Tukey classification of some sets arising in Banach spaces

Antonio Avilés, joint work with Grzegorz Plebanek and José Rodriguez

Universidad de Murcia, Author supported by MEyC and FEDER under project MTM2011- 25377

Bedlewo 2014

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow$ $f(A)$ unbounded

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow f(A)$ unbounded

1. P is Tukey reducible to Q if f exists.
   $P \preceq Q$.

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow f(A)$ unbounded

1. P is Tukey reducible to Q if f exists.
   $P \preceq Q$.
2. P and Q are Tukey equivalent if $P \preceq Q$ and $Q \preceq P$.
   $P \sim Q$

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow$ $f(A)$ unbounded

1. P is Tukey reducible to Q if f exists.
   $P \preceq Q$.
2. P and Q are Tukey equivalent if $P \preceq Q$ and $Q \preceq P$.
   $P \sim Q$

Example: $\mathbb{N}$

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow f(A)$ unbounded

1. P is Tukey reducible to Q if f exists.
   $P \preceq Q$.
2. P and Q are Tukey equivalent if $P \preceq Q$ and $Q \preceq P$.
   $P \sim Q$

Example: $\mathbb{N}$

1. $\mathbb{N} \preceq P$ iff P contains a sequence all of whose subsequences are unbounded.

Comparing the cofinal structure of partial orders

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets.

Definition

$f : P \longrightarrow Q$ is Tukey if $A \subset P$ unbounbed $\Rightarrow$ $f(A)$ unbounded

1. P is Tukey reducible to Q if f exists.
   $P \preceq Q$.
2. P and Q are Tukey equivalent if $P \preceq Q$ and $Q \preceq P$.
   $P \sim Q$

Example: $\mathbb{N}$

1. $\mathbb{N} \preceq P$ iff P contains a sequence all of whose subsequences are unbounded.
2. $P \preceq \mathbb{N}$ iff P has a countable cofinal set.

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset)$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset)$
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset)$
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N})$
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset)$
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N})$
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q})$
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset) \sim \{0\}$, ($E$ compact)
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N})$
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q})$
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset) \sim \{0\}$, ($E$ compact)
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N}) \sim \mathbb{N}$, ($E$ locally compact)
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q})$
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset) \sim \{0\}$, ($E$ compact)
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N}) \sim \mathbb{N}$, ($E$ locally compact)
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q}) \sim \mathbb{N}^{\mathbb{N}}$, ($E$ Polish)
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{K}(E)$

Let $\mathscr{K}(E) = \{L \subset E, L \text{ compact}\}$ ordered by $\subset$

Theorem(Fremlin)

Let E be a coanalytic separable metrizable space. Then, either

1. $\mathscr{K}(E) \sim \mathscr{K}(\emptyset) \sim \{0\}$, ($E$ compact)
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N}) \sim \mathbb{N}$, ($E$ locally compact)
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R} \setminus \mathbb{Q}) \sim \mathbb{N}^{\mathbb{N}}$, ($E$ Polish)
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem (Fremlin + Edgar,Wheeler)

If X is Banach space with separable dual, then

Tukey classification of $\mathscr{K}(B_X)$

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem (Fremlin + Edgar, Wheeler)

If X is Banach space with separable dual, then

1. $\mathscr{K}(B_X) \sim \{0\}$, if X is reflexive,

Tukey classification of $\mathscr{K}(B_X)$

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem (Fremlin + Edgar, Wheeler)

If X is Banach space with separable dual, then

1. $\mathscr{K}(B_X) \sim \{0\}$, if X is reflexive,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP,

Tukey classification of $\mathscr{K}(B_X)$

If X is a Banach space with separable dual, $(B_X, weak)$ is a coanalytic metrizable space.

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem (Fremlin + Edgar, Wheeler)

If X is Banach space with separable dual, then

1. $\mathscr{K}(B_X) \sim \{0\}$, if X is reflexive,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP,
3. $\mathscr{K}(B_X) \sim \mathscr{K}(\mathbb{Q})$ otherwise.

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable,

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable, under the axiom of analytic determinacy, either

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable, under the axiom of analytic determinacy, either

1. $\mathscr{K}(B_X) \sim \{0\}$,

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable, under the axiom of analytic determinacy, either

1. $\mathscr{K}(B_X) \sim \{0\}$,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable, under the axiom of analytic determinacy, either

1. $\mathscr{K}(B_X) \sim \{0\}$,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$
3. $\mathscr{K}(B_X) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

If X is separable, under the axiom of analytic determinacy, either

1. $\mathscr{K}(B_X) \sim \{0\}$,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$
3. $\mathscr{K}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{K}(B_X) \sim Fin(\mathbb{R})$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall\varepsilon \quad \exists\delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L$ weakly compact$\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall\varepsilon \quad \exists\delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L$ weakly compact$\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,

3. $\mathscr{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

$$\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta) \text{ is Tukey.}$$

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,

3. $\mathscr{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{AK}(B_X) \sim Fin(\mathbb{R})$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that
  $\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,
2. $\mathbb{N} \preceq \mathscr{AK}(B_X) \preceq \mathbb{N}^{\mathbb{N}}$
3. $\mathscr{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{AK}(B_X) \sim Fin(\mathbb{R})$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

Under the axiom of analytic determinacy, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,
2. $\mathscr{AK}(B_X) \sim \mathbb{N}$ or $\mathscr{AK}(B_X) \sim \mathbb{N}^{\mathbb{N}}$ (conjecture)
3. $\mathscr{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{AK}(B_X) \sim Fin(\mathbb{R})$

A few examples

$\mathscr{A}\mathscr{K}(B_X) \sim$

A few examples

$\mathscr{AK}(B_X) \sim$

1. $\{0\}$, if X is reflexive,

A few examples

$\mathscr{AK}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.

A few examples

$\mathscr{A}\mathscr{K}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.
3. $\mathscr{K}(\mathbb{Q})$ if X has separable dual but not PCP, like c_0.

A few examples

$\mathscr{A}\mathscr{K}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.
3. $\mathscr{K}(\mathbb{Q})$ if X has separable dual but not PCP, like c_0.
4. $Fin(\mathbb{R})$ if X has nonseparable dual but $\ell_1 \not\subset X$.

A few examples

$\mathscr{A}\mathscr{K}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.
3. $\mathscr{K}(\mathbb{Q})$ if X has separable dual but not PCP, like c_0.
4. $Fin(\mathbb{R})$ if X has nonseparable dual but $\ell_1 \not\subset X$.
5. $\mathbb{N}$ if X is nonreflexive SWCG space, like $L^1[0,1]$, $\ell_1(\ell_2)$.

A few examples

$\mathscr{AK}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.
3. $\mathscr{K}(\mathbb{Q})$ if X has separable dual but not PCP, like c_0.
4. $Fin(\mathbb{R})$ if X has nonseparable dual but $\ell_1 \not\subset X$.
5. $\mathbb{N}$ if X is nonreflexive SWCG space, like $L^1[0,1]$, $\ell_1(\ell_2)$.
6. $\mathscr{K}(\mathbb{Q})$ if $X = \ell_1(c_0)$.

A few examples

$\mathscr{A}\mathscr{K}(B_X) \sim$

1. $\{0\}$, if X is reflexive,
2. $\mathbb{N}^{\mathbb{N}}$ if X is nonreflexive with PCP, like J^*.
3. $\mathscr{K}(\mathbb{Q})$ if X has separable dual but not PCP, like c_0.
4. $Fin(\mathbb{R})$ if X has nonseparable dual but $\ell_1 \not\subset X$.
5. $\mathbb{N}$ if X is nonreflexive SWCG space, like $L^1[0,1]$, $\ell_1(\ell_2)$.
6. $\mathscr{K}(\mathbb{Q})$ if $X = \ell_1(c_0)$.
7. $Fin(\mathbb{R})$ if $X = C[0,1]$.

Axiom of analytic determinacy

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A in which Player I and Player II play alternatively natural numbers,

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A in which Player I and Player II play alternatively natural numbers,and Player I wins if the final sequence belongs to A.

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A in which Player I and Player II play alternatively natural numbers,and Player I wins if the final sequence belongs to A.

1. From an ultrafilter, one gets a set A such that none of the players a winning strategy on G_A.

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A in which Player I and Player II play alternatively natural numbers,and Player I wins if the final sequence belongs to A.

1. From an ultrafilter, one gets a set A such that none of the players a winning strategy on G_A.
2. If A is Borel, then one of the players has a winning strategy in G_A

Axiom of analytic determinacy

For a set $A \subset \mathbb{N}^{\mathbb{N}}$, consider the game G_A in which Player I and Player II play alternatively natural numbers,and Player I wins if the final sequence belongs to A.

1. From an ultrafilter, one gets a set A such that none of the players a winning strategy on G_A.
2. If A is Borel, then one of the players has a winning strategy in G_A

Axiom of Analytic Determinacy ($\mathbf{\Sigma^1_1 D}$)

If A is either analytic or coanalytic, then one player has a winning strategy in G_A.

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{K}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ ordered by $\subset$

Theorem 1 (APR)

($\mathbf{\Sigma^1_1 D}$) If X is separable Banach space, either

1. $\mathscr{K}(B_X) \sim \{0\}$,
2. $\mathscr{K}(B_X) \sim \mathbb{N}^{\mathbb{N}}$
3. $\mathscr{K}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{K}(B_X) \sim Fin(\mathbb{R})$

Tukey classification of $\mathscr{K}(B_X)$

Let $\mathscr{R}(A) = \{L \subset A, L$ relatively weakly compact$\}$ ordered by $\subset$

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

Tukey classification of $\mathscr{K}(B_X)$

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

In other models of set theory, there exists an unconditional basis A not fitting in the list.

Tukey classification of $\mathscr{K}(B_X)$

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

In other models of set theory, there exists an unconditional basis A not fitting in the list. More precisely, if there exists a coanalytic set of size ω_1, then there exists A with $\mathscr{R}(A) \sim Fin(\omega_1)$.

Tukey classification of $\mathscr{K}(B_X)$

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

By Grothendieck, $C \in \mathscr{R}(A)$ iff $\lim_n \lim_m x_n^*(x_m) = \lim_m \lim_n x_n^*(x_m)$ when $x_m \in C$.

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

By Grothendieck, $C \in \mathscr{R}(A)$ iff $\lim_n \lim_m x_n^*(x_m) = \lim_m \lim_n x_n^*(x_m)$ when $x_m \in C$. This allows to express $\mathscr{R}(A) = I^{\perp}$ where I is an analytic family of subsets of A.

Tukey classification of $\mathscr{K}(B_X)$

Theorem 1a

($\mathbf{\Sigma^1_1 D}$) If X is Banach, and $A \subset X$ is countable, either

1. $\mathscr{R}(A) \sim \{0\}$,
2. $\mathscr{R}(A) \sim \mathbb{N}$
3. $\mathscr{R}(A) \sim \mathbb{N}^{\mathbb{N}}$
4. $\mathscr{R}(A) \sim \mathscr{K}(\mathbb{Q})$
5. $\mathscr{R}(A) \sim Fin(\mathbb{R})$

By Grothendieck, $C \in \mathscr{R}(A)$ iff $\lim_n \lim_m x_n^*(x_m) = \lim_m \lim_n x_n^*(x_m)$ when $x_m \in C$. This allows to express $\mathscr{R}(A) = I^{\perp}$ where I is an analytic family of subsets of A.

$I^{\perp} = \{a \subset \mathbb{N} : \forall b \in I \;\, a \cap b \text{ is finite}\}$.

Combinatorial result behind Theorem 1

$I^{\perp} = \{a \subset \mathbb{N} : \forall b \in I \ a \cap b \text{ is finite}\}$

Theorem

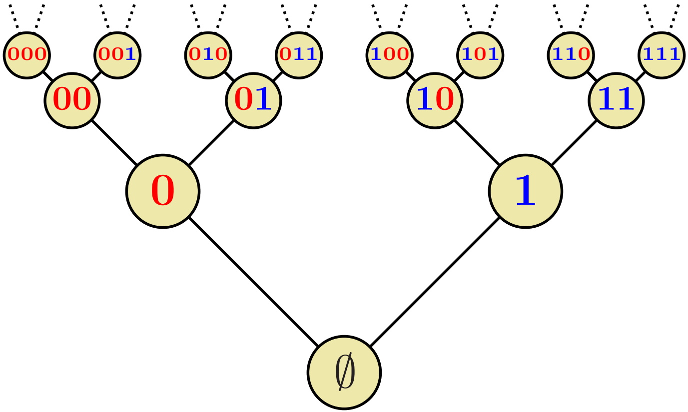
($\mathbf{\Sigma^1_1 D}$) If I is an analytic family of subsets of $\mathbb{N}$, then $I^{\perp}$ is Tukey equivalent to either $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$ or $Fin(\mathbb{R})$.

Proof:

1. By a modification of a result of Todorcevic, either we get a special copy of the dyadic tree (gives $Fin(\mathbb{R})$) or I and $I^{\perp}$ are countably separated.
2. By results of A. and Todorcevic, if I and $I^{\perp}$ are countably separated, we can identify $I^{\perp}$ with $\mathscr{K}(E)$ and then apply Fremlin's theorem.

The dyadic tree

The dyadic tree $2^{<\omega}$ is the set of finite sequences of 0's and 1's.

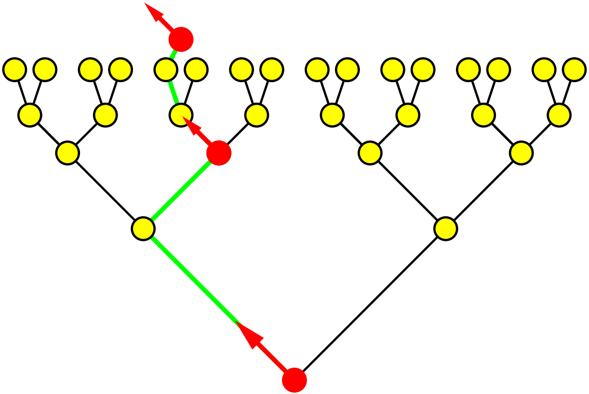


The dyadic tree

A 0-chain is a subset $\{x_1, x_2, \ldots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n{}^\frown 0^\frown y_n$

The dyadic tree

A 0-chain is a subset $\{x_1, x_2, \ldots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n^\frown 0^\frown y_n$

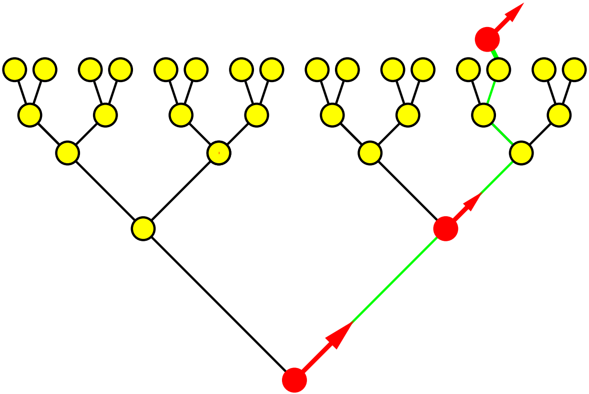


The dyadic tree

A 1-chain is a subset $\{x_1, x_2, \ldots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n{}^\frown 1^\frown y_n$

The dyadic tree

A 1-chain is a subset $\{x_1, x_2, \ldots\} \subset 2^{<\omega}$ in which $x_{n+1} = x_n^\frown 1^\frown y_n$



Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\mathbf{\Sigma^1_1 D}$) If I is analytic family of subsets of $\mathbb{N}$,

Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\mathbf{\Sigma^1_1 D}$) If I is analytic family of subsets of $\mathbb{N}$,

1. Either I and $I^\perp$ are countably separated,

Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\boldsymbol{\Sigma}^1_1\mathbf{D}$) If I is analytic family of subsets of $\mathbb{N}$,

1. Either I and $I^\perp$ are countably separated,
2. Or there exists an injective $u : 2^{<\omega} \longrightarrow \mathbb{N}$ which sends 0-chains to I and 1-chains to $I^\perp$

Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\mathbf{\Sigma^1_1 D}$) If I is analytic family of subsets of $\mathbb{N}$,

1. Either I and $I^\perp$ are countably separated,
2. Or there exists an injective $u : 2^{<\omega} \longrightarrow \mathbb{N}$ which sends 0-chains to I and 1-chains to $I^\perp$

- Condition (1) is equivalent to $I^\perp = \mathscr{R}(E)$ where E is separable metrizable space containing $\mathbb{N}$.

Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\mathbf{\Sigma^1_1 D}$) If I is analytic family of subsets of $\mathbb{N}$,

1. Either I and $I^\perp$ are countably separated,
2. Or there exists an injective $u : 2^{<\omega} \longrightarrow \mathbb{N}$ which sends 0-chains to I and 1-chains to $I^\perp$

- Condition (1) is equivalent to $I^\perp = \mathscr{R}(E)$ where E is separable metrizable space containing $\mathbb{N}$. This leads to Fremlin.

Non countably separated families

Theorem (essentially Todorcevic, via Feng)

($\mathbf{\Sigma^1_1 D}$) If I is analytic family of subsets of $\mathbb{N}$,

1. Either I and $I^\perp$ are countably separated,
2. Or there exists an injective $u : 2^{<\omega} \longrightarrow \mathbb{N}$ which sends 0-chains to I and 1-chains to $I^\perp$

- Condition (1) is equivalent to $I^\perp = \mathscr{R}(E)$ where E is separable metrizable space containing $\mathbb{N}$. This leads to Fremlin.
- Condition (2) leads to $I^\perp \sim I \sim Fin(2^\omega)$.

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$

Tukey classification of $\mathscr{AK}(B_X)$

- Let $\mathscr{AK}(B_X) = \{L \subset B_X, L \text{ weakly compact}\}$ endowed with multiple relations $L \leq_\varepsilon L'$ if $L \subset L' + \varepsilon B_X$.
- $\mathscr{AK}(B_X) \preceq \mathscr{AK}(B_Y)$ now means that there exist functions $f_\varepsilon : \mathscr{AK}(B_X) \longrightarrow \mathscr{AK}(B_Y)$ such that

  $\forall \varepsilon \quad \exists \delta \quad f_\varepsilon : (\mathscr{AK}(B_X), \leq_\varepsilon) \longrightarrow (\mathscr{AK}(B_Y), \leq_\delta)$ is Tukey.

Theorem 2 (APR)

($\mathbf{\Sigma^1_1 D}$) For X separable, either

1. $\mathscr{AK}(B_X) \sim \{0\}$,
2. $\mathbb{N} \preceq \mathscr{AK}(B_X) \preceq \mathbb{N}^{\mathbb{N}}$
3. $\mathscr{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$
4. $\mathscr{AK}(B_X) \sim Fin(\mathbb{R})$

Proof: Theorem 1 + Ramsey (Louveau, Milliken, A.-Todorcevic...)

Illustration of the use of Ramsey

- At a stage, we have $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{K}(B_X)$ and we want to prove $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{A}\mathscr{K}(B_X)$.

Illustration of the use of Ramsey

- At a stage, we have $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{K}(B_X)$ and we want to prove $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{A}\mathscr{K}(B_X)$.
- The proof of Theorem 1 provides an injection $u : \mathbb{Q} \longrightarrow B_X$ that identifies relatively compact subsets of $\mathbb{Q}$ and relatively weakly compact subsets of B_X.

Illustration of the use of Ramsey

- At a stage, we have $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{K}(B_X)$ and we want to prove $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{A}\mathscr{K}(B_X)$.
- The proof of Theorem 1 provides an injection $u : \mathbb{Q} \longrightarrow B_X$ that identifies relatively compact subsets of $\mathbb{Q}$ and relatively weakly compact subsets of B_X.
- We want to find a homogeneous $\delta > 0$ such that $u(S)$ is δ-far from weakly compact whenever $S \subset \mathbb{Q}$ is not rel. compact.

Illustration of the use of Ramsey

- At a stage, we have $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{K}(B_X)$ and we want to prove $\mathscr{K}(\mathbb{Q}) \preceq \mathscr{A}\mathscr{K}(B_X)$.
- The proof of Theorem 1 provides an injection $u : \mathbb{Q} \longrightarrow B_X$ that identifies relatively compact subsets of $\mathbb{Q}$ and relatively weakly compact subsets of B_X.
- We want to find a homogeneous $\delta > 0$ such that $u(S)$ is δ-far from weakly compact whenever $S \subset \mathbb{Q}$ is not rel. compact.
- We would color S acording to the δ necessary. Do we have a Ramsey theorem that allows to homogenize? The one recently found by A. and Todorcevic does the job.

Unconditional bases

Question

How to produce unconditional bases B such that $\mathscr{R}(B)$ is Tukey equivalent to any of $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$, $Fin(\mathbb{R})$?

Unconditional bases

Question

How to produce unconditional bases B such that $\mathscr{R}(B)$ is Tukey equivalent to any of $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$, $Fin(\mathbb{R})$?

Definition

An adequate family $\mathbb{A}$ is a family of sets such that $a \in \mathbb{A}$ iff all finite subsets of a belong to $\mathbb{A}$.

Unconditional bases

Question

How to produce unconditional bases B such that $\mathscr{R}(B)$ is Tukey equivalent to any of $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$, $Fin(\mathbb{R})$?

Definition

An adequate family $\mathbb{A}$ is a family of sets such that $a \in \mathbb{A}$ iff all finite subsets of a belong to $\mathbb{A}$.

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}$ of closed and discrete subsets of E

Unconditional bases

Question

How to produce unconditional bases B such that $\mathscr{R}(B)$ is Tukey equivalent to any of $\{0\}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathscr{K}(\mathbb{Q})$, $Fin(\mathbb{R})$?

Definition

An adequate family $\mathbb{A}$ is a family of sets such that $a \in \mathbb{A}$ iff all finite subsets of a belong to $\mathbb{A}$.

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}$.

Unconditional bases

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}_E$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}_E$.

Unconditional bases

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}_E$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}_E$.

From an adequate $\mathbb{A}$ one constructs an unconditional basis $B(\mathbb{A})$ with the norm

Unconditional bases

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}_E$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}_E$.

From an adequate $\mathbb{A}$ one constructs an unconditional basis $B(\mathbb{A})$ with the norm

$$\|\sum \lambda_n e_n\| = \sup_{a \in \mathbb{A}} \sum_{n \in A} |\lambda_n|$$

Unconditional bases

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}_E$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}_E$.

From an adequate $\mathbb{A}$ one constructs an unconditional basis $B(\mathbb{A})$ with the norm

$$\|\sum \lambda_n e_n\| = \sup_{a \in \mathbb{A}} \sum_{n \in A} |\lambda_n|$$

It happens that $\mathbb{R}(B(\mathbb{A}_E)) \sim \mathscr{K}(E)$ so one applies Fremlin.

Unconditional bases

Theorem (A., based on Talagrand)

Iff E is coanalytic, then there exists an adequate family $\mathbb{A}_E$ of closed and discrete subsets of E such that each infinite closed discrete subset of E contains an infinite set from $\mathbb{A}_E$.

From an adequate $\mathbb{A}$ one constructs an unconditional basis $B(\mathbb{A})$ with the norm

$$\|\sum \lambda_n e_n\| = \sup_{a \in \mathbb{A}} \sum_{n \in A} |\lambda_n|$$

It happens that $\mathbb{R}(B(\mathbb{A}_E)) \sim \mathscr{K}(E)$ so one applies Fremlin. From E with $|E| = \omega_1$, one gets B with $\mathscr{R}(B) \sim Fin(\omega_1)$