⊞-Infinite Divisibility of Free Multiplicative Convolutions with Wigner and Symmmetric Arcsin Measures.

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Abstract

This talk is based on the recent joint work "$\boxplus$-Infinite Divisibility of Free Multiplicative Convolutions with Wigner and Symmmetric Arcsin Measures." with V. Pérez-Abreu of CIMAT in Mexico.

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  This is also called the R–transform (by Speicher) of μ.
  If μ is bound supported, $\mathcal{C}_\mu^\boxplus(z) = \sum_{n=1}^\infty k_n z^n$, where $k_n(\mu)$ is free cumulant of μ.

Recall basic facts in Classical probability theory

$\mu \in \mathcal{P}(\mathbb{R})$ is $*$-ID
if $\forall n \in \mathbb{N}$, $\exists\ \mu_{1/n} \in \mathcal{P}(\mathbb{R})$ s.t. $\mu = \underbrace{\mu_{1/n} * \cdots * \mu_{1/n}}_{n\,times}$.

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Proposition (Lévy-Khintchine representation)

μ in $\mathcal{P}$ belongs to $\mathbf{I}^*$ if and only if

$$\mathcal{C}^*_\mu(t) = -\frac{1}{2}a_\mu z^2 + ib_\mu z + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x)\right)\nu_\mu(\mathrm{d}x), \quad z \in \mathbb{R},$$

where $b_\mu \in \mathbb{R}, a_\mu \geq 0$ and ν_μ, the Lévy measure, is such that $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu_\mu(dx) < \infty$.
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where $b_\mu \in \mathbb{R}_+$ and ν_μ, the Lévy measure, is such that $\nu_\mu((-\infty, 0]) = 0$ and $\int_{\mathbb{R}}(1 \wedge x)\nu_\mu(dx) < \infty$.
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We call this representation "regular " LK-representation for probability measure concentrated on cone.
$\mathbf{I}^*_{r+} := \{\mu \in \mathbf{I}^* \mid \mu$ **has RLK rep**$\}$.

Mixture of normal distribution

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$$\mathcal{C}^*_{\text{law of}X}(t) = \int_{\mathbb{R}} \left(e^{-\frac{t^2}{2}x} - 1\right) \nu_\sigma(dx) = \mathcal{K}_\sigma\left(\frac{t^2}{2}\right)$$

where ν_σ is Lévy measure of $\mathcal{L}(V)$ and $\mathcal{K}_\sigma(z)$ is log-Laplace transform of σ.

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One remark:
If $T(t)$ is Lévy process $\Rightarrow \mathcal{L}(T(t))$ is infinitely divisible for any $t \geq 0$. Therefore X is of type G.

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How should we understand "type G law" in free probablity world?
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- The image class of $*$–type G law by Bercovici–Pata bijection.
- Free multiplicative convolutions $\boxtimes$ with Wigner law.

One remark:
Now we know the S-transform of symmetric law.

- Finite variance case by Rao and Speicher.
- General symmetric case by Arizmendi and Pérez-Abreu.

LK–rep. for $\mathrm{I}^{\boxplus}$

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where $b_{\mu} \in \mathbb{R}$, $a_{\mu} \geq 0$ and ν_{μ}, the Lévy measure, is such that $\nu_{\mu}(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu_{\mu}(dx) < \infty$.
As in classical probability, the free triplet $(a_{\mu}, \nu_{\mu}, b_{\mu})$ is unique.

Important law in free probability

Semi-circle (in short SC) law with mean b and variance a is

$$\mathrm{w}_{b,a}(\mathrm{d}x) = \frac{1}{2\pi b}\sqrt{4a-(x-b)^2}\,1_{[b-2\sqrt{a},b+2\sqrt{a}]}\mathrm{d}x$$

$$\mathcal{C}^{\boxplus}_{\mathrm{w}_{b,a}}(z) = az^2 + bz.$$

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Marchenko-Pastur law $\mathrm{m_c}$ with parameter $c > 0$

$$\mathrm{m}_c(\mathrm{d}x)$$
$$= \begin{cases} \frac{1}{2\pi x}\sqrt{4c-(x-1-c)^2}\,1_{[(1-\sqrt{c})^2,(1+\sqrt{c})^2]}(x)\mathrm{d}x, & \text{if} \quad c \geq 1, \\ (1-c)\delta_0(dx) + \frac{1}{2\pi x}\sqrt{4c-(x-1-c)^2}\,1_{[(1-\sqrt{c})^2,(1+\sqrt{c})^2]}(x)\mathrm{d}x, \end{cases}$$

It corresponds to classical Poisson law, since it has free triplet $(0, c\delta_{\{1\}}, 0)$.

μ is **1/2–⊞–stable distribution** ($\nu(dx) = cx^{-\frac{3}{2}}1_{(0,\infty)}(x)dx.$), if the density $f(x)$ is

$$f(x) = \frac{c}{\pi}\frac{\sqrt{(x-b')-\frac{c^2}{4}}}{(x-b')^2} \qquad \left(x > \frac{c^2}{4}+b'\right)$$

This is **Beta of 2nd kind distribution** in CPT. (This is in both I^* and $\mathrm{I}^{\boxplus}$).

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μ_s : a symmetric Beta distribution (α, β) with parameter $s > 0$ if its density $f(x)$ is

$$f(s) = \frac{1}{2B(\alpha,\beta)\sqrt{s}} |x|^{\alpha-1} (2\sqrt{s} - |x|)^{\beta-1} \quad x \in (-2\sqrt{s}, 2\sqrt{s}).$$

If $\alpha = 1/2$ and $\beta = 3/2$, then we can calculate concrete free cumulant transform

$$\mathcal{C}_{\mu_s}(z) = \frac{1}{\sqrt{1-sz^2}} - 1 = \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 \right) a(x;s)dx,$$

$$\mu_s = \mathrm{a}_s \boxtimes \mathrm{m}.$$

Bercovici-Pata bijection

The Bercovici-Pata bijection $\Lambda : \mathrm{I}^* \longrightarrow \mathrm{I}^{\boxplus}$.
If $\mu \in \mathrm{I}^*$ has classical triplet (a_μ, ν_μ, b_μ) then $\Lambda(\mu) \in \mathrm{I}^{\boxplus}$ with free triplet (a_μ, ν_μ, b_μ).

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For $\mu_1, \mu_2 \in \mathrm{I}^*$,

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Example

- μ *is* $N(m, \sigma^2) \Rightarrow \Lambda(\mu)$ *is* $SC(m, \sigma)$.
- μ *is Po*$(\lambda) \Rightarrow \Lambda(\mu)$ *is* $\mathrm{fP}(\lambda)$.
- μ *is stable law* $\Rightarrow \Lambda(\mu)$ *is* ⊞*–stable law.*

Remark on regular rep.

A probability distribution $\sigma \in \mathrm{I}_{+}^{\boxplus}$ is (free) regular, if its LK–representation is given by

$$\mathcal{C}_{\sigma}^{\boxplus}(z) = b_{\sigma} z + \int_{\mathbb{R}_+} \left(\frac{1}{1 - zx} - 1 \right) \nu_{\sigma}(\mathrm{d}x), \quad z \in \mathbb{C}^{-}, \qquad (1)$$

where $b_{\sigma} \geq 0$, $\nu_{\sigma}((-\infty, 0]) = 0$ and $\int_0^{\infty} (1 \wedge x) \nu_{\sigma}(\mathrm{d}x) < \infty$.
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We denote by $\mathrm{I}_{r+}^{\boxplus}$ the class of all regular distribution in $\mathrm{I}^{\boxplus}$.

Induced measure

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Example

$\mathrm{w}^{(2)}(dx) = \mathrm{m}(dx)$. *This means if X is SC then X^2 is fP law.*
$\mathrm{a}^{(2)}(dx) = \mathrm{a}^{+}(dx)$.

Result 1

Theorem

$\mu \in \mathbf{I}_s^{\boxplus}$ if and only if there is $\sigma \in \mathbf{I}_{r+}^{\boxplus}$ such that

$$\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2). \tag{2}$$

Moreover, the relation between the Lévy measures of μ and σ is

$$\nu_\mu = \frac{1}{2}\left(\nu_\sigma^{(1/2)+} + \nu_\sigma^{(1/2)-}\right) \tag{3}$$

and

$$\nu_\sigma = \nu_\mu^{(2)}. \tag{4}$$

Free type G laws

The Lévy measure of a $*$–type G distribution μ is of the form

$$\nu_\mu(dx) = \int_{\mathbb{R}_+} \varphi(x,s)\rho_\mu(ds)dx \tag{5}$$

for some Lévy measure ρ_μ of a distribution in I_+^* and $\varphi(x,s)$ is the Gaussian density of mean zero and variance s.

Free type G laws $:= \Lambda(\{*\text{–type G}\})$
by Arzmendi, Barndorff-Nielsen and Pérez-Abreu to appear in Rev. Braz. Probab. Statist.

$$
\begin{aligned}
\mathbb{E}[e^{i\sqrt{V}Z}] &= \mathbb{E}[\mathbb{E}[e^{i\sqrt{V}Z}]|V] \\
&= \mathbb{E}[e^{-\frac{1}{2}Vt^2}] \\
&= \mathcal{K}_V\left(\frac{t^2}{2}\right) \\
&= \int_{\mathbb{R}_+}\left(e^{-\frac{1}{2}t^2 s}-1\right)\rho(ds) \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}}\left(e^{-itx}-1\right)\frac{1}{\sqrt{2\pi s}}e^{-\frac{x^2}{2s}}dx\rho(ds) \\
&= \int_{\mathbb{R}}\left(e^{-itx}-1\right)\int_{R_+}\frac{1}{\sqrt{2\pi s}}e^{-\frac{x^2}{2s}}\rho(ds)dx
\end{aligned}
$$

Proposition

Let μ be a free type G distribution with Lévy measure ν_μ. Let $\sigma \in \mathbf{I}^{\boxplus}_{r+}$ with Lévy measure ν_σ obtained from

$$\nu_\mu = \frac{1}{2}\left(\nu_\sigma^{(1/2)+} + \nu_\sigma^{(1/2)-}\right). \tag{6}$$

Then

$$\mathcal{C}^{\boxplus}_{\mu}(z) = \int_{\mathbb{R}_+} \mathcal{C}^{\boxplus}_{\mathrm{m}\boxtimes\varphi_s}(z)\rho_\mu(ds) \tag{7}$$

and

$$\mathcal{C}^{\boxplus}_{\sigma}(z) = \int_{\mathbb{R}_+} \mathcal{C}^{\boxplus}_{\mathrm{m}\boxtimes\varphi_s^{(2)}}(z)\rho_\mu(ds) \tag{8}$$

If V is Poisson, Λ(law of $\sqrt{V}Z) = \mathrm{m} \boxtimes \varphi_1$.

New classes mixture of wigner law and type W laws

Definition

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μ *is of type* W *if* $\exists\ \sigma \in \mathcal{P}_+$ *s.t.* $\mu = \overline{\sigma} \boxtimes \mathrm{w}$ *is symmetric* ⊞*–ID law.*

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Of courese the class is subset of all symmetric laws.

How large is these classes?

How to characterize?

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$\boldsymbol{\mu}$ is of type $\boldsymbol{W}$ if $\exists\ \boldsymbol{\sigma} \in \mathcal{P}_+$ *s.t.* $\boldsymbol{\mu} = \overline{\boldsymbol{\sigma}} \boxtimes \mathbf{w}$ *is symmetric $\boxplus$–ID law.*

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Not all symmetric distributions are multiplicative mixtures of the Wigner law, this is the case of the arcsine symmetric distribution $\mathbf{a}$ on $(-1,1)$, as shown by the following result.

Proposition

Let $\mathbf{a}$ be arcsine distribution on $(-1,1)$. There does not exist $\boldsymbol{\lambda} \in \mathcal{P}_+$ such that $\mathbf{a} = \boldsymbol{\lambda} \boxtimes \mathbf{w}$.

Main result

Theorem

Let $\overline{\boldsymbol{\sigma}} \in \mathcal{P}_{+}$.

$\boldsymbol{\mu} = \overline{\boldsymbol{\sigma}} \boxtimes \mathbf{w} \in \boldsymbol{I}^{\boxplus}_{sym} \Longleftrightarrow \boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}} \boxtimes \overline{\boldsymbol{\sigma}} \in \boldsymbol{I}^{\boxplus}_{r,+}$.

Moreover, $\boldsymbol{\mu}^{(2)} = \mathbf{m} \boxtimes \boldsymbol{\sigma}$.

Main result

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Let $\overline{\sigma} \in \mathcal{P}_+$.

$\mu = \overline{\sigma} \boxtimes \mathrm{w} \in I^{\boxplus}_{sym} \Longleftrightarrow \sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I^{\boxplus}_{r,+}$.

Moreover, $\mu^{(2)} = \mathrm{m} \boxtimes \sigma$.

This is much stronger statement than in classical probability. In classical case, we have only necessary condition!

Compare to classical case. $X^2 = VZ^2$ is always $*$–ID.

How is $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$?

We cite one distribution class by W. Młotkowski.

Fuss Catalan number

$$A_0(p,r) = 1$$

$$A_m(p,r) = \frac{r}{m!}\prod_{i=1}^{m-1}(mp + r - i) \quad \text{if} \quad m \geq 1.$$

$\mu_{(p,r)}$ **: a probability measure with the moments** $A_m(p,r)$.

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Fuss Catalan number

$$A_0(p,r) = 1$$

$$A_m(p,r) = \frac{r}{m!}\prod_{i=1}^{m-1}(mp + r - i) \quad \text{if} \quad m \geq 1.$$

$\mu_{(p,r)}$ **: a probability measure with the moments** $A_m(p,r)$**.**

Example

Take $p = 2$ *and* $r = 1$*, then it is fP law.*

Proposition (W. Młotkowski)

(a) The free cumulant sequence of $\mu_{(p,r)}$ *is* $\{A_m(p-r,r)\}_{m=1}^{\infty}$.

(b) If $0 \leq 2r \leq p$ *and* $r+1 \leq p$ *then* $\mu_{(p,r)}$ *is* ⊞*-ID.*

(c) $\mu_{(p_1,r)} \boxtimes \mu_{(1+p_2,1)} = \mu_{(p_1+rp_2,r)}$ *for* $r \neq 0$.

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(2) $\mu_{(3/2,1)} = \mu_{(5/4,1)} \boxtimes \mu_{(5/4,1)}$.
Both $\mu_{(3/2,1)}$ **and** $\mu_{(5/4,1)}$ **are not ⊞–ID.**

Example

- *Wigner law.*
- *Symmetric ⊞–stable law (Cauchy law etc.).*
- *Symmetric Beta* $(1/2, 3/2)$.

Type AS laws

Definition

For $\mu \in I^{\boxplus}_{sym}$, if there exists some $\lambda \in \mathcal{P}_+$ such that $\mu = \lambda \boxtimes \mathbf{a}$, we call μ type AS.

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Theorem

μ is type AS with λ iff there exists some $\sigma \in I^{\boxplus}_{r,+}$ such that $\lambda \boxtimes \lambda = m_2 \boxtimes \sigma$.

About type AS

Remark

In the above theorem, if $\boldsymbol{\sigma} \in \boldsymbol{I}_{\boldsymbol{r}+}^{\boxplus}$ *is* $\boxtimes$*–2 divisible,* $\boldsymbol{\lambda} = \overline{\boldsymbol{m_2}} \boxtimes \overline{\boldsymbol{\sigma}}$.

About type AS

Remark

In the above theorem, if $\sigma \in I_{r+}^{\boxplus}$ is ⊠–2 divisible, $\lambda = \overline{m_2} \boxtimes \overline{\sigma}$.

Example

- *If μ is distributed as symmetric beta $(1/2, 3/2)$, μ is free type AS. This is because $\mu = \mathbf{m} \boxtimes \mathbf{a}$.*
- *If μ is semicircle Marchenko-Pastur distribution (i.e. $\mu = \mathbf{w} \boxtimes \mathbf{m}$) is free type AS.*

$$S_{\mu}(z) = S_{\mathrm{a}}(z) S_{\mathrm{m}}(z) \frac{1}{\sqrt{z+2}},$$

.

Thank you for your attention.