Fourier Analysis for vector-measures

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Integration, Vector Measures and Related Topics
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Notation

Throughout X is a complex Banach space, G be a compact abelian group, $\mathscr{B}(G)$ for the Borel σ-algebra of G , m_G for the Haar measure of the group, $L^p(G)$ the space of mesurable functions such that $\int_G |f|^p dm_G < \infty$.

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Let $1 < p \leq \infty$. A measure ν is said to have bounded p-semivariation with respect to m_G if

$$\|\nu\|_{p,m_G} = \sup\left\{\left\|\sum_{A\in\pi} \alpha_A \nu(A)\right\|_X : \pi \text{ partition }, \|\sum_{A\in\pi} \alpha_A \chi_A\|_{L^{p'}(G)} \leq 1\right\}. \tag{1.1}$$

The case $p = \infty$ corresponds to $\|\nu(A)\| \leq C m_G(A)$ for $A \in \mathscr{B}(G)$ for some constant C and $\|\nu\|_{\infty,\lambda}$ is the infimum of such constants.

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The case $p = \infty$ corresponds to $\|\nu(A)\| \leq C m_G(A)$ for $A \in \mathscr{B}(G)$ for some constant C and $\|\nu\|_{\infty,\lambda}$ is the infimum of such constants. We use the notation $\mathscr{M}_p(G,X)$ and this space can be identify with $\mathscr{L}(L^{p'}(G),X)$ and $\|\nu\|_{p,m_G} = \|T_\nu\|_{L^{p'}(G),X}$.

Motivation (part 1)

$L(\nu)$ for the space of functions integrable with respect to a vector measure ν. If $f \in L^1(\nu)$ we denote

$$\nu_f(A) = \int_A f d\nu.$$

Then ν_f is a vector measure and $\|\nu_f\| = \|f\|_{L^1(\nu)}$. We write I_ν the integration operator, i.e. $I_\nu : L^1(\nu) \to X$ is defined by $I_\nu(f) = \nu_f(G) = \int_G f d\nu$

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Delgado y Miana (2009) introduced the notion of "norm integral translation invariant" vector measures, as those satisfying

$$\|I_\nu(\tau_a \phi)\| = \|I_\nu(\phi)\|, \phi \in \text{ simple function }, a \in G \tag{1.2}$$

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- **Is there any weaker condition than the "norm integral translation invariant" which still allows the convolution to be developed for functions on $L^1(\nu)$?**

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- **Is there any weaker condition than the "norm integral translation invariant" which still allows the convolution to be developed for functions on $L^1(\nu)$?**
- **Can one define convolution between general vector-measures and recover their results when applied to ν_f for $f \in L^1(\nu)$?**

Motivation (part 2)

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$$\hat{f}^{\nu}(\gamma) = \int_G f(t)\overline{\gamma(t)}d\nu(t), \gamma \in \Gamma \tag{1.3}$$

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Answering question (a): $\mathscr{M}_0(G,X) = \mathscr{M}_{ac}(G,X)$ if and only if X is finite dimensional.

Proposition

Let X be an infinite dimensional Banach space and $G = \mathbb{T}$. There exists a regular vector measure $\nu : \mathscr{B}(\mathbb{T}) \to X$ such that $\nu << m_{\mathbb{T}}$ and $\hat{\nu} \notin c_0(\mathbb{Z}, X)$.

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Some operators that play a role:

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Let $T:X\to Y$ be a completely continuous operator (i.e. it maps it maps weakly convergent sequences in X into norm convergent sequences in Y) and $\nu\in\mathscr{M}_{ac}(G,X)$. Then $T(\nu)\in\mathscr{M}_0(G,Y)$.

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Again the answer is NO! Let $G = \mathbb{T}$, $X = L^1(\mathbb{T})$ and $\nu(A) = \chi_A$. Clearly $T_\nu : C(\mathbb{T}) \to L^1(\mathbb{T})$ corresponds to the inclusion map. Hence $\hat{\nu}(n) = \phi_n$ where $\phi_n(t) = e^{int}$ and $\|\hat{\nu}(n)\| = 1$ for each $n \in \mathbb{Z}$.

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$$(RNP) \Longrightarrow (wRNP) \Longrightarrow (CCP) \Longrightarrow (RLP).$$

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- If $\nu \in \mathscr{M}_{p_1}(G,X)$, $g \in L^{p_2}(G)$ and $f \in L^{p_3}(\nu)$ with $\frac{1}{p_1}+\frac{1}{p_2} \leq 1$ and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \geq 1$ then $f *^\nu g \in P_r(G,X)$ for $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = \frac{1}{r'}$.

Invariance under homeomorphisms

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  (i) ν is semivariation H-invariant.
  (ii) $L^1(\nu) = L^1(\nu_H)$ isometrically.
  (iii) $\|T_{\nu_H} \circ M_f\| = \|T_\nu \circ M_f\|, \forall f \in C(G)$ where $M_f : C(G) \to C(G)$ the multiplication operator $g \to fg$.

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- Let $1 \leq p < \infty$ and let $\nu \in \mathscr{M}(G,X)$ be semivariation translation invariant with $\nu(G) \neq 0$. Then $L^p(\nu) \subset L^p(G)$ and

$$\|f\|_{L^p(G)} \leq \|f\|_{L^p(\nu)} \|\nu(G)\|^{-1/p}.$$

Semivariation invariant measures

Proposition

Let $\nu_{\mathbf{f}}(A) = \int_A \mathbf{f}(s)dm_G(s)$ *with* $\mathbf{f} \in L^\infty(G,X)$ *non constant function satisfying that* $\|\mathbf{f}(t)\| = 1, \quad t \in G$ *and there exists* $A \in \mathscr{B}(G)$ *and* $a \in G$ *for which* $\nu_{\mathbf{f}}(A) = 0, \quad \nu_{\mathbf{f}}(A+a) \neq 0$. *Then* $\nu_{\mathbf{f}}$ *is semivariation translation invariant but not norm integral translation invariant.*

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Proof.

Note that $\tau_t \nu_{\mathbf{f}} = \nu_{\tau_t \mathbf{f}}$ and $\tau_t \mathbf{f} \in L^\infty(G,X)$ for each $t \in G$. In particular $\tau_t \nu_{\mathbf{f}}$ is of bounded variation and $d|\tau_t \nu_{\mathbf{f}}| = \tau_t \|\mathbf{f}\| dm_G = dm_G$. Hence $L^1(\nu) = L^1(\tau_t \nu) = L^1(m_G)$ for any $t \in G$. Hence ν is semivariation translation invariant.

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$X = \mathbb{C}$, $G = \mathbb{T}$, $\mathbf{f}(s) = \chi_{[0,1/2)}(e^{2\pi i s}) - \chi_{[1/2,1)}(e^{2\pi i s})$, $A = \{e^{2\pi i s} : 1/4 \leq s < 3/4\}$ and $a = e^{i\pi/2}$) to have a particular example.

Final applications

If $L^1(\nu) \subset L^1(G)$ then we can define

$$f *_G g(t) = \int_G g(t-s)f(s)dm_G(s) = \int_G \tau_s g(t)f(s)dm_G(s)$$

for $f, g \in L^1(\nu)$.

Theorem

Let $1 \leq p < \infty$ and let $\nu \in \mathcal{M}(G,X)$ semivariation translation invariant. If $f \in L^1(G)$ and $g \in L^p(\nu)$ then $f *_G g \in L^p(\nu)$ with $\|f *_G g\|_{L^p(\nu)} \leq \|f\|_{L^1(G)} \|g\|_{L^p(\nu)}$.

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To extend to general functions, we use that $C(G)$ is dense in $L^1(\nu)$ and the fact that $L^1(\nu) \subset L^1(G)$. □

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