On existence of a wreath product of symmetric group and cyclic group in the symmetric group S_{p^n}

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Abstract

In this talk we show that if p is a prime and n is positive integer then in symmetric group S_{p^n} there exists a subgroup which is isomorphic tho a wreath product of the group $S_{\frac{k^p-k}{p}}$ and a cyclic group of order p. The construction is following. Every integer number $a \in \{0, \ldots, p^n - 1\}$ can be expressed as a n-tuple $(a_0, \ldots, a_n)$, where $a_0, \ldots, a_n \in \{0, \ldots, p-1\}$ (it is n-digits representation of a in basis p). Then every permutation from S_{p^n} acts on such tuples. Precisely for every permutation $\sigma \in S_{p^n}$ there exists functions $H_1, \ldots, H_n$, which map $\{0, \ldots, p-1\}^n$ into $\{0, \ldots, p-1\}$, such that $\sigma(a_0, \ldots, a_n) = (H_1(a_0, \ldots, a_n), \ldots, H_n(a_0, \ldots, a_n))$. We will shortly write $\sigma = (H_1, \ldots, H_n)$. Let us denote $\bar{a} = (a_0, \ldots, a_n)$ and $\bar{a}^\delta = (a_1, \ldots, a_n, a_0)$. Then multiplication of permutations $\sigma = (H_1, \ldots, H_n)$ and $\sigma' = (H_1', \ldots, H_n')$ is defined as follow:

$$\sigma\sigma'(\bar{a}) = (H_1(H_1'(\bar{a}), \ldots, H_n'(\bar{a}^{\delta^{n-1}})), \ldots, H_n(H_1'(\bar{a}), \ldots, H_n'(\bar{a}^{\delta^{n-1}})))$$

Let us consider a subgroup

$$G = \{\sigma = (H_1, \ldots, H_n) : \ H_1 = \ldots = H_n\}$$

So if $\sigma \in G$ then there exists a function H mapping $\{0, \ldots, p-1\}^n$ into $\{0, \ldots, p-1\}$, such that $\sigma = (H, \ldots, H)$. To describe a structure of G it is enough to describe a structure of the group $\bar{G}$ of invertible elements of the monoid which consists of all functions mapping $\{0, \ldots, p-1\}^n$ into $\{0, \ldots, p-1\}$ with the operation

$$HH'(\bar{a}) = H(H'(\bar{a}), H'(\bar{a}^\delta), \ldots, H'(\bar{a}^{\delta^{n-1}}))$$

Main Theorem *The group $\bar{G}$ is isomorphic to the semidirect product $S_n \leftthreetimes T$, where $T = S_{\frac{k^p-k}{p}} \wr C_p$ (C_p is a cyclic group of order p).*