Cohomology of categories and extensions of the generalized complexes of groups

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We extend the well-known classification of extensions of groups with non-abelian kernel in terms of third and second cohomolgy to the case of the generalized group complexes, originally introduced by A. Haefliger.

A generalized complex of groups or a twisted diagram of groups assings to every object d of an indexing category $\mathcal{D}$ a group $G(d)$ and to every morphism $d \longrightarrow d_0$ a homomorphism $G(d) \longrightarrow G(d_0)$, however it does not have to be completely functorial - it preserves composition only up to a compatible family of inner automorphisms. An extension of twisted diagrams of groups is a surjective homomorphism of twisted diagrams of groups defined on the same category $\mathcal{D}$. For example, the epimorphism of groups $SL_2\mathbb{Z} \twoheadrightarrow PSL_2\mathbb{Z}$ gives an extension of diagrams of groups

$$\mathbb{Z}_4 \leftarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_6 \quad \twoheadrightarrow \quad \mathbb{Z}_2 \leftarrow 1 \rightarrow \mathbb{Z}_3$$

Let Gr denote category of groups and Rep the category whose objects are groups but morphisms are representations i.e. $\mathrm{Rep}(G, H) := \mathrm{Hom}(G, H)/\mathrm{Inn}(H)$. Then any twisted diagram of groups $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ composed with projection $\mathrm{Gr} \longrightarrow \mathrm{Rep}$ gives a strict functor $\mathcal{D} \longrightarrow \mathrm{Rep}$. We shall begin with a question when a functor $F : \mathcal{D} \longrightarrow \mathrm{Rep}$ lifts to a twisted diagram of groups $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ and how many such liftings exist? An answer will be given in terms of cohomology of the small category $\mathcal{D}$ with coeffcients in certain functor to the category of abelian groups $Z_F : \mathcal{D} \longrightarrow \mathcal{A}b$. Precisely, to every functor $F : \mathcal{D} \longrightarrow \mathrm{Rep}$ one assignes in a natural way an obstruction element $o(F) \in H^3(\mathcal{D}; Z_F)$ such that $o(F) = 0$ if and only if the functor F has a lifting to a twisted diagram $F : \mathcal{D} \longrightarrow \mathrm{Gr}$. Moreover equivalence classes of such liftings are in bijective correspondence with elements of the group $H^2(\mathcal{D}; Z_F)$.

If $\mathcal{D}$ is a category defined by a group G then the above statement reduces to the classical case of extensions of groups.

To each twisted diagram $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ one associates its classifying category $\mathcal{B}_\mathcal{C}\mathcal{G}$ equiped with a projection functor $\mathcal{B}_\mathcal{C}\mathcal{G} \longrightarrow \mathcal{C}$.

I will show that there is a natural bijective correspondence between equivalence classes of epimorphisms $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ of twisted diagrams of groups and twisted diagrams defined over the category $\mathcal{B}_\mathcal{C}\mathcal{G}$.

Now, let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and let $N : \mathcal{B}_\mathcal{C}\mathcal{G} \longrightarrow \mathrm{Rep}$ be a functor. If an obstruction element $o(N) \in H^3(\mathcal{B}_\mathcal{C}\mathcal{G}; Z_N)$ vanishes then there is an epimorphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that the corresponding twisted diagram $\mathcal{B}_\mathcal{C}\mathcal{G} \longrightarrow \mathrm{Gr}$ is a lifting of N. Moreover, set of equivalence classes of such liftings is in natural bijectve correspondence with elements of $H^2(\mathcal{B}_\mathcal{C}\mathcal{G}; Z_N)$.