On panel-regular lattices in 2-dimensional buildings

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We consider lattices in reductive groups of rank 2 over a local, non-Archimedean field k. By Margulis' Arithmeticity, every lattice in such a group is arithmetic, i.e. can be constructed algebraically. But these lattices can also be studied via their actions on the associated 2-dimensional affine buildings.

For affine buildings, the only geometric constructions of lattices known to the author are the one by Cartwright-Mantero-Steger-Zappa for vertex-transitive lattices, as well as several constructions for chamber-transitive lattices leading to the classification of all chamber-transitive lattices by Kantor-Liebler-Tits. In this talk, we will present a new geometric method to construct panel-regular lattices in 2-dimensional buildings.

For buildings of type $\tilde{A}_2$, every vertex link is a projective plane. We first note that any vertex stabilizer in such a lattice must act point- and line-regularly on this plane. Such groups are called *Singer groups*. We can prove the following classification.

Theorem 1 *If Γ acts panel-regularly on a building X of type $\tilde{A}_2$, there is a explicit presentation of Γ in terms of Singer groups of the vertex link projective plane and so-called difference sets of these Singer groups.*

Conversely, given three Singer groups and three ordered difference sets, we can give an explicit construction of the associated lattice and building X.

This can be proved as follows. We use the given data to construct a complex of groups, whose local developments are projective planes. By a theorem of Bridson and Haefliger, this complex is then developable and its universal cover is a building of type $\tilde{A}_2$ by a recognition theorem due to Charney and Lytchak.

By using cyclic Singer groups, we obtain a class of lattices with very simple presentations. However, it is still unclear whether the associated buildings are classical or exotic. By adapting a technique of Cartwright-Mantero-Steger-Zappa, we show that these buildings cannot be associated to $\mathrm{PSl}_3(\mathbb{Q}_p)$.

Theorem 2 *For every prime power q and for any three choices of ordered difference sets $D_i = \{d^i_j : 0 \leq j \leq q\}$, the group*

$$\Gamma = \langle x_1, x_2, x_3 : x_1^{q^2+q+1} = x_2^{q^2+q+1} = x_3^{q^2+q+1} = 1,\ x_1^{d^1_j} x_2^{d^2_j} x_3^{d^3_j} = x_1^{d^1_k} x_2^{d^2_k} x_3^{d^3_k} \qquad \forall j, k \rangle$$

is a cocompact lattice in a building X of type $\tilde{A}_2$. The building X is not associated to $\mathrm{PSl}_3(k)$ for any field k with characteristic zero.

Finally, a very important feature of this construction is that it extends to buildings of type $\tilde{B}_2$ by using structural results on Singer quadrangles developed by Shult, K. Thas and de Winter. It might be possible to extend the construction to buildings of type $\tilde{G}_2$, provided the existence of Singer hexagons.