Topological and measure properties of some self-similar sets

Taras Banakh, Artur Bartoszewicz, Małgorzata Filipczak
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Theorem (Kakeya)

For any sequence $x \in l_1 \setminus c_{00}$

1. $E(x)$ *is a perfect compact set.*
2. *If* $|x_n| > \sum_{i>n} |x_i|$ *for almost all* n, *then* $E(x)$ *is homeomorphic to the ternary Cantor set.*
3. *If* $|x_n| \leq \sum_{i>n} |x_i|$ *for almost all* n, *then* $E(x)$ *is a finite union of closed intervals. In the case of non-increasing sequence* x, *the last inequality is also necessary for* $E(x)$ *to be a finite union of intervals.*

Theorem

For any sequence $x \in l_1 \setminus c_{00}$, $E(x)$ is one of the following sets:

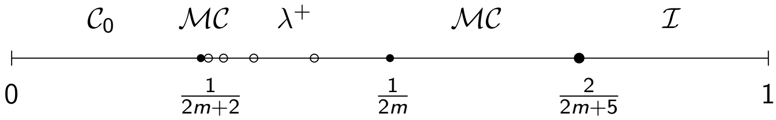
1. *a finite union of closed intervals;*
2. *homeomorphic to the Cantor set;*
3. *homeomorphic to the set $T = E(c)$ for the sequence $c = \left(\frac{5+(-1)^n}{4^n}\right)_{n=1}^{\infty}$.*

We call a sequence *multigeometric* if it is of the form

$$(k_0, k_1, \dots, k_m, k_0 q, k_1 q, \dots, k_m q, k_0 q^2, k_1 q^2, \dots, k_m q^2, k_0 q^3 \dots)$$

for some positive numbers $k_0, \dots, k_m$ and $q \in (0, 1)$.

$$
\begin{aligned}
a &= (8, 7, 6, 5, 4; \tfrac{1}{10}) \\
b &= (7, 6, 5, 4, 3; \tfrac{2}{27}) \\
c &= (3, 2; \tfrac{1}{4}) \\
d &= (3, 2, 2, 2; \tfrac{19}{109}).
\end{aligned}
$$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$$\Sigma=\left\{\sum_{n=0}^{m} k_n\varepsilon_n : (\varepsilon_n)_{n=0}^{m}\in\{0,1\}^{m+1}\right\}$$

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In particular, given a finite subset $\Sigma \subset \mathbb{R}$ of cardinality $|\Sigma| \geq 2$, we will write it as $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ for real numbers $\sigma_1 < \dots < \sigma_s$. Then we have

$$\operatorname{diam}(\Sigma) = \sigma_s - \sigma_1,\ \ \delta(\Sigma) = \min_{i<s}(\sigma_{i+1} - \sigma_i),$$
$$\Delta(\Sigma) = \max_{i<s}(\sigma_{i+1} - \sigma_i).$$

Theorem

Let $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ *for some real numbers* $\sigma_1 < \dots < \sigma_s$. *Then:*

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3. $K(\Sigma; q)$ *contains an interval if* $q \geq i(\Sigma)$*;*
4. *If* $d = \frac{\delta(\Sigma)}{\mathrm{diam}(\Sigma)} < \frac{1}{3+2\sqrt{2}}$ *and* $\frac{1}{|\Sigma|} < \frac{\sqrt{d}}{1+\sqrt{d}}$*, then for almost all* $q \in \left(\frac{1}{|\Sigma|}, \frac{\sqrt{d}}{1+\sqrt{d}}\right)$ *the set* $K(\Sigma; q)$ *has positive Lebesgue measure and the set* $K(\Sigma; \sqrt{q})$ *contains an interval;*

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4. *If* $d = \frac{\delta(\Sigma)}{\mathrm{diam}(\Sigma)} < \frac{1}{3+2\sqrt{2}}$ *and* $\frac{1}{|\Sigma|} < \frac{\sqrt{d}}{1+\sqrt{d}}$, *then for almost all* $q \in \left(\frac{1}{|\Sigma|}, \frac{\sqrt{d}}{1+\sqrt{d}}\right)$ *the set* $K(\Sigma; q)$ *has positive Lebesgue measure and the set* $K(\Sigma; \sqrt{q})$ *contains an interval;*
5. $K(\Sigma; q)$ *is a Cantor set of zero Lebesgue measure if* $q < \frac{1}{|\Sigma|}$ *or, more generally, if* $q^n < \frac{1}{|\Sigma_n|}$ *for some* $n \in \mathbb{N}$ *where* $\Sigma_n = \left\{ \sum_{k=0}^{n-1} a_k q^k : (a_k)_{k=0}^{n-1} \in \Sigma^n \right\}$.

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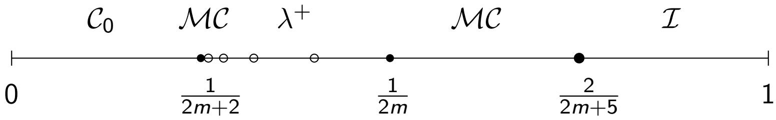
6. *If* $\Sigma \supset \{a, a+1, b+1, c+1, b+|\Sigma|, c+|\Sigma|\}$ *for some real numbers* $a, b, c \in \mathbb{R}$ *with* $b \neq c$, *then there is a strictly decreasing sequence* $(q_n)_{n\in\omega}$ *with* $\lim_{n\to\infty} q_n = \frac{1}{|\Sigma|}$ *such that the sets* $K(\Sigma; q_n)$ *has Lebesgue mesure zero.*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Multigeometric sequences of the form

$$(k+m, \ldots, k+1, k; q)$$

with $m \geq k$ we will call *Ferens-like sequences*. The achievement set $E(x)$ for a Ferens-like sequence coincides with the self-similar set $K(\Sigma; q)$ for the set

$$\Sigma = \{0, k, k+1, \ldots, n-k, n\},$$

where $n = (m+1)(2k+m)/2$. Sets $K(\Sigma; q)$ with Σ of this form will be called *Ferens-like fractals*.

$$x_q = (4, 3, 2; q)$$
$$\Sigma = \{0, 2, 3, 4, 5, 6, 7, 9\}$$

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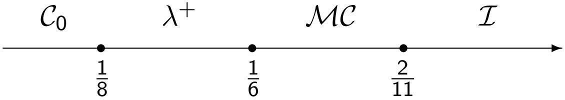

$$
\begin{aligned}
x_q &= (6,5,4,3;q)\\
\Sigma &= \{0,3,\dots,15,18\}\\
|\Sigma| &= 15
\end{aligned}
$$

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x_q &= (6, 5, 4, 3; q) \\
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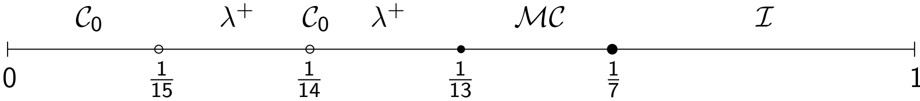

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