Free Lévy processes in large and small time limits

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Classical limit theorem for random walks

Let $\{X_i\}$ be iid random variables ($\mathbb{R}$-valued) and

$$S_n = X_1 + \cdots + X_n.$$

Question

When does $a_n S_n + b_n$ converge in law as $n \to \infty$ for some deterministic sequences $a_n > 0$ and $b_n \in \mathbb{R}$?

The answer is well known (Lévy, Khintchine,...):

(1) the possible limit distributions of $a_n S_n + b_n$ are stable distributions and delta measures;

(2) Given a stable distribution μ and a_n, b_n, a necessary and sufficient condition for the convergence $a_n S_n + b_n \Rightarrow \mu$ can be given in terms of X_1, a_n, b_n.

Reference: Gnedenko & Kolmogorov's book

Limit theorem for Lévy processes

The continuous-time version of random walk is Lévy processes. A stochastic process $\{X_t\}_{t\geq 0}$ is called an (additive) Lévy process if

- $X_0 = 0$ a.s.,
- $t \mapsto X_t$ is right continuous with finite left limits,
- $X_t \to X_s$ in law if $t \to s$,
- The law of $X_t - X_s$ is equal to that of X_{t-s} for every $0 \leq s \leq t$.
- For all $0 = t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

Remark

If $\{X_t\}_{t\geq 0}$ is a Lévy process then the discrete time process $S_n^{(\delta)} := X_{n\delta}, n = 0, 1, 2, 3, \ldots,$ is a random walk since

$$S_n^{(\delta)} = Y_1 + Y_2 + \cdots + Y_n,$$

where $Y_n = X_{n\delta} - X_{(n-1)\delta} \stackrel{\mathrm{d}}{=} X_\delta$ are iid random variables.

Limit theorem for Lévy processes

For an (additive) Lévy process $\{X_t\}$ on $\mathbb{R}$, we consider the following question.

Question

When does $a(t)X_t + b(t)$ converge in law as $t \to \infty$ for some deterministic functions $a(t) > 0$ and $b(t) \in \mathbb{R}$?

Theorem (Bertoin 96, Doney & Maller 02, de Weert 03)

(1) *the possible limit distributions of $a(t)X_t + b(t)$ are stable distributions and delta measures;*

(2) *given a stable distribution μ and functions a, b, a necessary and sufficient condition for the convergence $a(t)X_t + b(t) \Rightarrow \mu$ is known in terms of X_1 and a, b.*

We can also discuss the convergence as $t \to 0$. Then similar results hold (Maller & Mason 09)

Free Lévy processes

- In free probability, we have free (additive) Lévy processes. They can be realized as large dimensional limits of some Hermitian matrix-valued, unitarily invariant Lévy processes [Perez & Perez-Abreu & Rocha-Arteaga]

- There is a homeomorphism (Bercovici-Pata bijection) between classical ID distributions and free ID distributions, so the complete analogy holds for limits of free Lévy processes. Namely:

Theorem

Let $\{X_t\}_{t\geq 0}$ be a free Lévy process.

(1) *The possible limit distributions of $a(t)X_t + b(t)$ are free stable distributions and delta measures;*

(2) *given a free stable distribution μ and functions a, b, a necessary and sufficient condition for convergence $a(t)X_t + b(t) \Rightarrow \mu$ can be written in terms of X_1, a, b.*

Multiplicative LP

Classical multiplicative Lévy processes $\{M_t\}$ on the multiplicative group $(0, \infty)$ can also be defined via

- $M_0 = 1$ a.s.,
- $t \mapsto M_t$ is right continuous with finite left limits,
- $M_t \to M_s$ in law if $t \to s$,
- The law of $M_s^{-1} M_t$ is equal to that of M_{t-s} for every $0 \leq s \leq t$,
- For all $0 = t_0 < t_1 < \cdots < t_n$, the random variables

  $$M_{t_0}^{-1} M_{t_1}, M_{t_1}^{-1} M_{t_2}, \ldots, M_{t_{n-1}}^{-1} M_{t_n}$$

  are independent.

but it eventually means $X_t := \log M_t$ is an additive LP on $\mathbb{R}$.

Limit theorem for multiplicative LP

For a multiplicative Lévy process (M_t), let $X_t = \log M_t$. Then

$$\log e^{b(t)}(M_t)^{a(t)} = a(t)X_t + b(t).$$

Thus the limit theorems for $b(t)(M_t)^{a(t)}$ for deterministic functions $a(t) > 0, b(t) > 0$ follow from the additive case.

Theorem

(1) *the possible limit distributions of $b(t)(M_t)^{a(t)}$ (as $t \to \infty$ or $t \to 0$) are delta measures and log stable distributions (the law of e^S where S is a stable random variable);*

(2) *given a log stable distribution μ and functions $a, b > 0$, a necessary and sufficient condition for convergence of $b(t)(M_t)^{a(t)}$ can be written in terms of the law of M_1, a, b.*

Multiplicative free LP

Biane (1998) defined positive free multiplicative Lévy processes $\{M_t\}$ via

- $M_t \geq 0$ (possibly unbounded, affiliated with a finite vN algebra) and $M_0 = 1$,
- $M_t \to M_s$ in law if $t \to s$,
- The law of $M_s^{-1/2} M_t M_s^{-1/2}$ is equal to that of M_{t-s} for every $0 \leq s \leq t$,
- For all $0 = t_0 < t_1 < \cdots < t_n$, the random variables

$$M_{t_0}^{-1/2} M_{t_1} M_{t_0}^{-1/2}, \ldots, M_{t_{n-1}}^{-1/2} M_{t_n} M_{t_{n-1}}^{-1/2}$$

  are free independent.

By the non-commutativity, $X_t := \log M_t$ may not be an additive free LP.

S-transform

Let φ denote a state on a W^*-algebra. For $X > 0$ (possibly unbounded) let

$$\psi_X(z) = \varphi\left(\frac{zX}{1 - zX}\right), \qquad z < 0.$$

Then ψ_X is strictly increasing and maps $(-\infty, 0)$ onto $(-\alpha, 0)$ for some $\alpha > 0$. Let

$$S_X(z) = \frac{1+z}{z}\psi_X^{-1}(z), \qquad z \in (-\alpha, 0)$$

For a multiplicative free LP $\{M_t\}_{t\geq 0}$ it holds that

$$S_{M_t}(z) = e^{tv(z)}$$

for some function v (infinitesimal generator). The S-transform is the main tool to analyze limit theorems for $\{M_t\}$.

Limit theorems for multiplicative free LP in large time

Theorem ((Special case of) Tucci 10, Haagerup & Moeller 13)

Let $\{M_t\}$ be a multiplicative free LP. Then

$$(M_t)^{1/t} \Rightarrow \nu \qquad (t \to \infty),$$

where $\nu([0,x]) = S_{M_1}^{-1}(1/x) + 1$. ($S_X$ is the S-transform of X)

In particular, the map " $\underbrace{\mu_{M_1}}_{\text{law of } M_1} \mapsto \nu$ " is injective, because

$$\nu = \nu' \quad \Rightarrow \quad S_{M_1}(x) = S_{M_1'}(x) \quad \Rightarrow \quad \mu_{M_1} = \mu_{M_1'}$$

The limit distributions are not universal

Multiplicative FLP in small times

Some examples from our results

Theorem (Arizmendi-H.)

Let $\{N_t\}$ *be a multiplicative free LP such that* $S_{N_t}(z) = e^{t(-z)^{\alpha-1}}$, $1 < \alpha \leq 2$. *Then*

$$(N_t)^{t^{-1/\alpha}} \overset{\mathrm{d}}{\Rightarrow} e^{Z_\alpha}, \qquad t \to 0,$$

where Z_α *has a one-sided free* α*-stable law. In particular,* Z_2 *follows the standard semicircle law* $\frac{1}{2\pi}\sqrt{4-x^2}dx$.

Theorem (Arizmendi-H.)

Let $\lambda \geq 1$. *Let* $\{N_t\}$ *be a multiplicative free LP such that* $S_{N_t}(z) = \frac{1}{(\lambda+z)^t} = e^{-t\log(\lambda+z)}$, *(*$N_t \sim$ *the Marchenko-Pastur law). Then*

$$t(N_t)^{1/t} \overset{\mathrm{d}}{\Rightarrow} \mathrm{DH}, \qquad t \to 0.$$

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$$t(N_t)^{1/t} \overset{\mathrm{d}}{\Rightarrow} \mathrm{DH}, \qquad t \to 0.$$

[Dykema & Haagerup 04]

- DH has moments $\frac{n^n}{(n+1)!}$ & support $[0, e]$ & an implicit density
- Let $\{t_{ij}\}_{1\leq i<j\leq N}$ be indep. complex Gaussian, mean 0 and var. $1/n$;

$$T_N := \begin{pmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1,N-1} & t_{1N} \\ 0 & 0 & t_{23} & \cdots & t_{2,N-1} & t_{2N} \\ 0 & 0 & 0 & \cdots & t_{3,N-1} & t_{3N} \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & t_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then the mean empirical eigenvalue distr. of $T_N^* T_N \Rightarrow \mathrm{DH}$ $(N \to \infty)$.

By computation of the densify functions we found that:

Proposition

If X follows the free 1-stable law supported on $(-\infty, 1]$ then

$$e^X \sim \mathrm{DH}.$$

- This means that the empirical eigenvalue distribution of $\log(T_N^* T_N)$ converges to the free 1-stable law.
- Recall that the empirical eigenvalue distribution of $T_N + T_N^*$ converges to the semicircle law $\frac{1}{2\pi}\sqrt{4 - x^2}$ on $[-2, 2]$ (free 2-stable).

Question

- *Do other free stable distributions have RM models made of T_N?*
- *Is there any natural connection between the upper-triangular Gaussian RM and positive free multiplicative LPs?*

Summary

- For classical additive LPs (X_t), the limit distr. of $a(t)X_t + b(t)$ ($t \to \infty$ or 0), if exists, is stable.
- For free additive LPs (Y_t), the limit distr. of $a(t)Y_t + b(t)$ ($t \to \infty$ or 0), if exists, is free stable.
- For classical multiplicative LPs (M_t), the limit distr. of $e^{b(t)}(M_t)^{a(t)}$ ($t \to \infty$ or 0), if exists, is log stable (the law of e^Z, where $Z \sim$ stable).
- For free multiplicative LPs (N_t), the limit distr. of $(N_t)^{1/t}$ $(t \to \infty)$ always exists and is not universal.
- For some free multiplicative LPs (N_t) and functions a, b, the limit distr. of $e^{b(t)}(N_t)^{a(t)}$ $(t \to 0)$ is log free stable.

Conjecture (after our examples)

For free multiplicative LPs (N_t), the limit distr. of $e^{b(t)}(N_t)^{a(t)}$ $(t \to 0)$, if exists, must be log free stable.

Some ideas for better understanding

- For large time, non-commutativity of multiplicative free LPs is not negligible because we take products many times.

- For small time, we can guess that the contribution of non-commutativity is small and so we can expect that classical and free limit theorems are similar.

- How about matrices, e.g. 2×2 positive matrix-valued Lévy processes?

Idea of the proof

Theorem (Arizmendi-H.)

Let $\lambda \geq 1$*. Let* $\{N_t\}$ *be a multiplicative free LP such that* $S_{N_t}(z) = \frac{1}{(\lambda+z)^t} = e^{-t\log(\lambda+z)}$*, (*$N_t \sim$ *the Marchenko-Pastur law). Then*

$$t(N_t)^{1/t} \overset{\mathrm{d}}{\Rightarrow} \mathrm{DH}, \qquad t \to 0.$$

Lemma (Haagerup-Moeller 13)

Let μ *be a probability measure on* $(0,\infty)$*. Then*

$$\int_{(0,\infty)} x^\alpha \, \mu(dx) = \frac{1}{B(1-\alpha, 1+\alpha)} \int_{(0,1)} \left(\frac{1-x}{x} S_\mu(x-1)\right)^{-\alpha} dx$$

for $\alpha \in (-1,1)$ *as an equality in* $[0,\infty]$*, where* $B(p,q)$ *is the Beta function. Note that*

$$\frac{1}{B(1-\alpha, 1+\alpha)} = \frac{\sin \pi\alpha}{\pi\alpha}.$$

Let μ_X be the law of X. Suppose N_1 follows Marchenko-Pastur with $\lambda > 1$. Then, for $\alpha \in (0, t)$,

$$\begin{aligned}
&\int_{(0,\infty)} x^{\alpha}\,\mu_{tN_t^{1/t}}(dx) \\
&= t^{\alpha} \int_{(0,\infty)} x^{\alpha/t}\,\mu_{N_t}(dx) \\
&= \frac{t^{\alpha}}{B(1-\alpha/t, 1+\alpha/t)} \int_{(0,1)} \left(\frac{1-x}{x} S_{N_t}(x-1) \right)^{-\alpha/t} dx \\
&= \frac{t^{\alpha}}{B(1-\alpha/t, 1+\alpha/t)} \int_{(0,1)} \left(\frac{1-x}{x} \frac{1}{(x+\lambda-1)^{t}} \right)^{-\alpha/t} dx \\
&= t^{\alpha}(\lambda-1)^{\alpha}{}_2F_1(-\alpha, \alpha/t+1; 2; -(\lambda-1)^{-1}).
\end{aligned}$$

The last expression makes sense for all $\alpha > 0$. By analytic continuation, for all $\alpha > 0$, we have

$$\int_{(0,\infty)} x^{\alpha}\,\mu_{tN_t^{1/t}}(dx) = t^{\alpha}(\lambda-1)^{\alpha}{}_2F_1(-\alpha, \alpha/t+1; 2; -(\lambda-1)^{-1}).$$

For all $\alpha > 0$ we have

$$\int_{(0,\infty)} x^\alpha \, \mu_{tN_t^{1/t}}(dx) = t^\alpha (\lambda - 1)^\alpha {}_2F_1(-\alpha, \alpha/t + 1; 2; -(\lambda - 1)^{-1}).$$

The RHS converges to

$$\frac{\alpha^\alpha}{\Gamma(\alpha + 2)} \quad \text{as } t \to 0$$

by using asymptotic behavior of hypergeometric function. The limit value for $\alpha = n \in \mathbb{N}$ is

$$n^n/(n + 1)!,$$

which is the n-th moment of the Dykema-Haagerup distribution. This shows

$$t(N_t)^{1/t} \Rightarrow \mathrm{DH}.$$

Unitary free BM

We can obtain rather general limit theorems for unitary free LPs. For example:

Proposition (Arizmendi-H.)

For free unitary BM $\{U_t\}$ *we have*

$$(U_t)^{[1/\sqrt{t}]} \overset{\mathrm{d}}{\Rightarrow} e^{iS}, \qquad t \to 0,$$

where $S \sim$ *standard semicircle law.*

$\{U_t\}$: unitary free BM, S: semicircular element

Proof. Biane 97 obtained the formula

$$\mathbb{E}[U_t^m] = e^{-\frac{mt}{2}} \sum_{k=0}^{m-1} (-1)^k \frac{t^k}{k!} m^{k-1} \binom{m}{k+1}, \qquad m \geq 1.$$

If we take $m = n[1/\sqrt{t}]$ then as $t \to 0$ we have

$$\mathbb{E}\left[\left(U_t^{[1/\sqrt{t}]}\right)^n\right] \sim e^{-\frac{n\sqrt{t}}{2}} \sum_{k=0}^{n[1/\sqrt{t}]-1} (-1)^k t^k \frac{(nt^{-1/2})^{2k}}{k!(k+1)!}$$

$$\to \sum_{k=0}^{\infty} (-1)^k \frac{n^{2k}}{k!(k+1)!} = \frac{J_1(2n)}{n} = \mathbb{E}[e^{inS}],$$

where J_1 is the Bessel function of the 1st kind. So we have proved that

$$U_t^{[1/\sqrt{t}]} \overset{\mathrm{d}}{\Rightarrow} e^{iS}, \qquad t \downarrow 0.$$

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