \quad v(cx) = \alpha(c)v(c)$$

$$< v(x), v(x') >= \alpha(< x, x' >)$$

$$v \otimes 1 \circ v = 1 \otimes \Delta \circ v$$

$$v(X_v)1 \otimes \mathcal{A} \quad \text{dense in} \quad \mathcal{C} \otimes \mathcal{A}$$

Assuming the ergodic G–action has been constructed, how to construct the induced bimodules?

Example G a group. Start with a restricted representation $v \restriction_K$. If $\zeta \in X_{v\restriction_K}$ then $v(g^{-1})\zeta(g) \in H_v \otimes C(K\backslash G)$ hence

$$\mathrm{Ind}(v \restriction_K) \simeq v \otimes \text{right translation}$$

Hence given an ergodic action $(\mathcal{C}, \alpha)$ of a quantum group,

$$v \otimes \alpha \quad \text{replaces} \quad \mathrm{Ind}(v \restriction_K).$$

It is a representation of G on the free right module $H_v \otimes \mathcal{C}$.

Which left module structure? The trivial left $\mathcal{C}$–action

$$c(\psi \otimes c') := \psi \otimes cc'$$

is not a good choice in noncommutative cases:

- If G is a group, $u \otimes \alpha$ is a bimodule representation. However,

$$(u \otimes \alpha, u' \otimes \alpha) \subset \mathcal{B}(H_u, H_{u'}) \otimes Z(\mathcal{C}).$$

- If G is a quantum group, $u \otimes \alpha$ is not even a bimodule representation.

Due to noncommutativity, it is not easy to complete directly the G–module structure of $u \otimes \alpha$ to a 'good' G–bimodule structure. In some cases there isn't any (already for $\mathcal{C} = M_3$, $G = SU(2)$).

Main results

If a tensor C^*–category with an embedable subcategory $\mathcal{A} \subset \mathcal{T}$ are given, then an embedding of $\mathcal{A}$ gives, by Woronowicz duality, a compact quantum group G.

Hence we may start with a pair of tensor *–functors

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

Theorem (Roberts–P, 06–09) If μ surjective on objects, there are

- an ergodic action $(\mathcal{C}, \alpha)$ of G,
- induced bimodule G–representations $\text{Ind}(\mu_u) \simeq u \otimes \alpha$ over $\mathcal{C}$ with a 'good' left action,
- a faithful Frobenius tensor *–functor with full image

$$\text{Ind} : \mathcal{T} \to \text{Bimod}(G)$$

‘Good’ left actions

$$c\xi = \xi c, \qquad \xi \in (H_u \otimes \mathcal{C})^{u\otimes\alpha}, \quad c \in \mathcal{C}.$$

We call such bimodule structures full, as they give rise to a full functor:

$$u \otimes \alpha \in \text{Bimod}(G) \to u \otimes \alpha \in \text{Mod}(G)$$

$$(v \otimes \alpha, v' \otimes \alpha)_{Bimod} = (v \otimes \alpha, v' \otimes \alpha)_{Mod},$$

a property trivially satisfied in commutative case.

Constructing G

Universal quantum groups

Van Daele and Wang and associated with an invertible matrix F quantum groups $A_u(F)$ and $A_o(F)$. (The latter needs $F\overline{F} = \pm I$.)

They play a universality role analogous to that of $U(n)$ and $O(n)$ among compact Lie groups.

Any compact matrix quantum group in the sense of Woronowicz is a subgroup of some $A_u(F)$ (Van Daele–Wang, 95)

Banica studied representation theory of $A_o(F)$ and $A_u(F)$. In particular,

The representation theories of $A_o(F)$ and $SU(2)$ have the same fusion rules (Banica, 96)

Theorem (Roberts–P, 09)
Let $\mathcal{T}$ be a tensor C^*–category with a distinguished generating object ρ. If ρ is selfconjugate object with $d(\rho) \geq 2$ then the induction machine applies for $G = A_o(F)$ where F is any invertible matrix s.t.

$$\mathrm{Tr}(FF^*) = d(\rho), \qquad F\overline{F} = \pm I,$$

with $\pm$ chosen according to whether ρ is real or pseudoreal

In detail, there are:

- an ergodic C^*–action $(\mathcal{C}, \alpha)$ of $A_o(F)$,
- induced full bimodule representations of $A_o(F)$ on all $u \otimes \alpha$,
- a Frobenius tensor *–isomorphism

$$\mathcal{T} \to \mathrm{Bimod}(A_o(F))$$

Real or pseudoreal objects:

An object ρ of a tensor C^*–category s.t. there is $R \in (\iota, \rho^2)$ s.t.

$$R^* \otimes 1 \circ 1 \otimes R = \pm 1$$

Any selfconjugate irreducible object is either real or pseudoreal.

Jones index theory suggests to consider categories with two non-selfconjugate generating objects ρ, $\overline{\rho}$.

Theorem (Roberts–P, 09)
In the non-selfconjugate case, if ρ and $\overline{\rho}$ are generating objects of a tensor C^*–category $\mathcal{T}$ with $d(\rho) \geq 2$, similar conclusions hold where the quantum group is now the unitary version $A_u(F)$.

Examples

- (Quantum quotients) If K is a quantum subgroup of a compact quantum group G and $\mu : v \in \text{Rep}(G) \to v \upharpoonright_K \in \text{Rep}(K)$ is the restriction functor then

$$\mathcal{C} = C(K\backslash G), \quad X_v = \text{Ind}(v \upharpoonright_K) \simeq H_v \otimes C(K\backslash G),$$

is a free right module, in agreement with classical Swan's theorem.

Left module structure is non-trivial in the noncommutative case:

$$< \psi_i \otimes I, c\psi_j \otimes I >= \sum_h v_{hi}^* c v_{hj}.$$

- (II_1 Subfactors) If $N \subset M$ is a proper inclusion of II_1 subfactors with finite Jones index, the squared object

$$\overline{X} \otimes_M X = {}_N M_N$$

has a real structure defined by the *–involution:

$$R = \sum_i u_i \otimes u_i^*,$$

with (u_i) a module basis for M_N. Hence we have an ergodic C^*–action of $A_o(F)$, with

$$\mathrm{Trace}(F^* F) = [M : N], \quad F\overline{F} = I.$$

The spectral spaces of defining representation are

$$L_{u^{\otimes r}} = N' \cap M_{r-1}.$$

- In particular, we may choose $S_{-\mu}U(2)$ with $\mu + \mu^{-1} = [M : N]$.
- The ergodic actions arising from amenable inclusions in the sense of Popa for $[M : N] \notin \mathbb{N}$ are never quotient,
- If $[M : N] = n \in \mathbb{N}$, we may choose $A_o(n)$, of Kac type, and in this case we find an ergodic action with a tracial invariant state.

Problems:

- Study the ideal structure of these C^*–algebras. In particular, of that arising from the Jones subfactor.
- In general, these ergodic C^*–algebras are not expected to have traces for actions of non-Kac-type quantum groups. Determine the type of the associated von Neumann completions in the GNS representation of the invariant state. When do we get factors?

Gaps

However, not all ergodic actions arise from pairs of tensor functors

$$\mathrm{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

The simplest gaps are the ergodic actions of

- $G = SU(2)$ on M_n, $n \geq 2$,
- finite groups on M_n with full spectrum but low multiplicities

We may include all ergodic actions replacing tensor functors

$$\mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

with weaker kinds of inclusions, called quasitensor functors μ.

Rather surprisingly, all the induction machine extends, the only difference being that Ind(μ_u) is finite projective rather than free. (Roberts–P, 09)

As a consequence, there may be different induction theories on the same noncommutative ergodic space, varying the choice of μ.

The above result sheds light on the problem of recognizing which non-permutation symmetric tensor categories are embedable into the Hilbert spaces. The following is a first step.

Theorem (Roberts–P, 09)
A tensor C^*–category with a distinguished pseudoreal generating object ρ with $d(\rho) = 2$ is embedable into the Hilbert spaces, as a full subcategory of Rep(K), with K a closed subgroup of $SU(2)$.

Remark: It is proved by classifying the possible full bimodule representations of $SU(2)$. This relies also on the work of Wassermann that will be recalled soon.

Structure of proof

- Start from
$$\mathrm{Rep}(G) \longleftarrow \mathcal{A} \longrightarrow \mathcal{T}$$
where $\mathcal{A}$ has conjugation, and construct the G–ergodic C^*–algebra $(\mathcal{C}, \alpha)$. This is done via a spectral approach to ergodic actions,
- construct the induced G–bimodule representations and, when $\mathcal{A} \to \mathcal{T}$ is surjective on objects, construct the Frobenius functor
$$\mathcal{T} \to \mathrm{Bimod}(G),$$
- Consider universal categories: $TL_{\pm d}$ (Temperley–Lieb categories) and $\mathcal{T}_d$, and realize examples of pairs of functors:
$$\mathrm{Rep}(A_o(F)) \longleftarrow TL_{\pm d} \longrightarrow \mathcal{T}$$
$$\mathrm{Rep}(A_u(F)) \longleftarrow \mathcal{T}_d \longrightarrow \mathcal{T}$$

Ergodic actions

Given an ergodic action

$$\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$$

of a compact quantum group $G = (\mathcal{A}, \Delta)$,
and a d–dimensional representation u of G, the associated spectral space is

$$L_u := (H_u \otimes \mathcal{C})^{u \otimes \alpha} =$$

$$\{(c_i) : c_i \in \mathcal{C},\ \alpha(c_i) = \sum_k c_k \otimes u^*_{k,i}\}$$

The L_u's are Hilbert spaces: $\quad (c, d) := \sum c_i^* d_i$.

Entries of the L_u's span a dense *–subalgebra $\mathcal{C}_{\text{spectral}}$ (Podles).

Example K subgroup of G, $\quad \mathcal{C} = C(K\backslash G)$,

$$L_u \simeq \{K - \text{fixed vectors in } H_u\}$$

via

$$k \in \{K - \text{fixed vectors in } H_u\} \to \sum \psi_i \otimes < u(g)\psi_i, k >$$

Spectrum and multiplicities

$$\text{spec}(\alpha) := \{u \text{ irreducible rep: } L_u \neq 0\} \qquad \text{mult}(u) := \dim(L_u).$$

This is not enough to reconstruct the ergodic C^*–action:

Mackey observed that an example due to Todd provides two non-conjugate subgroups K_1 and K_2 of a finite group G with isospectral G–actions on $K_1\backslash G$ and $K_2\backslash G$.

Facts:

- If G is a group, the invariant state of the ergodic C^*–algebra is a trace and

$$\text{mult}(u) \leq \dim(u)$$

  (Høegh–Krohn, Landstad, Størmer, 81)

- $SU(2)$ acts ergodically only on type I von Neumann algebras (Wassermann, 88)

- If G is a quantum group ergodic theory is much richer: Haar measure is not tracial (Woronowicz, 87), there are examples on type *III* factors (Wang, 99). Since the first examples, ergodic actions of quantum groups have been constructed from the spectral spaces (e.g. Podles quantum spheres)
- mult(u) $\leq$ q-dim(u)(Boca, 95)
- Tomatsu, 06, has classified certain ergodic actions of $S_\mu U(2)$ embedable in the translation action on itself
- Bichon–De Rijdt–Vaes, 06, constructed more ergodic actions of $S_\mu U(2)$) where the multiplicity of an irreducible is bigger than its dimension.

Remark: BDV's method combines Woronowicz' Krein reconstruction with the existence of tensorial isomorphisms Rep($A_o(F)$) $\to$ Rep($A_o(F')$) for different quantum groups.

However, BDV construct ergodic actions with tensorial spectral spaces:

$$L_{u \otimes v} \simeq L_u \otimes L_v.$$

In general, if u and v are spectral representations of an ergodic action,

$$(c_i) \in L_u, \quad (d_j) \in L_v \Rightarrow (d_j c_i) \in L_{u \otimes v}$$

Thus we have isometries

$$S_{u,v} : L_u \otimes L_v \to L_{u \otimes v},$$

In general, $S_{u,v}$ are not unitary: If G is a group,

$$L_{u \otimes v} \simeq L_u \otimes L_v \quad \Leftrightarrow \quad \text{mult}(u) = \dim(u) \quad \text{for all} \quad u$$

Although $u \to L_u$ is not tensor, there are coherent rules that govern the behaviour of L under tensor products. Rules later recognized analogous to Popa's commuting squares appearing in Jones index theory. This eventually led to construction of ergodic actions of compact quantum groups from a subfactor or a tensor category.

Theorem (Roberts–P, 06)
Given spectral spaces L_u, L_v, L_w of an ergodic action, the following diagram commutes (associativity of L)

$$\begin{array}{ccc} L_u \otimes L_v \otimes L_w & \xrightarrow{1\otimes S} & L_u \otimes L_{v\otimes w} \\ \downarrow{\scriptstyle S\otimes 1} & & \downarrow{\scriptstyle S} \\ L_{u\otimes v} \otimes L_w & \xrightarrow{S} & L_{u\otimes v\otimes w} \end{array}$$

and it is a commuting square in the sense of Popa:

$$E^{L_{u\otimes v\otimes w}}_{L_{u\otimes v}\otimes L_w}(L_u \otimes L_{v\otimes w}) = L_u \otimes L_v \otimes L_w$$

Furthermore these rules lead to a duality theorem for ergodic actions analogous to Woronowicz duality.

Dual object of an ergodic action

is L as a functor

$$L : \text{Rep}(G) \to \text{Hilbert spaces}$$

together with the associated natural transformation:

$$S_{u,v} : L_u \otimes L_v \to L_{u \otimes v}.$$

An abstract pair (L, S) with

$$L : \mathcal{S} \to \mathcal{T}, \qquad S_{u,v} \in L_u \otimes L_v \to L_{u \otimes v}$$

of a *–functor between tensor C^*–categories and a natural transformation satisfying the above commuting square condition is called a quasitensor functor

Example: $\mathcal{T}$ tensor C^*–category,

$$\rho \in \mathcal{T} \to (\iota, \rho) \in \text{Hilbert spaces}$$

Duality Theorem (Roberts–P, 06)

- The spectral functor (L, S) allows to reconstruct $(\mathcal{C}_{\text{spectral}},\ \alpha,\ \text{invariant state},\ \text{finite maximal } C^*\text{–norm})$
- Any quasitensor functor (L, S) between

$$L : \text{Rep}(G) \to \text{Hilbert spaces}$$

  is the spectral functor of an ergodic action of G on a unital C^*–algebra.

Corollary

The spectral fuctor (L, S) is a complete invariant for ergodic C^*–actions of compact quantum groups over amenable algebras:

$$\mathcal{C}_{\text{red}} = \mathcal{C}_{\text{max}}$$

Examples of amenability: classical homogeneous spaces, $S_\mu U(d)$ (Nagy),
of non-amenability: $A_o(F)$, $n \geq 3$ (Skandalis) and $A_u(F)$ (Banica)

Idea of the construction of the ergodic C^*–algebra $\mathcal{C}$

Turning to a tensor C^*–category $\mathcal{T}$, with

$$\mathrm{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

we may roughly think of composing

$$\mathrm{Rep}(G) \xrightarrow{\tau^{-1}} \mathcal{A} \xrightarrow{\mu} \mathcal{T} \longrightarrow \text{Hilbert spaces}$$

and find a quasitensor functor giving rise to the desired ergodic action $(\mathcal{C}, \alpha)$ of G.

Remark: Injectivity of μ or τ is unimportant, although on arrows is guaranteed by the fact that any tensor C^*–category with conjugation is simple (PR, 08)

Embedding universal categories

The universal tensor *–category $TL_{\pm d}$ with objects $\mathbb{N}_0$ and arrows generated by a single $R \in (0, 2)$,

$$R^* \otimes 1 \circ 1 \otimes R = \pm 1,$$

$$d = R^* R$$

is the Temperley-Lieb category, the categorical counterpart of the TL algebras

(often defined without reference to the *–operation).

- *$TL_{\pm d}$ is simple except for $d = 2\cos \pi/m$, when it has a single non-zero proper tensor ideal $\mathcal{I}$.*
- *$TL_{\pm d}$ for $d \geq 2$ and $TL_{\pm d}/\mathcal{I}$ for $d = 2\cos \pi/m$, are tensor C^*–categories*

(Goodman–Wenzl, 03).

It is natural to look for a generalization to the non self-conjugate case.

Consider the universal tensor *–category $\mathcal{T}_d$ generated by two objects x, $\overline{x}$ and two arrows $R \in (\iota, \overline{x} \otimes x)$, $\overline{R} \in (\iota, x \otimes \overline{x})$ s.t.

$$\overline{R}^* \otimes 1_x \circ 1_x \otimes R = 1_x, \quad R^* \otimes 1_{\overline{x}} \circ 1_{\overline{x}} \otimes \overline{R} = 1_{\overline{x}},$$

$$R^* R = \overline{R}^* \overline{R} = d.$$

Goodman–Wenzl's theorem extends to the categories $\mathcal{T}_d$ (Roberts–P, 08)

- For $d = 2\cos\pi/m$, the quotient categories of $TL_{\pm d}$ and $\mathcal{T}_d$ can not be embedded into the Hilbert spaces, as these values are not taken by Woronowicz quantum dimension

- For $d \geq 2$, $TL_{\pm d}$ and $\mathcal{T}_d$ are embedable. More precisely (Yamagami, 06):

  For any $F \in M_n$ satisfying

$$\mathrm{Tr}(FF^*) = \mathrm{Tr}((FF^*)^{-1}) = d,$$

  (and $F\overline{F} = \pm I$, resp.) there is an isomorphism

$$\mathcal{T}_d \to \mathrm{Rep}(A_u(F))$$

$$(TL_{\pm d} \to \mathrm{Rep}(A_o(F)), \textit{resp.})$$

  Remark: many non-isomorphic quantum groups with defining representations of varying integral dimensions have isomorphic representation categories

Outlook

- Jones index theory and ergodic theory are explicitly connected by quasitensor functors. Insight gained from subfactor theory might prove useful in the problem of classifying the amenable ergodic C^*–actions of $S_\mu U(2)$.
- In a work in progress, we are considering induction theory for a pair of quasitensor functors:

$$\mathcal{S} \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

  Precise conditions on μ and τ are needed.
- Tensor C^*–categories with conjugation are sources of noncommutative compact homogeneous spaces with good properties. The description of these spaces naturally emphasizes a spectral viewpoint. Regard these spaces and induced bimodules as the starting point of a construction of geometric objects. **Task:** Try to relate explicitly to Connes' spectral triples.