Filter convergence and decompositions for vector lattice-valued measures

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Outline

1. Some background
   - Vector lattices
   - Filter convergence
   - Decompositions
2. Convergence theorems
   - The σ-additive case
   - The finitely additive case
3. The (*SCP*) property

Vector lattices

Let $(\mathbf{X}, +, \cdot, \leq)$ be a real vector space, endowed with a compatible ordering $<$. If $\mathbf{X}$ is stable under finite suprema (and infima) then it is called a

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- $\mathbf{X}$ is **weakly σ-distributive**

  (i.e. $0 = \bigwedge_\phi \bigvee_i a_{i,\phi(i)}$ holds true, for each double sequence $(a_{i,j})$ such that $a_{i,j} \downarrow_j 0$ for every integer i *(regulator), and ϕ runs among all mappings from $\mathbb{N}$ to $\mathbb{N}$)*

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(o)-sequence

Any decreasing sequence $(p_n)_n$ in X, such that $\inf_n p_n = 0$.

(o)-convergence

A sequence $(a_n)_n$ in $\mathbf{X}$ is said to be (o)**-convergent** to $a \in \mathbf{X}$ whenever an (o)-sequence $(p_n)_n$ exists, such that

$$|a_n - a| \leq p_n$$

for all n. If this happens, $(p_n)_n$ will be called a *regulating* (o)-sequence for $(a_n)_n$.

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Lemma

(see [a]) Let $(r_n)_n$ be any (o)-sequence in a super-Dedekind complete vector lattice X. For every positive element $u \in X^+$ there exists an increasing mapping $\omega : \mathbb{N} \to \mathbb{N}$ such that

$$N \mapsto u \wedge \Big(\sum_{n=N}^{\infty} r_{\omega(n)}\Big)$$

defines an (o)-sequence in X.

[a] A. BOCCUTO, D. C., *A survey of decomposition and convergence theorems for l-group-valued measures*, Atti Sem. Mat. Fis. Univ. Modena e Reggio Emilia, **53**, (2005), 243-260.

Filter convergence

Let Z be any fixed set.
A family $\mathcal{F}$ of subsets of Z is called a **filter** of Z iff

- $\emptyset \notin \mathcal{F}$
- $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$
- $A \in \mathcal{F}, B \supset A \Rightarrow B \in \mathcal{F}$.

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Given any filter $\mathcal{F}$ of subsets of Z, the **dual ideal** of $\mathcal{F}$ is

$$\mathcal{I}_{\mathcal{F}} := \{F^c : F \in \mathcal{F}\}.$$

If $\{z\} \in \mathcal{I}_{\mathcal{F}}$ for all $z \in Z$, $\mathcal{F}$ is a **free** filter.

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Examples: $Z = \mathbb{N}$

- **Statistical filter:** $\mathcal{F} := \{J \subset \mathbb{N} : \lim_n \frac{|J \cap [0,n]|}{n} = 1\}$
- **Countably generated filters:** $\mathcal{I}_{\mathcal{F}}$ is generated by a countable partition of $\mathbb{N}$.
- **(free) Ultrafilters**

Definition

A sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbf{X}$ $(o_{\mathcal{F}})$**-converges to** $x \in \mathbf{X}$ $(x_k \overset{o_{\mathcal{F}}}{\to} x)$ iff there exists an (o)-sequence $(\sigma_p)_p$ in $\mathbf{X}$ such that the set

$$\{k \in \mathbb{N} : |x_k - x| \leq \sigma_p\}$$

is an element of $\mathcal{F}$ for each $p \in \mathbb{N}$.

If this is the case, then $(\sigma_p)_p$ is said to be a **regulator** for $(o_{\mathcal{F}})$-convergence of $(x_k)_k$.

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Lemma

(see [a]) Let $(\sigma^j_p)_p$ be an (o)-sequence for all $j \in \mathbb{N}$, and assume that the set $\{\sigma^j_p : p \in \mathbb{N}, j \in \mathbb{N}\}$ is bounded in $\mathbf{X}$. Then there exists an (o)-sequence $(r_n)_n$ such that, for every j and every n there exists p satisfying $\sigma^j_p \leq r_n$.

[a] B. Riecan-T.Neubrunn *Integral, Measure and Ordering*, Kluwer, Ister Science, Dordrecht/Bratislava (1997).

- $\forall$ filter $\mathcal{F}$ in Z, a subset $H \subset Z$ is **stationary** if $Z \notin \mathcal{I}_{\mathcal{F}}$, i.e. if and only if $H \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

[1] A. AVILES LOPEZ, B. CASCALES SALINAS, V. KADETS, A. LEONOV, *The Schur l_1-theorem for filters*, J. Math. Phys. Anal. Geom. **3** (2009), 383-398.

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- The filter $\mathcal{F}$ is **block-respecting** if, for every stationary set H and every block $\{D_k : k \in \mathbb{N}\}$ of H there exists a stationary set $J \subset H$ such that $card(J \cap D_k) \leq 1$ for all k. ($\forall$ infinite $I \subset Z$ a **block** of I is any partition $\{D_k, k \in \mathbb{N}\}$ of I, obtained with *finite* sets D_k in Z).

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- The filter $\mathcal{F}$ is said to be **diagonal** if for every sequence $(A_n)_n$ in $\mathcal{I}_{\mathcal{F}}$ and every stationary set $I \subset Z$, there exists a stationary set $J \subset I$ such that $J \cap A_n$ is finite for all $n \in \mathbb{N}$.

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- In the paper [1] the **simplified Schur** property has been proved to be equivalent to the **block-respecting** property.

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Examples

- The statistical filter is diagonal but not block-respecting.
- Any countably generated filter has both properties.

[1] A. AVILES LOPEZ, B. CASCALES SALINAS, V. KADETS, A. LEONOV, *The Schur l_1-theorem for filters*, J. Math. Phys. Anal. Geom. **3** (2009), 383-398.

Notations and Definitions:

- $\Omega \equiv$ any abstract space.
- $\mathcal{H} \equiv$ any algebra of subsets of Ω.
- $\mathcal{A} \equiv$ any σ-algebra in Ω.
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- m is ***s*-bounded** if $\exists$ an (o)-sequence (p_n) in $\mathbf{X}$ such that, for every pairwise disjoint sequence $(F_k)_k$ from $\mathcal{H}$ and every integer n, it is possible to find an index $k(n)$ satisfying

$$\bigvee_{k \geq k(n)} |m(F_k)| \leq p_n. \tag{1}$$

In this case we say that (p_n) **regulates** s-boundedness of m.

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- **Variations**:

$$v^+(m)(H) = \sup_{A \in \mathcal{H}} m(A \cap H), \quad v^-(m) := v^+(-m), \quad v(m) = v^+ + v^-.$$

- $m : \mathcal{H} \to \mathbf{X}$ is σ**-additive** if $\exists$ an (o)-sequence (p_n) in $\mathbf{X}$ such that, for every decreasing sequence $(F_k)_k$ from $\mathcal{H}$ with empty intersection, and every integer n it is possible to find an index $k(n)$ satisfying

$$\bigvee_{A \in \mathcal{H}} |m(A \cap F_{k(n)})| \leq p_n. \tag{2}$$

Also in this case, (p_n) **regulates** σ-additivity of m.

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Theorem

(see [a]) Let $m : \mathcal{A} \to \mathbf{X}$ be any s-bounded finitely additive measure, defined on a σ-algebra $\mathcal{A}$, and let $(H_n)_n$ be any pairwise disjoint family from $\mathcal{A}$. Then there exists a sub-sequence $(H_{n_k})_k$ such that m is σ-additive in the σ-algebra generated by the sets H_{n_k} (Same regulating (o)-sequence).

[a] A. BOCCUTO, D. C., *Convergence and decompositions for l-group-valued set functions*, Commentationes Matematicae, **44**, (1) (2004), 11-37.

Lebesgue decompositions

Let $m : \mathcal{H} \to \mathbf{X}$ and $\nu : \mathcal{H} \to \mathbb{R}_0^+$ be bounded finitely additive measures.

- (**absolute continuity**): $m \ll \nu$ when the following setting defines an (o)-sequence in $\mathbf{X}$:

$$p_n := \sup\{|m(A)| : A \in \mathcal{H}, \nu(A) \leq \frac{1}{n}\}, \qquad n \in \mathbb{N}.$$

- (**singularity**): $m \perp \nu$ if $\exists\ (A_k)_k$ in $\mathcal{H}$ and an (o)-sequence $(q_k)_k$ in $\mathbf{X}$ such that $\lim_k \nu(A_k) = 0$ and, for every k

$$\sup\{|m(E \setminus A_k)| : E \in \mathcal{H}\} \leq q_k.$$

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Theorem

(see [a]) Let $m : \mathcal{H} \to \mathbf{X}$ and $\nu : \mathcal{H} \to \mathbb{R}_0^+$ be two s-bounded finitely additive measures on an algebra $\mathcal{H}$. Then there exist (unique) $\mathbf{X}$-valued measures $m^<$ and $m^\perp$, mutually singular, such that

$$m^< \ll \nu, \quad m^\perp \perp \nu, \; m^< + m^\perp = m$$

[a] A. BOCCUTO, D. C., *A survey of decomposition and convergence theorems for l-group-valued measures*, Atti Sem. Mat. Fis. Univ. Modena e Reggio Emilia, **53**, (2005).

Sobczyk-Hammer decompositions

Definition

Let $m : \mathcal{H} \to \mathbf{X}_0^+$ be any finitely additive measure. We say that m is **continuous** if there exists an (o)-sequence $(p_n)_n$ in $\mathbf{X}$ and a sequence $(\pi_n)_n$ of finite partitions of Ω, $\pi_n = \{J_1, ..., J_{k_n}\}$ such that for each n we have

$$\sup_{i=1\ldots,k_n} m(J_i) \le p_n.$$

We also say that m is **atomic** if there exist no nonzero continuous finitely additive measure $\mu : \mathcal{H} \to \mathbf{X}_0^+$ such that $\mu \le m$.
In case m is not a positive measure, but is bounded, then it will be said to be **continuous** if $v(m)$ is.

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Theorem

(see [a]) Let $m : \mathcal{H} \to \mathbf{X}_0^+$ be s-bounded and finitely additive. Then $\exists$ mutually singular finitely additive measures m^s and m^a, both $\ll m$, such that

$$m^s \textbf{ is continuous } \; m^a \textbf{ is atomic }, \quad m^s + m^a = m.$$

Convergence: countably additive case

Let **any** (free) filter $\mathcal{F}$ be fixed in $\mathbb{N}$.

Theorem

Let $(m_n)_n$ be any sequence of bounded $\mathbf{X}$*-valued σ-additive measures defined on a measure space $(\Omega, \mathcal{A}, \nu)$ such that the sequence $(m_n)_n$* **is pointwise $(o_{\mathcal{F}})$-convergent to a σ-additive measure m.**
Then the sequences $(m_n^{<})_n, \ (m_n^{\perp})_n, \ (m_n^a)_n, \ (m_n^s)_n$ **converge in the same way to $m^{<}, \ m^{\perp}, \ m^a, \ m^s$ respectively.**

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If $\mathcal{F}$ is **block-respecting and diagonal**, we have (**Schur**):

Theorem

(see [a]) Assuming that the measures m_n are defined on $\mathcal{P}(\mathbb{N})$, are uniformly bounded and $(o_{\mathcal{F}})$-convergent to 0, then the sequence $n \mapsto \sum_{k \in \mathbb{N}} |m_n(\{k\})|$ is $(o_{\mathcal{F}})$-convergent to 0.

[a] A. BOCCUTO, X.DIMITRIOU, N. PAPANASTASSIOU, *Schur lemma and limit theorems in lattice groups with respect to filters,* Math. Slovaca **62** (6) (2012), 1145-1166.

Uniform *s*-boundedness

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Definition

We say that a sequence $(m_n)_n$ of $\mathbf{X}$-valued finitely additive measures, defined on a algebra $\mathcal{H}$, is **uniformly s-bounded** if $\exists$ an (o)-sequence $(r_k)_k$ in $\mathbf{X}$ such that, for every disjoint sequence (H_j) from $\mathcal{H}$ and for every $k \in \mathbb{N}$ an index $j(k)$ can be found, such that

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Theorem

Assume that the f.a. measures $(m_n)_n$, defined on $\mathcal{H}$, are uniformly s-bounded *and pointwise $(o_{\mathcal{F}})$-convergent to some finitely additive measure m. Assume also that $\nu : \mathcal{H} \to \mathbb{R}_0^+$ is a fixed finitely additive measure. Then the sequences*

$$(m_n^{<})_n, \ (m_n^{\perp})_n, \ (m_n^{a})_n, \ \ (m_n^{s})_n$$

converge in the same way to $m^{<}$, $m^{\perp}$, m^{a}, m^{s} respectively.

Ideal s-boundedness

Uniform s-boundedness can be relaxed, as follows.

Definition

Given a sequence $\{m_j : j \in \mathbb{N}\}$ of s-bounded finitely additive measures on $\mathcal{H}$, and a filter $\mathcal{F}$ in $\mathcal{P}(\mathbb{N})$, we say that the measures $\{m_j : j \in \mathbb{N}\}$ are **ideally uniformly s-bounded** if there exists an (o)-sequence $(r_k)_k$ such that, for any family $(H_l)_l$ of pairwise disjoint sets in $\mathcal{H}$, any integer k and any element I of the dual ideal of $\mathcal{F}$, there exists an integer $l(k)$ such that

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Assume now that the filter $\mathcal{F}$ is **block-respecting and diagonal**.

Theorem

Let $(m_n)_n$ be an equibounded sequence of **ideally uniformly *s*-bounded** *finitely additive measures, defined on a σ-algebra $\mathcal{A}$, and taking values in* $\mathbf{X}$*. If the measures m_n are $(o_{\mathcal{F}})$-convergent to an s-bounded finitely additive measure m, then the measures are uniformly s-bounded.*

The (SCP) Property

Definition

Let $\mathcal{H}$ be an algebra of subsets of an abstract set Ω. We say that $\mathcal{H}$ enjoys the *property (SCP)* if, for every sequence $(H_k)_k$ of pairwise elements from $\mathcal{H}$, there exists a subsequence $(H_{k_r})_r$ whose union belongs to $\mathcal{H}$.

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Let $\mathcal{H}$ be an algebra of subsets of an abstract set Ω. We say that $\mathcal{H}$ enjoys the *property (SCP)* if, for every sequence $(H_k)_k$ of pairwise elements from $\mathcal{H}$, there exists a subsequence $(H_{k_r})_r$ whose union belongs to $\mathcal{H}$.

Theorem

(see [a]) Assume that $(m_n)_n$ is an equibounded sequence of **ideally uniformly s-bounded** *finitely additive measures, defined on an* **algebra $\mathcal{H}$ enjoying** (SCP)*, and taking values in* **X***. Assume that the filter* $\mathcal{F}$ **is countably generated***, and that the measures m_n are $(o_{\mathcal{F}})$-convergent to an s-bounded finitely additive measure m. Moreover, let $\nu : \mathcal{H} \to \mathbb{R}_0^+$ be any positive finitely additive measure. Then the measures $(m_n)_n$ are* **uniformly s-bounded** *and the sequences $(m_n^<)_n$, $(m_n^\perp)_n$, $(m_n^a)_n$, $(m_n^s)_n$* **converge** *in the same way to $m^<$, $m^\perp$, m^a, m^s respectively, where absolute continuity and singularity are meant w.r.t. ν.*

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[a] D. C., A.R. SAMBUCINI *Filter convergence and decompositions for vetcor lattice-valued measures*, in press in Mediterranean J. Math. (2014)

THANK YOU!!!

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