Matrix operators on symmetric Kähler domains.
References: A. Boussejra and H. Airault,

1 The basic tools

The representation

$$T_g F(Z) = k_g'(Z)^\gamma F(k_g(Z))$$

$$Z \in D \to k_g(Z) \in D, \quad g \in G$$

Hua's classical matrix

$$Z = (z_{ij}) \to \partial_Z = (\frac{\partial}{\partial z_{uj}})$$

Matrix multiplication
Let A, B be two square matrices and denote

$$\mathcal{V}_1 = AB - BA, \qquad \mathcal{V}_2 = i\,(AB + BA)$$
$$\mathcal{V}_3 = B - ABA, \qquad \mathcal{V}_4 = i\,(B + ABA)$$

Then $trace\,(\mathcal{V}_1\overline{\mathcal{V}}_1 - \mathcal{V}_2\overline{\mathcal{V}}_2 + \mathcal{V}_3\overline{\mathcal{V}}_3 + \mathcal{V}_4\overline{\mathcal{V}}_4) =$

$$2\,trace\,((I - A\overline{A})\overline{B}(I - \overline{A}A)B)$$

and $\mathcal{V}_1^2 - \mathcal{V}_2^2 + \mathcal{V}_3^2 + \mathcal{V}_4^2 = 0$.
Define

$$\mathcal{H}(\mathcal{V}_1) = 0, \qquad \mathcal{H}(\mathcal{V}_2) = i\,Id$$
$$\mathcal{H}(\mathcal{V}_3) = A, \qquad \mathcal{H}(\mathcal{V}_4) = -\,i\,A$$

then $trace\,(\mathcal{H}(\mathcal{V}_2)\mathcal{V}_2 + \mathcal{H}(\mathcal{V}_3)\mathcal{V}_3 + \mathcal{H}(\mathcal{V}_4)\mathcal{V}_4) = 0$ and $trace\,(\overline{\mathcal{H}(\mathcal{V}_2)}\mathcal{V}_2 + \overline{\mathcal{H}(\mathcal{V}_3)}\mathcal{V}_3 + \overline{\mathcal{H}(\mathcal{V}_4)}\mathcal{V}_4) =$

$$2\,trace\,[(I - \overline{A}A)\,B\,A]$$

2 Objectives

(1) The construction of measures on Kähler manifolds. We have already the volume measure (if the manifold is finite dimensional).

(2) Obtain Laplacian and Ornstein-Uhlenbeck operators on a Kähler manifold D from the infinitesimal holomorphic representation of a group G on D.

(3) Study the measures on D which are invariant with respect to the Laplacian and Ornstein-Uhlenbeck operators. The volume measure is an invariant measure for the Laplacian.

(4) Provide an approach to the construction of invariant measures on infinite dimensional manifolds.

3 Our present approach:

3.1 The vector fields of the infinitesimal holomorphic representation of G are the coefficients of Hua's matrix operators

We give examples of Kähler manifolds D and a group G of holomorphic transformations of D where the vector fields of the infinitesimal holomorphic representation of G are the coefficients of Hua's matrix operators and can be calculated with matrix products.

In this talk, I shall restrict to the manifold D of symmetric matrices Z such that $I - Z\overline{Z} > 0$ and to submanifolds of D.

Theorem 1 The vector fields of the infinitesimal representation are given by the matrix products

$$(i) \qquad \mathcal{V}_{\mathbf{t}} = (Z\partial_Z^\epsilon - \partial_Z^\epsilon Z) + i\,(Z\partial_Z^\epsilon + \partial_Z^\epsilon Z)$$

$$(ii) \qquad \mathcal{V}_{\mathbf{p}} = (\partial_Z^\epsilon - Z\partial_Z^\epsilon Z) + i\,(\partial_Z^\epsilon + Z\partial_Z^\epsilon Z)$$

where ∂_Z^ϵ is a Hua type matrix operator which has to be defined. The Laplacian and Ornstein-Uhlenbeck operators on D can be calculated by making products of the matrices

$$\mathcal{V}_1 = (Z\partial_Z^\epsilon - \partial_Z^\epsilon Z) = (a_{jk}),\ \mathcal{V}_2 = i\,(Z\partial_Z^\epsilon + \partial_Z^\epsilon Z) = (b_{jk})$$

$$\mathcal{V}_3 = (\partial_Z^\epsilon - Z\partial_Z^\epsilon Z) = (\phi_{jk}),\ \mathcal{V}_4 = i\,(\partial_Z^\epsilon + Z\partial_Z^\epsilon Z) = (\psi_{jk})$$

and their complex conjugates.

$\mathcal{V}_1$ is antisymmetric, thus $\sum_{j,k} a_{jk}\overline{a_{jk}} = -\,trace(\mathcal{V}_1\overline{\mathcal{V}_1})$,

$\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_4$ are symmetric, thus

$$\sum_{j,k} b_{jk}\overline{b_{jk}} = trace(\mathcal{V}_2\overline{\mathcal{V}_2})$$

$$\sum_{j,k} \phi_{jk}\overline{\phi_{jk}} + \sum_{j,k} \psi_{jk}\overline{\psi_{jk}} = trace(\mathcal{V}_3\overline{\mathcal{V}_3}) + trace(\mathcal{V}_4\overline{\mathcal{V}_4})$$

then

$$trace(\mathcal{V}_1\overline{\mathcal{V}_1}) - trace(\mathcal{V}_2\overline{\mathcal{V}_2}) + trace(\mathcal{V}_3\overline{\mathcal{V}_3}) + trace(\mathcal{V}_4\overline{\mathcal{V}_4})$$
$$= 2\,trace\,[(I - Z\overline{Z})\overline{\partial_Z}(I - \overline{Z}Z)\partial_Z]$$

It links Hua's expression of the Laplacian

$$trace\,[(I - Z\overline{Z})\overline{\partial_Z}(I - \overline{Z}Z)\partial_Z]$$

to the vector fields of the infinitesimal holomorphic representation of G.

3.2 The vector field $\mathcal{V} = \; trace\,((I - \overline{Z}Z)\partial^{\epsilon}_{Z}\, Z)$ in the Ornstein-Uhlenbeck operator

∂^{ϵ}_{Z} is the Hua type matrix operator to be defined.

How to obtain $\mathcal{V}$ from the operators

$$(i) \qquad \mathcal{V}_{\mathbf{t}} = (Z\partial^{\epsilon}_{Z} - \partial^{\epsilon}_{Z}Z) \,+\, i\,(Z\partial^{\epsilon}_{Z} + \partial^{\epsilon}_{Z}Z)$$

$$(ii) \qquad \mathcal{V}_{\mathbf{p}} = (\partial^{\epsilon}_{Z} -\, Z\partial^{\epsilon}_{Z}Z) + i\,(\partial^{\epsilon}_{Z} +\, Z\partial^{\epsilon}_{Z}Z)$$

To a matrix operator $\mathcal{T} = (v_{ij})$ where v_{ij} are vector fields, we associate the matrix of functions

$$\mathcal{H}(\mathcal{T}) = (h(v_{ij}))$$

where

$$h(v)(z) = \frac{d}{dt}_{|t=0} k'_{g_t}(z)$$

is the derivative of the holomorphic jacobian of the map $z \longrightarrow k_{g_t}(z)$, t is a real parameter, $g_t \in G$, $g_0 = e$ and $v = \frac{d}{dt}_{|t=0} g_t$.

We find

$\mathcal{H}(Z\partial^{\epsilon}_{Z} - \partial^{\epsilon}_{Z}Z) = 0,\;\; \mathcal{H}(Z\partial^{\epsilon}_{Z} + \partial^{\epsilon}_{Z}Z) = \; Id,$

$\mathcal{H}(\partial^{\epsilon}_{Z} \,-\, Z\partial^{\epsilon}_{Z}Z) = \; Z, \qquad \mathcal{H}(\partial^{\epsilon}_{Z} \,+\, Z\partial^{\epsilon}_{Z}\, Z) = -\,Z$

Let $\mathcal{U} = \overline{\mathcal{H}(Z\partial^{\epsilon}_{Z} + \partial^{\epsilon}_{Z}Z)}\,(Z\partial^{\epsilon}_{Z} + \partial^{\epsilon}_{Z}Z) +$

$$\overline{\mathcal{H}(\partial^{\epsilon}_{Z} \,-\, Z\partial^{\epsilon}_{Z}Z)}\,(\partial^{\epsilon}_{Z} \,-\, Z\partial^{\epsilon}_{Z}Z) \,+\, \overline{\mathcal{H}(\partial^{\epsilon}_{Z} \,+\, Z\partial^{\epsilon}_{Z}\, Z)}\,(\partial^{\epsilon}_{Z} \,+\, Z\partial^{\epsilon}_{Z}\, Z)$$

then

$$trace\,\mathcal{U} = \; trace\,[(I - \overline{Z}Z)\partial^{\epsilon}_{Z}\, Z] = \mathcal{V}$$

3.3 The representation of the group of holomorphic transformations $k_g : Z \longrightarrow \; k_g(Z)$ on D and the holomorphic Jacobian determinant $k'_g(Z)$

Let D be the manifold of $n \times n$ symmetric matrices Z such that $I - Z\overline{Z} > 0$. The group G of matrices $g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$ operates transitively on D with

$$W = k_g(Z) = (AZ + B)(\overline{B}Z + \overline{A})^{-1}$$

$$= (AZ + B)\, M^{-1} \qquad \text{where } M(Z) = \overline{B}Z + \overline{A}$$

We have

$$I - \overline{k_g(Z)}k_g(Z) = (M^*)^{-1}(I - \overline{Z}Z)M^{-1}$$

The volume element on D is

$$dv = \frac{dc_1\overline{dc_1}\cdots}{\det(I - \overline{Z}Z)^{n+1}}$$

and the holomorphic Jacobian is such that

$$|k'_g(Z)|^2 = \frac{\phi(k_g(Z))}{\phi(Z)} \quad \text{with} \quad \phi(Z) = (\det(I - \overline{Z}Z))^{n+1}$$

This does not stay true for a submanifold $\mathcal{M}$ of D. We shall give an example of $\mathcal{M}$ in the set of 3×3 matrices such that

$$|k'_g(Z)|^2 = \frac{\phi(k_g(Z))}{\phi(Z)} \quad \text{with} \quad \phi(Z) = \lambda_1(Z)^2\lambda_2(Z)^3\lambda_3(Z)^3$$

where $\lambda_1(Z)$, $\lambda_2(Z)$, $\lambda_3(Z)$ are the three eigenvalues of $I - \overline{Z}Z$.

Since $I - \overline{k_g(Z)}k_g(Z) = (M^*)^{-1}(I - Z^*Z)M^{-1}$, we deduce

$$\det(I - \overline{W}W) = \frac{\det(I - \overline{Z}Z)}{|\det M(Z)|^2}$$

We can write this identity in the following way

$$\Pi_j \ \lambda_j(W) = \frac{\Pi_j \ \lambda_j(Z)}{\Pi_j \ |\nu_j(Z)|^2}$$

where $\lambda_1(Z)$, $\lambda_2(Z)$, $\cdots$, $\lambda_n(Z)$ are the eigenvalues of $I - \overline{Z}Z$ and $\nu_1(Z)$, $\nu_2(Z)$, $\cdots$, $\nu_n(Z)$ are the eigenvalues of $M(Z)$.

For our example of Kähler submanifold $\mathcal{M}$ of 3×3 symmetric matrices in D, we prove

$$\lambda_1(W) = \frac{\lambda_1(Z)}{|\nu_1(Z)|^2} \quad \text{and} \quad \lambda_2(W)\lambda_3(W) = \frac{\lambda_2(Z)\lambda_3(Z)}{|\nu_2(Z)|^2|\nu_3(Z)|^2}$$

3.4 Find invariant measures on the Kähler submanifolds $\mathcal{M}$ of D.

Consider the subgroup $G_{\mathcal{M}}$ of G and acting on $\mathcal{M}$. Denote $Z = (z_1, z_2, \cdots)$ the independent variables in Z and $W = (w_1, w_2, \cdots)$ the independent variables in W and denote $k'_g(Z)$ the holomorphic Jacobian determinant of the map

$$k_g; \quad Z = (z_1, z_2, \cdots) \to W = (w_1, w_2, \cdots)$$

1- We wish to find probability measures μ on $\mathcal{M}$ such that for $g \in G_{\mathcal{M}}$,

$$\int_{\mathcal{M}} |k'_g(Z)|^\gamma |F(k_g(Z))|^2 \mu_\gamma(dZ) = \int_{\mathcal{M}} |F(Z)|^2 \mu_\gamma(dZ)$$

2- One of the difficulties for submanifolds is to determine the Jacobian determinant $k'_g(Z)$ when the number p of independent variables in the $n \times n$ symmetric matrix Z is less than $n(n+1)/2$. For a submanifold $\mathcal{M}$ of D, let $G_{\mathcal{M}}$ be the subgroup of G and acting on $\mathcal{M}$.

(1) Is there any relation between $k'_g(Z)$, $\det(\overline{B}Z + \overline{A})$ and $\dfrac{\det(I - \overline{W}W)}{\det(I - \overline{Z}Z)}$ as it was the case for D?

(2) The calculation of the holomorphic Jacobian $k'_g(Z)$ is delicate.

(3)- How to relate the independent vector fields

$$\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots$$

to the independent vector fields $\dfrac{\partial}{\partial w_1}, \dfrac{\partial}{\partial w_2}, \cdots$, that is how to calculate the holomorphic Jacobian matrix of the map $k_g(Z)$?

4 The condition $dZ = M^{tr}[dW]M$

Let $W = (AZ + B)(\overline{B}Z + \overline{A})^{-1}$ then

$$dW = (M^{-1})^{tr}(dZ)M^{-1} \quad \text{with} \quad M = \overline{B}Z + \overline{A}$$

If the coefficients of the matrix Z are all distinct, we have independent variables and

$$\partial_W = M \partial_Z \, M^{tr}$$

where $\partial_W = (\dfrac{\partial}{\partial w_{jk}})$ and $\partial_Z = (\dfrac{\partial}{\partial z_{jk}})$ are the classical Hua's matrices operators.

The relation $\partial_W = M\partial_Z \, M^{tr}$ is no more valid if the coefficients of Z are not independent variables. In the case of the domain of symmetric matrices Z, it remains true if we replace ∂_Z and ∂_W respectively by the operators matrices

$$\widehat{\partial_Z} = ((1 + \delta_{jk})\frac{\partial}{\partial z_{jk}}) \quad \text{and} \quad \widehat{\partial_W} = ((1 + \delta_{jk})\frac{\partial}{\partial w_{jk}})$$

When there are less than $n(n+1)/2$ independent variables, we have to make another choice for $\widehat{\partial_Z}$ and $\widehat{\partial_W}$ and it depends on the domain.

Remark In the case of the domain of symmetric matrices Z, in order to obtain the Laplacian as a trace of product of matrices, Hua makes a rescaling of the coefficients of Z, keeping the matrix symmetric, instead of changing ∂_Z. This amounts to the same.

4.1 From $dZ = M^{tr}[dW]M$ to $\overset{\diamond\diamond}{\partial}_W = M\overset{\diamond\diamond}{\partial}_Z M^{tr}$

The matrix operator $\overset{\diamond\diamond}{\partial}_Z$ depends on the manifold $\mathcal{M}$.

We say that $\mathcal{M}$ is a linear submanifold of D if it consists of matrices $Z \in D$ such that $ZJ_1 = J_2Z$ where J_1 and J_2 are two real invertible matrices.

Let Z and W in the same linear submanifold $\mathcal{M}$ of D and assume that $dZ = M^{tr}[dW]M$. The following lemma can be applied to some examples of linear submanifolds of D.

Lemma. Assume that (C) is satisfied:

(C) :There is a partition $A_1 \cup A_2 \cup \cdots \cup A_k$ of the indices of the coefficients of the matrices W and Z such that when $(j,p) \in A_r$ then $z_{jp} = b_r$ and $w_{jp} = c_r$. If $r \neq s$, then b_r and b_s (c_r and c_s) are independent variables. Let $n_{jp} = n_r$ be the cardinal of A_r. Then $\overset{\diamond\diamond}{\partial}_W = M\overset{\diamond\diamond}{\partial}_Z M^{tr}$ with the matrix operator $\overset{\diamond\diamond}{\partial}_Z = (\frac{1}{n_{jp}}\frac{\partial}{\partial z_{jp}})$.

5 The Kähler manifold of symmetric matrices Z such that $I - Z\overline{Z} > 0$ and submanifolds.

A Kähler manifold (D,ω) is a complex manifold D and ω is a smooth differential (1,1)-form on D which can be written

$$\omega = i\sum_{j,k}\frac{\partial^2 \log K}{\partial c_j \partial \overline{c_k}} dc_j \wedge d\overline{c_k}, \qquad c = (c_1, c_2, \cdots) \in D$$

where the Kähler potential K is a real positive function. Then ω is closed ($d\omega = 0$).

To ω is associated the riemannian metric

$$ds^2 = -\sum_{j,k}\frac{\partial^2 \log K}{\partial c_j \partial \overline{c_k}} dc_j \, d\overline{c_k}$$

5.1 We take as Kähler manifold D the set of symmetric matrices such that $I - Z\overline{Z} > 0$

with Kähler potential

$$K(Z,\overline{Z}) = \det(I - \overline{Z}Z)$$

In that case, the independant coefficients in $Z = (z_{ij})$ are

$$(c_1, c_2, \cdots) = (z_{ij})_{i \leq j}$$

For the manifold D, the Bergman kernel is

$$[\det(I - \overline{Z}Z)]^{\alpha}$$

see for example Hua's book. Thus for D, the Kähler metric defined with the potential $K(Z, \overline{Z}) = \det(I - \overline{Z}Z)$ is the same as the Bergman metric.

5.2 We take as Kähler manifold $\mathcal{M}$ a submanifold of D. In $\mathcal{M}$, we still have symmetric matrices such that $I - Z\overline{Z} > 0$, but not all

On $\mathcal{M}$, we shall take again as Kähler potential $K(Z, \overline{Z}) = \det(I - \overline{Z}Z)$. But for a submanifold, $[\det(I - \overline{Z}Z)]^{\alpha}$ may be different from the Bergman potential as shown by our examples.

Recall: Bergman kernel, let $f : \Omega_1 \to \Omega_2$, f holomorphic, then

$$\det J_{\mathbf{C}} f(z) K_{\Omega_2}(f(z), f(\xi)) \overline{\det J_{\mathbf{C}} f(\xi)} = K_{\Omega_1}(z, \xi)$$

In our example of $\mathcal{M}$ of 3×3 matrices, we prove

$$|k_g'(Z)|^2 = \frac{\phi(k_g(Z))}{\phi(Z)} \quad \text{with} \quad \phi(Z) = \lambda_1(Z)^2 \lambda_2(Z)^3 \lambda_3(Z)^3$$

where $\lambda_1(Z)$, $\lambda_2(Z)$, $\lambda_3(Z)$ are the three eigenvalues of $I - \overline{Z}Z$. Then $[\det(I - \overline{Z}Z)]^{\alpha}$ is not the Bergman kernel and the metric obtained with

$$K(Z, \overline{Z}) = \det(I - \overline{Z}Z)$$

is different from the Bergman metric.

5.3 The (1,1)-form ω and the metric when the Kähler potential is $K(Z, \overline{Z}) = \det(I - \overline{Z}Z)$

In that case, the Kähler differential (1,1)-form ω and the metric have nice expressions thanks to the relation

$$\log\det(I - \overline{Z}\,Z) = \mathit{trace}\ \log(I - \overline{Z}\,Z)$$

Then (see for example BSM 2012-HA)

$$p_{jk} = \frac{\partial^2}{\partial c_j \partial \overline{c_k}} \log\det(I - \overline{Z}Z)$$

$$= -\,\mathit{trace}[(I - \overline{Z}Z)^{-1} \frac{\partial\,\overline{Z}}{\partial \overline{c_k}} \times (I - Z\overline{Z})^{-1} \frac{\partial Z}{\partial c_j}]$$

Since $dZ = (dz_{ij})$ is given by $dZ = \sum_k \frac{\partial Z}{\partial c_k} dc_k$, the closed two-form ω and the riemannian metric ds^2 are

$$\omega = i \;\; trace\,[(I - \overline{Z}Z)^{-1} d\overline{Z} \wedge (I - Z\overline{Z})^{-1} dZ]$$

$$ds^2 = -\sum_{j,k} p_{jk} dc_j \, d\overline{c_k} = \; trace\,[(I - \overline{Z}Z)^{-1} d\overline{Z}(I - Z\overline{Z})^{-1} dZ]$$

5.4 The (1,1)-form ω and the metric for a submanifold

For a submanifold $\mathcal{M}$ of D, we take the same Kähler potential

$$K(Z, \overline{Z}) = \det(I - \overline{Z}Z)$$

but there are less independent variables:

then dZ is different, it gives a closed form $\omega_{\mathcal{M}}$ which is different from ω and a metric $ds^2_{\mathcal{M}}$ different from the ds^2 on D and also different from the Bergman metric on $\mathcal{M}$.

For submanifolds $\mathcal{M}$, formulas

$$\omega_{\mathcal{M}} = i \;\; trace\,[(I - \overline{Z}Z)^{-1} d\overline{Z} \wedge (I - Z\overline{Z})^{-1} dZ]$$

$$= i \;\; trace\,[\overline{\Omega} \wedge \Omega]$$

and $ds^2_{\mathcal{M}} = -\sum_{j,k} p_{jk} dc_j \, d\overline{c_k} = \; trace\,[(I - \overline{Z}Z)^{-1} d\overline{Z}(I - Z\overline{Z})^{-1} dZ]$

are still valid but with a different dZ.

Assume that there is a group $G_{\mathcal{M}}$ of transformations $Z \to W$ on $\mathcal{M}$ such that

$$dZ = M^{tr} dW \, M$$

and $I - \overline{W}W = (M^*)^{-1}(I - \overline{Z}Z)M^{-1}$ with $W = k_g(Z)$, the closed $(1,1)$-form $\omega_{\mathcal{M}}$ and the metric $ds^2_{\mathcal{M}}$ are invariant under the transformations of the group $G_{\mathcal{M}}$.

The volume element is obtained with

$$\omega_{\mathcal{M}} \wedge \omega_{\mathcal{M}} \wedge \cdots$$

From $dZ = M^{tr} dW \, M$, we deduce operators matrices $\overset{\diamond}{\partial}_W$ and $\overset{\diamond}{\partial}_Z$ such that

$$\overset{\diamond}{\partial}_W = M \overset{\diamond}{\partial}_Z \, M^{tr}$$

Thus the operator

$$\Delta_{\mathcal{M}} = \; trace\,[(I - Z\overline{Z})\overline{\overset{\diamond}{\partial}_Z}(I - \overline{Z}Z)\overset{\diamond}{\partial}_Z]$$

is invariant under the transformations of $G_{\mathcal{M}}$.

For our examples of submanifolds $\mathcal{M}$, $\Delta_{\mathcal{M}}$ is the Laplacian associated to the metric

$$ds^2_{\mathcal{M}} = trace\,[(I - \overline{Z}Z)^{-1}d\overline{Z}(I - Z\overline{Z})^{-1}dZ]$$

but $ds^2_{\mathcal{M}}$ is not the Bergman metric corresponding to the group of transformations $G_{\mathcal{M}}$.

6 The group G of holomorphic transformations of D and the operator T_g on holomorphic functions on D.

$$W = k_g(Z) = (AZ + B)(\overline{B}Z + \overline{A})^{-1} \quad g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \in G$$

with $\overline{A}^{tr} A - B^{tr}\overline{B} = I$ and $\overline{A}^{tr} B = B^{tr}\overline{A}$.

$$T_g F(Z) = k'_g(Z)^{\gamma} F(k_g(Z)), \qquad \gamma \in \mathbf{R}$$

Since

$$|k'_g(Z)|^2 = \frac{\phi(k_g(Z))}{\phi(Z)} \quad \text{with} \quad \phi(Z) = (\det(I - Z^*Z))^{\alpha}$$

the operator T_g is unitary in $Hol^2_{\mathbf{C}}(d\mu)$ for the measure

$$d\mu_{\gamma} = [\det(I - \overline{Z}Z)]^{\alpha(\gamma-1)} d\lambda(Z)$$

The volume element on D is up to a multiplicative constant, equal to

$$d\mu_0 = \frac{d\lambda(Z)}{\phi(Z)}$$

6.1 The Lie algebra g of G is the real vector space,

$$\mathbf{g} = \{\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \text{ with } \alpha + \alpha^* = 0 \text{ and } \beta = \beta^{tr}\}$$

$$\mathbf{g} = \{\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \alpha^* + \alpha = 0\} \oplus \{\begin{pmatrix} 0 & \beta \\ \overline{\beta} & 0 \end{pmatrix}, \beta = \beta^{tr}\}$$

$$\mathbf{g} = \mathbf{t} \oplus \mathbf{p} \quad \text{(Cartan decomposition)}$$

6.2 The holomorphic vector fields on D.

Let $X = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbf{g}$. Denote $W_t(X) = g_t$ a curve on G such that

$$g_t = \begin{pmatrix} A_t & B_t \\ \overline{B_t} & \overline{A_t} \end{pmatrix} \in G, \; g_0 = I \; \text{ and } \; \frac{d}{dt}_{|t=0} g_t = X$$

$$\text{where} \quad A_t = I + t\alpha + o(t^2) \quad \text{and} \quad B_t = t\beta + o(t^2)$$

Then

$$W_t = k_{g_t}(Z) = (Z + t(\alpha Z + \beta) + o(t^2))(I - t(\overline{\alpha} + \overline{\beta} Z) + o(t^2))$$

$$\frac{d}{dt}_{|t=0} k_{g_t}(Z) = \frac{d}{dt}_{|t=0} (A_t Z + B_t)(\overline{B_t} Z + \overline{A_t})^{-1} = \alpha Z - Z\overline{\alpha} + \beta - Z\overline{\beta} Z$$

Let F be a complex-valued holomorphic function on D, ($\frac{\partial F}{\partial \overline{z}} = 0$). The holomorphic vector fields $V(X)$ are given by

$$(V(X)F)(Z) = \frac{d}{dt}_{|t=0} F(k_{g_t}(Z))$$

$$\frac{d}{dt}_{|t=0} F(W_t) = \sum_{p \leq q} \frac{\partial F}{\partial z_{pq}} \frac{d}{dt}_{|t=0} (W_t)_{pq} \quad \textit{where} \quad W_t = k_{g_t}(Z)$$

7 A basis (e_j) of $\mathbf{t}$ and the vector fields $V(e_j)$.

The Lie algebta $\mathbf{t}$ has dimension n^2 as real vector space.

Basis: $H_j = \begin{pmatrix} d_j & 0 \\ 0 & \overline{d_j} \end{pmatrix}$ where $d_j = iE_{jj}, \; j = 1, .., n$,

$$X_{jk} = \begin{pmatrix} E_{jk} - E_{kj} & 0 \\ 0 & E_{jk} - E_{kj} \end{pmatrix}$$

$$Y_{jk} = \begin{pmatrix} i(E_{jk} + E_{kj}) & 0 \\ 0 & -i(E_{jk} + E_{kj}) \end{pmatrix}, \quad 1 \leq j < k \leq n$$

Vector fields:

$$V(H_j) = [(d_j Z + Z d_j)\partial_Z]_{jj} = \, iz_{jj} \frac{\partial}{\partial z_{jj}} + i(Z\partial_Z^{tr})_{jj}$$

$$V(X_{jk}) = (Z\partial_Z^{tr})_{kj} - (Z\partial_Z^{tr})_{jk} + z_{jk} \frac{\partial}{\partial z_{jj}} - z_{kj} \frac{\partial}{\partial z_{kk}}, \quad j < k$$

$$V(Y_{jk}) = i\,[(Z\partial_Z^{tr})_{kj} + (Z\partial_Z^{tr})_{jk} + z_{kj} \frac{\partial}{\partial z_{jj}} + z_{jk} \frac{\partial}{\partial z_{kk}}\,], \quad j < k$$

8 The vector fields $V(H_j)$, $V(X_{jk})$, $V(Y_{jk})$ as coefficients of a Hua's matrix operator

To the matrix $Z = (z_{ij}) \in D$, we associate the Hua's matrix operator,

$$\widehat{\partial_Z} = \frac{1}{2}((1+\delta_{ij})\frac{\partial}{\partial z_{ij}}) \quad \text{where} \quad \delta_{ij} = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad \delta_{jj} = 1$$

then

$$Z\widehat{\partial_Z} - \widehat{\partial_Z}\, Z = -\frac{1}{2}\begin{pmatrix} 0 & V(X_{12}) & V(X_{13}) & \cdots \\ -V(X_{12}) & 0 & V(X_{23}) & \cdots \\ -V(X_{13}) & -V(X_{23}) & 0 & \cdots \\ & \cdots & & \end{pmatrix}$$

$$Z\widehat{\partial_Z} + \widehat{\partial_Z}\, Z = -\frac{i}{2}\begin{pmatrix} 2V(H_1) & V(Y_{12}) & V(Y13) & \cdots \\ V(Y_{12}) & 2V(H_2) & V(Y_{23}) & \cdots \\ V(Y_{13}) & V(Y_{23}) & 2V(H_3) & \cdots \\ & \cdots & & \end{pmatrix}$$

8.1 Corollaries

$$\sum_{e_j \in \mathbf{t}} B(e_j, e_j)^{-1} V(e_j)^2 = 2\, trace\,(Z\widehat{\partial_Z} Z\widehat{\partial_Z})$$

and

$$D_{\mathbf{t}} =: \sum_{e_j \in \mathbf{t}} B(e_j, e_j)^{-1} V(e_j)\overline{V(e_j)} = -2\sum_{j,k} (Z\widehat{\partial_Z})_{kj}\, (\overline{Z}\widehat{\partial_{\overline{Z}}})_{kj}$$

$$= -2 trace\,(\overline{Z} Z \widehat{\partial_Z}\widehat{\partial_{\overline{Z}}})$$

where $B(X,Y) = trace\,(XY)$ is the Killing form on $\mathbf{g}$.

9 A basis (e_j) of the Lie subalgebra p and the vector fields $V(e_j)$.

The basis: $A_j = \begin{pmatrix} 0 & a_j \\ a_j & 0 \end{pmatrix}$, $\quad B_j = \begin{pmatrix} 0 & b_j \\ -b_j & 0 \end{pmatrix}$ where $a_j = diag(0, ..., 0, 1, 0, ...; 0)$, $b_j = ia_j$, $j = 1, ..., n$ and A_{jk}, B_{jk} with $(1 \leq j < k \leq n)$:

$$A_{jk} = \begin{pmatrix} 0 & E_{jk} + E_{kj} \\ E_{jk} + E_{kj} & 0 \end{pmatrix},$$

$$B_{jk} = \begin{pmatrix} 0 & i(E_{jk} + E_{kj}) \\ -i(E_{jk} + E_{kj}) & 0 \end{pmatrix}$$

9.1 The vector fields

For $j < k$, let $\beta_{jk} = E_{jk} + E_{kj}$ and $a_j = \frac{1}{2}(E_{jj} + E_{jj}) = \frac{1}{2}\beta_{jj}$. Then

$$V(A_{jk}) = \frac{\partial}{\partial z_{jk}} - \sum_{p\le q}(Z\beta_{jk}Z)_{pq}\frac{\partial}{\partial z_{pq}}$$

$$V(B_{jk}) = i\,(\frac{\partial}{\partial z_{jk}} + \sum_{p\le q}(Z\beta_{jk}Z)_{pq}\frac{\partial}{\partial z_{pq}})$$

$$V(A_j) = \frac{\partial}{\partial z_{jj}} - \frac{1}{2}\sum_{p\le q}(Z\beta_{jj}Z)_{pq}\frac{\partial}{\partial z_{pq}}$$

$$V(B_j) = i\,(\frac{\partial}{\partial z_{jj}} + \frac{1}{2}\sum_{p\le q}(Z\beta_{jj}Z)_{pq}\frac{\partial}{\partial z_{pq}})$$

9.2 The vector fields $V(A_j)$, $V(B_j)$, $V(A_{jk})$, $V(B_{jk})$ as coefficients of a Hua's matrix operator

For the symmetric matrices $\mathcal{V}^A = \widehat{\partial_Z} - Z\widehat{\partial_Z}\,Z$ and $\mathcal{V}^B = i\,(\widehat{\partial_Z} + Z\widehat{\partial_Z}\,Z)$, it holds

$$\widehat{\partial_Z} - Z\widehat{\partial_Z}\,Z = \begin{pmatrix} 2V(A_1) & V(A_{12}) & V(A_{13}) & \cdots \\ V(A_{12}) & 2V(A_2) & V(A_{23}) & \cdots \\ V(A_{13}) & V(A_{23}) & 2V(A_3) & \cdots \\ \cdots & & & \end{pmatrix}$$

$$i\,(\widehat{\partial_Z} + Z\widehat{\partial_Z}\,Z) = \begin{pmatrix} 2V(B_1) & V(B_{12}) & V(B_{13}) & \cdots \\ V(B_{12}) & 2V(B_2) & V(B_{23}) & \cdots \\ V(B_{13}) & V(B_{23}) & 2V(B_3) & \cdots \\ \cdots & & & \end{pmatrix}$$

9.3 Corollaries

$$\frac{1}{4}\sum_{j<k}V(A_{jk})^2 + V(B_{jk})^2 + \frac{1}{2}\sum_j V(A_j)^2 + V(B_j)^2$$
$$= -\frac{1}{2}\,trace\,(Z\widehat{\widehat{\partial_Z}}\,Z\widehat{\widehat{\partial_Z}})$$

$$\frac{1}{4}\sum_{j<k}V(A_{jk})\overline{V(A_{jk})} + V(B_{jk})\overline{V(B_{jk})} + \frac{1}{2}\sum_j V(A_j)\overline{V(A_j)} + V(B_j)\overline{V(B_j)}$$
$$= \frac{1}{4}\,trace\,(\widehat{\widehat{\partial_Z}}\widehat{\widehat{\partial_{\overline{Z}}}} + \overline{Z}\,Z\widehat{\widehat{\partial_Z}}Z\overline{Z}\widehat{\widehat{\partial_{\overline{Z}}}})$$

Notation: We have put $\widehat{\widehat{\partial_Z}} = 2\widehat{\partial_Z}$.

9.4 Identification of $\sum_j B(e_j,e_j)^{-1}l(e_j)\overline{l(e_j)}$ with the Laplacian

$$\sum_j B(e_j,e_j)^{-1}l(e_j)\overline{l(e_j)} = \frac{1}{4}\,trace\,[(I-\overline{Z}Z)\widehat{\widehat{\partial_Z}}(I-Z\overline{Z})\widehat{\widehat{\partial_{\overline{Z}}}}]$$

where $\widehat{\widehat{\partial_Z}} = \partial_Z + D_Z$ and $D_Z = diagonal\,[\frac{\partial}{\partial z_{11}},\cdots,\frac{\partial}{\partial z_{jj}},\cdots]$ is a diagonal matrix operator.

10 Linear Kähler submanifolds of D.

We say that $\mathcal{M}$ is a linear submanifold of D if it consists of matrices $Z\in D$ such that $ZJ_1 = J_2Z$ where J_1 and J_2 are two real invertible matrices. We have considered three examples:

Example 1. $Z\in\mathcal{M}$ if $JZ = ZJ$ where
$J = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$. Then $Z = \begin{pmatrix} z_1 & z_2 & z_4\\ z_2 & z_3 & z_2\\ z_4 & z_2 & z_1\end{pmatrix}$.

Example 2. $Z\in\mathcal{M}_-$ if $JZ = ZJ$ where
$J = \begin{pmatrix} 0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$. Then $Z = \begin{pmatrix} z_1 & z_2 & z_3\\ z_2 & z_1 & -z_3\\ z_3 & -z_3 & z_4\end{pmatrix}$.

In Examples 1 and 2, $J_1 = J_2 = J$ and $J^2 = Id$.

Example 3. Extending Example 1, let $Z\in\mathcal{M}$ if $Z = \begin{pmatrix} z_1 & z_2 & z_4\\ z_2 & z_3 & \lambda\,z_2\\ z_4 & \lambda z_2 & \lambda^2\,z_1\end{pmatrix}$,
$J = \begin{pmatrix} 0 & 0 & \lambda\\ 0 & 1 & 0\\ \frac{1}{\lambda} & 0 & 0\end{pmatrix}$, then $J^{tr}ZJ = Z$ and λ is fixed.

11 Condition $dZ = M^{tr}[dW]M$ for linear submanifolds of D.

Example 1. $Z = \begin{pmatrix} z_1 & z_2 & z_4\\ z_2 & z_3 & z_2\\ z_4 & z_2 & z_1\end{pmatrix}$, $dZ = M^{tr}[dW]M$ implies $M = \begin{pmatrix} m_1 & m_2 & m_4\\ \kappa & m_3 & \kappa\\ m_4 & m_2 & m_1\end{pmatrix}$
and $\widehat{\widehat{\partial}}_W = M\widehat{\widehat{\partial}}_Z M^{tr}$ with the matrix operator

$$\widehat{\widehat{\partial}}_Z = \begin{pmatrix} \frac{1}{2}\frac{\partial}{\partial z_1} & \frac{1}{4}\frac{\partial}{\partial z_2} & \frac{1}{2}\frac{\partial}{\partial z_4}\\ \frac{1}{4}\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} & \frac{1}{4}\frac{\partial}{\partial z_2}\\ \frac{1}{2}\frac{\partial}{\partial z_4} & \frac{1}{4}\frac{\partial}{\partial z_2} & \frac{1}{2}\frac{\partial}{\partial z_1}\end{pmatrix}$$

Example 2. $Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_1 & -z_3 \\ z_3 & -z_3 & z_4 \end{pmatrix}$, $\quad M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_2 & m_1 & -m_3 \\ \delta & -\delta & m_4 \end{pmatrix}$

$$\widehat{\widehat{\partial}}_W = M \widehat{\widehat{\partial}}_Z M^{tr} \quad \text{with} \quad \widehat{\widehat{\partial}}_Z = \begin{pmatrix} \frac{1}{2}\frac{\partial}{\partial z_1} & \frac{1}{2}\frac{\partial}{\partial z_2} & \frac{1}{4}\frac{\partial}{\partial z_3} \\ \frac{1}{2}\frac{\partial}{\partial z_2} & \frac{1}{2}\frac{\partial}{\partial z_1} & -\frac{1}{4}\frac{\partial}{\partial z_3} \\ \frac{1}{4}\frac{\partial}{\partial z_3} & -\frac{1}{4}\frac{\partial}{\partial z_3} & \frac{\partial}{\partial z_4} \end{pmatrix}$$

Example 3. $Z = \begin{pmatrix} z_1 & z_2 & z_4 \\ z_2 & z_3 & \lambda\, z_2 \\ z_4 & \lambda z_2 & \lambda^2\, z_1 \end{pmatrix}$, $dZ = M^{tr}[dW]M$ implies $M = \begin{pmatrix} m_1 & \lambda\, m_2 & \lambda^2\, m_4 \\ \delta_2 & m_3 & \lambda\delta_2 \\ m_4 & m_2 & m_1 \end{pmatrix}$ and $\widehat{\widehat{\partial}}_W = M \widehat{\widehat{\partial}}_Z\, M^{tr}$ with $\widehat{\widehat{\partial}}_Z = constant \begin{pmatrix} 2\frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & 2\frac{\partial}{\partial z_4} \\ \frac{\partial}{\partial z_2} & 4\frac{\partial}{\partial z_3} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} \\ 2\frac{\partial}{\partial z_4} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} & \frac{2}{\lambda^2}\frac{\partial}{\partial z_1} \end{pmatrix}$.

In Examples 1 and 2, we know the group of holomorphic transformations on $\mathcal{M}$ and its Lie algebra. With the same methods that we have used on D, we find the vector fields of the infinitesimal representation and we verify that they are given by $Z\widehat{\widehat{\partial}}_Z - \widehat{\widehat{\partial}}_Z\, Z$, $\quad i(Z\widehat{\widehat{\partial}}_Z + \widehat{\widehat{\partial}}_Z\, Z)$, $\widehat{\widehat{\partial}}_Z - Z\widehat{\widehat{\partial}}_Z\, Z$ and $\quad i(\widehat{\widehat{\partial}}_Z + Z\widehat{\widehat{\partial}}_Z\, Z)$.

Example 1, $ZJ = JZ$,

$Z = \begin{pmatrix} z_1 & z_2 & z_4 \\ z_2 & z_3 & z_2 \\ z_4 & z_2 & z_1 \end{pmatrix}$, $\widehat{\widehat{\partial}}_Z = constant \begin{pmatrix} 2\frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & 2\frac{\partial}{\partial z_4} \\ \frac{\partial}{\partial z_2} & 4\frac{\partial}{\partial z_3} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} \\ 2\frac{\partial}{\partial z_4} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} & \frac{2}{\lambda^2}\frac{\partial}{\partial z_1} \end{pmatrix}$.

The group $G_{\mathcal{M}}$ is the set of matrices $\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$ such that $\overline{A}^{tr} A - B^{tr}\overline{B} = I$, $\overline{A}^{tr} B = B^{tr}\overline{A}$, $JA = AJ$, $JB = BJ$.

$$Z\widehat{\widehat{\partial}}_Z - \widehat{\widehat{\partial}}_Z\, Z = \begin{pmatrix} 0 & -\frac{V(\alpha)}{4} & 0 \\ \frac{V(\alpha)}{4} & 0 & \frac{V(\alpha)}{4} \\ 0 & -\frac{V(\alpha)}{4} & 0 \end{pmatrix}$$

$$i(Z\widehat{\widehat{\partial}}_Z + \widehat{\widehat{\partial}}_Z\, Z) = \begin{pmatrix} \frac{V(\gamma(1))}{2} & \frac{V(\gamma(2))}{4} & \frac{V(\gamma(4))}{2} \\ \frac{V(\gamma(2))}{4} & V(\gamma(3)) & \frac{V(\gamma(2))}{4} \\ \frac{V(\gamma(4))}{2} & \frac{V(\gamma(2))}{4} & \frac{V(\gamma(1))}{2} \end{pmatrix}$$

$$\widehat{\widehat{\partial}}_Z \; - \; Z\widehat{\widehat{\partial}}_Z\, Z = \begin{pmatrix} \frac{V(\beta(1))}{2} & \frac{V(\beta(2))}{4} & \frac{V(\beta(4))}{2} \\ \frac{V(\beta(2))}{4} & V(\beta(3)) & \frac{V(\beta(2))}{4} \\ \frac{V(\beta(4))}{2} & \frac{V(\beta(2))}{4} & \frac{V(\beta(1))}{2} \end{pmatrix}$$

$$i(\widehat{\widehat{\partial}}_Z \; + \; Z\widehat{\widehat{\partial}}_Z\, Z) = \begin{pmatrix} \frac{V(i\beta(1))}{2} & \frac{V(i\beta(2))}{4} & \frac{V(i\beta(4))}{2} \\ \frac{V(i\beta(2))}{4} & V(i\beta(3)) & \frac{V(i\beta(2))}{4} \\ \frac{V(i\beta(4))}{2} & \frac{V(i\beta(2))}{4} & \frac{V(i\beta(1))}{2} \end{pmatrix}$$

Then $\sum_j \frac{1}{B(e_j,e_j)} l(e_j)\overline{l(e_j)} = trace\,((I - \overline{Z}Z)\widehat{\widehat{\partial}}_Z\,(I - Z\overline{Z})\widehat{\widehat{\partial}}_{\overline{Z}})$ where $l(e_j)$ are the the vector fields of the infinitesimal representation.

Recall:

To a matrix operator $\mathcal{T} = (v_{ij})$ where v_{ij} are vector fields, we associate the matrix of functions

$$\mathcal{H}(\mathcal{T}) = (h(v_{ij})) \quad \text{where} \quad h(v)(z) = \frac{d}{dt}_{|t=0} k'_{g_t}(z)$$

is the derivative of the holomorphic jacobian of $z \longrightarrow k_{g_t}(z)$, t is a real parameter, $g_t \in G_{\mathcal{M}}$, $g_0 = e$ and $v = \frac{d}{dt}_{|t=0} g_t$.

For the manifold $\mathcal{M}$,

$\mathcal{H}(Z\hat{\hat{\partial}}_Z - \hat{\hat{\partial}}_Z Z) = 0$,

$\mathcal{H}(Z\hat{\hat{\partial}}_Z + \hat{\hat{\partial}}_Z Z) = 3\, Id$,

$\mathcal{H}(\hat{\hat{\partial}}_Z - Z\hat{\hat{\partial}}_Z) = -3Z + \frac{z_1 - z_4}{2}(\beta(1) - \beta(4))$,

$\mathcal{H}(\hat{\hat{\partial}}_Z + Z\hat{\hat{\partial}}_Z\, Z) = 3\, Z - \frac{z_1 - z_4}{2}(\beta(1) - \beta(4))$

where $\beta(1) - \beta(4) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

Let $\mathcal{U} = \overline{\mathcal{H}(Z\partial^\epsilon_Z + \partial^\epsilon_Z Z)}\,(Z\partial^\epsilon_Z + \partial^\epsilon_Z Z)$
$-\overline{\mathcal{H}(\partial^\epsilon_Z - Z\partial^\epsilon_Z Z)}\,(\partial^\epsilon_Z - Z\partial^\epsilon_Z Z)$
$-\overline{\mathcal{H}(\partial^\epsilon_Z + Z\partial^\epsilon_Z\, Z)}\,(\partial^\epsilon_Z + Z\partial^\epsilon_Z\, Z)$

then $trace\,(\mathcal{U}) = 6\, trace\,[(I - \overline{Z}Z)\partial^\epsilon_Z\, Z]$
$+ (\overline{z_1} - \overline{z_4})\, trace\,[(\beta(1) - \beta(4))\, Z\, \partial^\epsilon_Z Z] = \mathcal{V}$.

We deduce $\mathcal{V} := \sum_j B(e_j, e_j)^{-1}\overline{h(e_j)}l(e_j) =$

$$trace\,[(I - \overline{Z}Z)\hat{\hat{\partial}}_Z\, Z] - constant\,(\overline{z_1} - \overline{z_4})(z_1 - z_4)^2(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_4})$$

12 Example 3: We do not know the group of transformations

In Example 3, $Z = \begin{pmatrix} z_1 & z_2 & z_4 \\ z_2 & z_3 & \lambda\, z_2 \\ z_4 & \lambda z_2 & \lambda^2\, z_1 \end{pmatrix}$, let W be of the same kind. By identifying coefficients, we can calculate the conditions on the matrix M in order that $dZ = M^{tr}[dW]M$. This implies that $M = \begin{pmatrix} m_1 & \lambda\, m_2 & \lambda^2\, m_4 \\ \delta_2 & m_3 & \lambda\delta_2 \\ m_4 & m_2 & m_1 \end{pmatrix}$.

Then the condition

$$dZ = M^{tr}[dW]M$$

permits by identification of coefficients, to calculate $\overset{\diamond}{\partial}_Z$ such that

$$\overset{\diamond}{\partial}_W = M\overset{\diamond}{\partial}_Z\, M^{tr}$$

We find $\overset{\diamond}{\partial}_Z = constant \begin{pmatrix} 2\frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & 2\frac{\partial}{\partial z_4} \\ \frac{\partial}{\partial z_2} & 4\frac{\partial}{\partial z_3} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} \\ 2\frac{\partial}{\partial z_4} & \frac{1}{\lambda}\frac{\partial}{\partial z_2} & \frac{2}{\lambda^2}\frac{\partial}{\partial z_1} \end{pmatrix}$.

Without knowing a group of holomorphic transformations on $\mathcal{M}$, with the matrix operators $Z\overset{\diamond}{\partial}_Z - \overset{\diamond}{\partial}_Z Z$,

$i(Z\overset{\diamond}{\partial}_Z + \overset{\diamond}{\partial}_Z Z)$, $\overset{\diamond}{\partial}_Z - Z\overset{\diamond}{\partial}_Z Z$ and $\overset{\diamond}{\partial}_Z + Z\overset{\diamond}{\partial}_Z Z$, we can construct holomorphic vector fields $l(e_j)$. With the $l(e_j)$ obtained as the coefficients of the matrix operators do we have

$$\sum_j \frac{1}{B(e_j, e_j)} l(e_j)\overline{l(e_j)} = trace\,((I - \overline{Z}Z)\overset{\diamond}{\partial}_Z\,(I - Z\overline{Z})\overset{\diamond}{\partial}_{\overline{Z}}\,)$$

12.1 The volume element on $\mathcal{M}$ in Example 1.

Let $Z = \begin{pmatrix} z_1 & z_2 & z_4 \\ z_2 & z_3 & z_2 \\ z_4 & z_2 & z_1 \end{pmatrix}$ and let $\lambda_1(Z) = (u_1 - u_4)(Z)$, $\lambda_2(Z)$, $\lambda_3(Z)$ be the eigenvalues of $I - \overline{Z}Z = \begin{pmatrix} u_1 & \overline{u_2} & u_4 \\ u_2 & u_3 & u_2 \\ u_4 & \overline{u_2} & u_1 \end{pmatrix}$, it holds

$$|k'_{g_t}(Z)|^2 = \frac{\phi(k_{g_t}(Z))}{\phi(Z)} \quad \text{with} \quad \phi(Z) = \lambda_1(Z)^2\lambda_2(Z)^3\lambda_3(Z)^3$$

The volume element on $\mathcal{M}$ is

$$dv = \lambda_1(Z)^{-2}\lambda_2(Z)^{-3}\lambda_3(Z)^{-3}\, dz_1\overline{dz_1}\cdots\, dz_4\overline{dz_4}$$

The representation $T_gF(Z) = F(k_g(Z))$ is unitary in $Hol^2_{\mathbf{C}}(dv)$.

13 Question

What happens if we take the metric associated to the Bergman kernel in a general context. Let G denote the identity component of the group of all biholomorphic mappings of the domain and let $J(g, z)$ be the Jacobian determinant of the biholomorphic map $z \to g.z$. Do we have a similar expression for the Laplacian with the holomorphic vector fields of the representation? and what about the anti-holomorphic (holomorphic) gradient vector field and the complex Ornstein-Uhlenbeck operators on the domain?