Topological characterization of boundaries of free products of groups

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This report describes some results from [1].

Recall that Gromov boundary of a hyperbolic group is a compact metrisable space. Not much is known about explicit topological spaces that can occur as Gromov boundary of a hyperbolic group.

Each hyperbolic group splits over finite subgroups into a graph of groups with all vertex groups finite or 1-ended hyperbolic. Such a splitting is called *terminal* since its factors do not split further. In view of this, the problem of understanding Gromov boundaries of hyperbolic groups consists of the following two parts:

(1) to understand boundaries of 1-ended hyperbolic groups (this class coincides with the class of hyperbolic groups whose Gromov boundaries are connected);
(2) to understand boundaries of ∞-ended hyperbolic groups in terms of boundaries of factors (i.e. vertex groups) in their terminal splittings.

The results presented below provide a satisfactory answer to part (2) of the above problem.

Theorem 1. *Let $\mathcal{X} = (X_i)_{i\in I}$ be a nonempty countable (finite or infinite) family of nonempty metric compacta. Suppose that a space Y satisfies the following conditions:*

(1) *Y is compact metrisable;*
(2) *Y contains a family of pairwise disjoint subspaces $X_{i,\lambda} : i \in I, \lambda \in \Lambda_i$ such that each index set Λ_i is countable infinite, and for each $i \in I$ and any $\lambda \in \Lambda_i$ the subspace $X_{i,\lambda}$ is homeomorphic to X_i;*
(3) *the family $(X_{i,\lambda})_{i,\lambda}$ is null, i.e. diameters of the sets converge to 0;*
(4) *each subspace $X_{i,\lambda}$ is boundary in Y, i.e. its complement is dense;*
(5) *any two distinct points of Y not contained in the same $X_{i,\lambda}$ can be separated by a closed-open subset $H \subset Y$ which is $(X_{i,\lambda})_{i,\lambda}$-saturated, i.e. such that for each $i \in I$ and each $\lambda \in \Lambda_i$ either $X_{i,\lambda} \subset H$ or $X_{i,\lambda} \cap H = \emptyset$.*

Then Y exists, and is unique up to homeomorphism.

Notation. Denote the unique space Y as above with $\widetilde{\sqcup}\mathcal{X}$ or $\widetilde{\sqcup}(X_i : i \in I)$ and call it the *dense amalgam* of the family $\mathcal{X}$.

Theorem 2. *Let $\mathcal{G}$ be a graph of groups with finite edge groups and with hyperbolic vertex groups, at least one of which is infinite. Suppose also that the fundamental group $\Gamma = \pi_1(\mathcal{G})$ is ∞-ended. Then Γ is hyperbolic, and $\partial\Gamma \cong \widetilde{\sqcup}\mathcal{X}$, where $\mathcal{X}$ is the family of Gromov boundaries of infinite vertex groups of $\mathcal{G}$.*

REFERENCES

[1] J. Świątkowski, *The dense amalgam of metric compacta and topological characterization of boundaries of free products of groups*, preprint 2014, arXiv:1410.4989.