Transition Probability of States on $*$-Algebras

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Contents

Consider a quantum system realized on a Hilbert space $\mathcal{H}$, states are unit vector φ, ψ of $\mathcal{H}$

Definition: Max Born (1925/26)

Transition probability: $|\langle\varphi, \psi\rangle|^2$

Max Born

Max Born (1882–1970)

- born in Breslau
- introduced (1925) the relation $PQ - QP = -i\hbar I$
- M. Born and P. Jordan (1925) found a representation of this relation by infinite matrices
- Professor in Göttingen (1921-1933), went to England
- 1939: British citizenship
- Nobelpreis for Physics 1954

Definitions

In what follows: A is a **complex unital $*$-algebra**

(a complex algebra with unit element 1 and involution $a \to a^+$)

Definition: $*$-representation

Let $(\mathcal{D}, \langle\cdot,\cdot\rangle)$ be a unitary space. A **$*$-representation** of A on $\mathcal{D}$ is an algebra homomorphism $\pi \to L(\mathcal{D})$ satisfying $\pi(1) = I_{\mathcal{D}}$ and

$$\langle\pi(a)\varphi, \psi\rangle = \langle\varphi, \pi(a^+)\psi\rangle, \quad a \in A, \ \varphi, \psi \in \mathcal{D}.$$

We write $\mathcal{D}(\pi) = \mathcal{D}$ and call $\mathcal{D}$ the **domain of π.**
$\mathrm{Rep}\, A$: **family of all $*$-representations of A**

Definition: vector state

If π is a $*$-representation of A and $\varphi \in \mathcal{D}(\pi)$, $\|\varphi\| = 1$, then

$$f_\varphi(a) := \langle \pi(a)\varphi, \varphi \rangle, \quad a \in A,$$

is called a **vector state of π.**

Note that $f_\varphi(a^+a) = \|\pi(a)\varphi\|^2 \geq 0$ and $f_\varphi(1) = \|\varphi\|^2 = 1$.

Definition: state

State: linear functional f such that $f(1) = 1$ and $f(a^+a) \geq 0$, $a \in A$.
$\mathcal{S}(A)$: set of states of A

By the GNS construction, each state f on A is a vector state of some representation π_f: $\quad f(a) = \langle \pi_f(a)\varphi_f, \varphi_f \rangle$, $a \in A$.

Transition probability and Bures distance

Let $\pi \in \mathrm{Rep}A$ and $f \in \mathcal{S}(A)$.

$S(\pi, f)$: set of vectors $\varphi \in \mathcal{D}(\pi)$ such that $f(a) = \langle \pi(a)\varphi, \varphi \rangle, \quad a \in A$.

Definition: Transition probability (Uhlmann 1976)

Transition probability of two states $f, g \in \mathcal{S}(A)$ is

$$P_A(f,g) := \sup_{\pi \in \mathrm{Rep}A} \sup_{\varphi \in \mathcal{S}(\pi,f), \psi \in \mathcal{S}(\pi,g)} |\langle \varphi, \psi \rangle|^2.$$

Definition: Bures distance (Bures 1969)

Bures distance of $f, g \in \mathcal{S}(A)$ is

$$d_A(f,g) := \inf_{\pi \in \mathrm{Rep}A} \inf_{\varphi \in \mathcal{S}(\pi,f), \psi \in \mathcal{S}(\pi,g)} \| \varphi - \psi \|.$$

Quantum information theory: $\sqrt{P_A(f,g)}$ - **fidelity**.
Bures distance and the transition probability are related by

$$d_A(f,g) = 2 - 2\sqrt{P_A(f,g)}, \quad f,g \in \mathcal{S}(A).$$

d_A is a metric on $\mathcal{S}(A)$.

Basic properties: A unital C^*-algebra.

(i) $0 \leq P_A(f,g) = P_A(g,f) \leq 1$.
(ii) $P_A(f,g) = 1$ if and only if $f = g$.
(iii) $P_A(f,g) = 0$ if and only if f and g are orthogonal.
(iv) $\lambda P_A(f_1,g) + (1-\lambda)P_A(f_2,g) \leq P_A(\lambda f_1 + (1-\lambda)f_2, g)$ for $\lambda \in [0,1]$.

Theorem 1: P. M. Alberti (1983)

Let A be a **unital C^*-algebra**. For $f, g \in \mathcal{S}(A)$ we have

$$P_A(f,g) = \inf\{f(a)g(a^{-1}) : a \in A, a^{-1} \in A, a \geq 0\}$$

Interwining space:
$I(\pi_f, \pi_g) = \{T \in \mathbf{B}(\mathcal{H}(\pi_f), \mathcal{H}(\pi_g)) : T\pi_f(a) = \pi_g(a)T, \ a \in A\}$

Theorem 2: P.M. Alberti (1983)

For states f, g on a **unital C^*-algebra** A we have

$$P_A(f,g) = \sup \ \{|\langle T\varphi_f, \varphi_g\rangle|^2 : T \in I(\pi_f, \pi_g), \|T\| \leq 1\}.$$

Now let A be a **unital $*$-algebra** and $f, g \in \mathcal{S}(A)$.

Definition: Transition form

A linear functional on A is called a **transition form** from f to g if

$$|F(b^+ a)|^2 \leq f(a^+ a) g(a^+ a), \quad a \in A.$$

$\mathcal{G}(f, g)$: set of transition forms

If $\varphi \in S(\pi, f), \psi \in S(\pi, g)$, then $F(a) := \langle \pi(a)\varphi, \psi \rangle$ is a transition form:

$$\begin{aligned} |F(b^+ a)|^2 &= |\langle \pi(a)\varphi, \pi(b)\psi \rangle|^2 \\ &\leq \|\pi(a)\varphi\|^2 \, \|\pi(b)\psi\|^2 = f(a^+ a) g(b^+ b). \end{aligned}$$

Theorem 3: A. Uhlmann (1985)

$$P_A(f, g) = \sup \ \{ |F(1)|^2 : F \in \mathcal{G}(f, g) \}.$$

Technical interlude

Let π be a $*$-representation. Then we define

$$\mathcal{D}(\overline{\pi}) := \cap_{a\in A}\mathcal{D}(\overline{\pi(a)})\ ,\quad \overline{\pi}(a) := \overline{\pi(a)}\lceil\mathcal{D}(\overline{\pi}),\quad a\in A,$$
$$\mathcal{D}(\pi^*) := \cap_{a\in A}\mathcal{D}(\pi(a)^*)\ ,\quad \pi^*(a) := \pi(a^+)^*\lceil\mathcal{D}(\pi^*),\quad a\in A.$$

Definition:

A $*$-representation π of a $*$-algebra A is called

- **self-adjoint** if $\pi = \pi^*$, equivalently, $\mathcal{D}(\pi) = \mathcal{D}(\pi^*)$,
- **essentially self-adjoint** if $\pi^* = (\pi^*)^*$, equivalently, $\mathcal{D}(\pi^{**}) = \mathcal{D}(\pi^*)$.

For symmetric operators we have $\overline{T} = T^{**}$,
for $*$-representations $\overline{\pi} \neq \pi^{**}$ in general.

Usually, essential self-adjointness was defined by $\overline{\pi}\ (=\overline{\pi}^*) = \pi^*$,
but it seems to be better to define $\pi^* = (\pi^*)^*$.

Technical interlude- continued

Example: One-dimensional moment problem

Let $A = \mathbb{C}[x]$ be the $*$-algebra of polynomials in one variable.

$f \in \mathcal{S}(A)$ corresponds to a **moment sequence** $s = (s_n)$, $s_n = f(x^n)$.
Assume that $f(p^2) > 0$, $p \in \mathbb{C}[x]$.

Suppose μ is a representing measure of s: $\quad s_n = \int x^n d\mu(x), n \in \mathbb{N}_0$.

Let $\mathcal{D} = \mathbb{C}[x]$ with scalar product $\langle p, q \rangle = \int p(x)\overline{q(x)}\, d\mu, p, q \in \mathbb{C}[x]$

There is a $*$-representation of A on $\mathcal{D} = \mathbb{C}[x]$ s.t. $\pi(p)q = p \cdot q$.

$\overline{\pi}$ is self-adjoint if and only if $(\,\overline{\pi(x)}\,)^n$ is self-adjoint for all $n \in \mathbb{N}$.

π is **essentially self-adjoint** iff $\overline{\pi(x)}$ is self-adjoint iff s is **determinate**.

Crucial assumption: π is **essentially self-adjoint**!

Definition: GNS representation π_f

Let $f \in \mathcal{S}(A)$. There exists a unique $*$-representation π_f of A s. t.

$$\mathcal{D}(\pi_f) = \pi_f(A)\varphi_f \quad \text{and} \quad f(a) = \langle \pi_f(a)\varphi_f, \varphi_f \rangle, a \in A.$$

Theorem 4: "supremum is a maximum"

Let $f, g \in \mathcal{S}(A)$ and let π be a $*$-representation such that f and g are vector functionals of π. Suppose π_f, π_g, π are **essentially self-adjoint**.

There exist $\varphi, \psi \in \mathcal{D}(\pi)$ s.t. $f(a) = \langle \pi(a)\varphi, \varphi\rangle$ and $g(a) = \langle \pi(a)\psi, \psi\rangle$,

$$P_A(f, g) = |\langle \varphi, \psi\rangle|^2.$$

The proof uses a deep result of P.M. Alberti on transition probabilities for von Neumann algebras.

The supremum in the definition on $P_A(f, g)$ is a maximum!

This holds for **each** essentially self-adjoint rep π for which f and g are vector functionals. For instance, $\pi = (\pi_f)^{**} \oplus (\pi_g)^{**}$.

Theorem 5:

Let $f, g \in \mathcal{S}(A)$. Suppose the GNS representations π_f and π_g are essentially self-adjoint. Suppose there exist a positive linear functional h on A and elements $b, c \in A$ such that $c^+ b \in \sum A^2$ and

$$f(a) = h(b^+ab) \quad \text{and} \quad g(a) = h(c^+ac), \quad a \in A.$$

Then

$$P_A(f,g) = h(c^+b)^2.$$

Applications

Theorem 6: "states define by trace class operators"

Let π be an irreducible $*$-representation of A.
Suppose $s, t \in \mathbf{B}_1(\mathcal{H}(\pi))_+$ define states on A by

$$f_s(a) = \mathrm{Tr}\, \pi(a)s, \quad f_t(a) = \mathrm{Tr}\, \pi(a)t \quad \text{for} \quad a \in A.$$

Suppose π_{f_s}, π_{f_t}, and π are essentially self-adjoint. Then

$$P_A(f_s, f_t) = (\mathrm{Tr}\, |t^{1/2} s^{1/2}|)^2 = (\, \mathrm{Tr}\, (s^{1/2}\, t\, s^{1/2})^{1/2}\,)^2. \tag{1}$$

Formula (1) was discovered by Araki (1972) for von Neumann algebras.

Example: Schrödinger representation of the Weyl algebra

Let $A = \mathbb{C}\langle p = p^+, q = q^+ : pq - qp = -i\rangle$ be the Weyl algebra.
The Schrödinger representation π of A,

$$(\pi(q)\varphi)(x) = x\varphi(x),\ (\pi(p)\varphi)(x) = -\mathrm{i}\varphi'(x),\quad \varphi \in \mathcal{S}(\mathbb{R}),$$

is irreducible, self-adjoint. Suppose $s, t \in \mathbf{B}_1(L^2(\mathbb{R}))_+$ define states f_s, f_t on A such that π_{f_s} and π_{f_t} are essentially self-adjoint. Then

$$P_A(f_s, f_t) = (\mathrm{Tr}\,|t^{1/2}s^{1/2}|)^2 = (\,\mathrm{Tr}\,(s^{1/2}ts^{1/2})^{1/2}\,)^2. \tag{2}$$

Specialize $s = \varphi \otimes \varphi$ and $t = \psi \otimes \psi$ with $\varphi, \psi \in \mathcal{S}(\mathbb{R})$.
Then $f_s(a) = \langle\rho(a)\varphi, \varphi\rangle$, $f_t(a) = \langle\rho(a)\psi, \psi\rangle$ and (2) yields

$$P_A(f_s, f_t) = |\langle\varphi, \psi\rangle|^2.$$

Theorem 7: "states defined by integrals"

X: locally compact Hausdorff space,
A: $*$-subalgebra of $C(X)$ which contains 1 and separates points of X.
μ: Radon measure on X such that $A \subseteq L^1(X, \mu)$
$\eta, \xi \in L^\infty(X, \mu)_+$ of norm 1. Define states f_η and f_ξ on A by

$$f_\eta(a) = \int_X a(x)\eta(x)\, d\mu(x), \quad f_\xi(a) = \int_X a(x)\xi(x)\, d\mu(x), \;\; a \in A.$$

Suppose π_{f_η} and π_{f_ξ} are essentially self-adjoint. Then

$$P_A(f_\eta, f_\xi) = \left(\int_X \eta(x)^{1/2}\xi(x)^{1/2}\, d\mu(x) \right)^2.$$

Example: One-dimensional moment problem

Recall states f on the $*$-algebra $A = \mathbb{C}[x]$ correspond to **moment sequences**: $s = (s_n)$, $s_n = f(x^n)$. Assume that $f(p^2) > 0$, $p \in \mathbb{C}[x]$.

Suppose μ is a representing measure of s: $s_n = \int x^n d\mu(x), n \in \mathbb{N}_0$.

Let $\mathcal{D} = \mathbb{C}[x]$ with scalar product $\langle p, q \rangle = \int p(x)\overline{q(x)}\, d\mu, p, q \in \mathbb{C}[x]$ and π_μ the $*$-representation of A on $\mathcal{D} = \mathbb{C}[x]$ s.t. $\pi(p)q = p \cdot q$.

How to determine $P_A(f, g)$?

μ: Radon measure on $\mathbb{R}$. Define $f_\eta(a) = \int a(x)\eta d\mu$, $f_\xi(a) = \int a(x)\xi d\mu$. If $\eta d\mu$, $\xi d\mu$ are **determinate**, $\pi_{f_{\mu_\eta}}$, $\pi_{f_{\mu_\xi}}$ are essentially self-adjoint.

$$P_A(f_\eta, f_\xi) = \left(\int_X \eta(x)^{1/2}\xi(x)^{1/2}\, d\mu(x) \right)^2 .$$

For **indeterminate measures**, this formula does not hold in general.

Open problem: What is $P_A(f, g)$ for indeterminate moment functionals ?

A counter-example

Let A be the Weyl algebra and π is the Schrödinger representation.

For $\eta \in \mathcal{S}(\mathbb{R}), \|\eta\| = 1$, define the vector state f_η by $f_\eta(a) = \langle \pi(a)\eta, \eta \rangle$.

$(*)$ *There are disjoint open intervals* $J_j(\eta) = (\alpha_j, \beta_j)$, $j = 1, \ldots, r$, *s. t.* $\eta(t) \neq 0$ *for* $t \in J(\eta) := \cup_j J_j(\eta)$ *and* $\eta^{(n)}(t) = 0$ *for* $t \in \mathbb{R}/J(\eta)$, $n \in \mathbb{N}_0$.

Theorem 6:

Suppose unit vectors $\varphi, \psi \in C_0^\infty(\mathbb{R})$ satisfy condition $(*)$. Then

$$P_A(f_\varphi, f_\psi) = \left(\sum_{k,j} \left| \int_{\mathcal{J}_k(\varphi) \cap \mathcal{J}_j(\psi)} \varphi(x)\overline{\psi(x)}\, dx \right| \right)^2. \qquad (3)$$

- If $\mathcal{J}(\varphi)$ and $\mathcal{J}(\psi)$ are single intervals, then $P_A(f_\varphi, f_\psi) = |\langle \varphi, \psi \rangle|^2$.
- Suppose $\mathcal{J}_j(\varphi) = \mathcal{J}_j(\psi)$ and $\varphi(x) = \epsilon_j \psi(x)$ on $\mathcal{J}_j(\varphi)$, $\epsilon_j \in \{1, -1\}$.

(3) yields $P_A(f_\varphi, f_\psi) = \|\varphi\|^4$. We can choose ϵ_j such that $\langle \varphi, \psi \rangle = 0$.