On some properties of modular function spaces

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Aim of this talk is to introduce modular function spaces as defined by Kozlowoski, to present some examples and prove admissibility of the spaces.

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- The theory of Banach function spaces [Luxemburg, Zaanen].
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Here we consider a class of modular spaces given by modulars not of a particular form but having much more convenient properties than the abstract modulars can possess.

Definitions and notations

[W.M. Kozlowski, Modular function spaces, 1988]

X nonempty set - $\mathcal{P}$ nontrivial δ-ring of subsets of X
- Σ the smallest σ-algebra of subsets of X s.t. $\mathcal{P} \subset \Sigma$.
Assume $E \cap A \in \mathcal{P}$ for $E \in \mathcal{P}$ and $A \in \Sigma$, and $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \subset X_{n+1}$ and $X_n \in \mathcal{P}$ for any $n \in N$.

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$(W, \|\cdot\|)$ Banach space.

- $\mathcal{E}$ the linear space of all $\mathcal{P}$-simple functions.
- $M(X, W)$ the set of all measurable functions. $f : X \to W$ *measurable* if $s_n(x) \to f(x)$ for any $x \in X$, with $\{s_n\}$ $\mathcal{P}$-simple functions

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(1) $\rho(0, E) = 0$ for every $E \in \Sigma$,

(2) $\rho(f, E) \leq \rho(g, E)$ whenever $\|f(x)\| \leq \|g(x)\|$ for all $x \in E$ and any $f, g \in \mathcal{E}$ $(E \in \Sigma)$;

(3) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$;

(4) $\rho(\alpha, A) \to 0$ as $\alpha \in R_+$ decreases to 0 for every $A \in \mathcal{P}$, where

$$\rho(\alpha, A) = \sup\{\rho(r\chi_A, A) : r \in W, \|r\| \leq \alpha\};$$

(5) there is $\alpha_0 \geq 0$ such that $\sup_{\beta>0} \rho(\beta, A) = 0$ whenever $\sup_{\alpha>\alpha_0} \rho(\alpha, A) = 0$;

(6) $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$ for every $\alpha > 0$, that is $\rho(\alpha, A_n) \to 0$ for any sequence $\{A_n\} \subset \mathcal{P}$ decreasing to $\emptyset$.

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For $f \in M(X, W)$

$$\rho(f, E) = \sup\{\rho(g, E) : g \in \mathcal{E}, \|g(x)\| \leq \|f(x)\| \ x \in E\}.$$

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Then the functional $\rho : M(X, W) \to [0, +\infty]$ defined by $\rho(f) = \rho(f, X)$ is a semimodular, that is

- $\rho(\lambda f) = 0$ for any $\lambda > 0$ iff $f = 0$ ρ-a.e.;
- $\rho(\alpha f) = \rho(f)$ if $|\alpha| = 1$ and $f \in M(X, W)$;
- $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$ $(\alpha, \beta \geq 0)$ and $f, g \in M(X, W)$.

Given the semimodular ρ we consider the modular space

$$L_\rho = \{f \in M(X, W) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0\},$$

and we endowed it by the F-norm

$$\|f\|_\rho = \inf \left\{ \lambda > 0 : \ \rho\left(\frac{f}{\lambda}\right) \leq \lambda \right\}.$$

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Let E_ρ the closed subspace of L_ρ defined by

$E_\rho = \{f \in M(X, W) : \rho(\alpha f, \cdot)$ is order continuous for every $\alpha > 0\}$.

For any set S in E_ρ we denote by clS the closure of S with respect to $\|\cdot\|_\rho$. We recall that $E_\rho = cl\mathcal{E}$.

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$$\sup_n \rho(2f_n, A_k) \to 0 \qquad k \to \infty,$$

whenever $\{f_n\}_n \subset M(X, W)$, $A_k \in \Sigma$, $A_k \to \emptyset$ and $\sup_n \rho(f_n, A_k) \to 0$ as $k \to \infty$.

Examples

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(X, Σ, μ) measure space. Let $\phi : X \times R^+ \to R^+$ be a ϕ_1-function

1. $\phi(x, \cdot)$ is continuous for every $x \in X$
2. $\phi(x, 0) = 0$ for every $x \in X$
3. $\phi(x, u) \to \infty$ as $u \to \infty$
4. $\phi(\cdot, u)$ is locally integrable for every $u \geq 0$
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$$\rho(f, E) = \int_E \phi(x, |f(x)|) d\mu$$

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Orlicz Δ_2-condition holds:
There exists $K > 0$ and an integrable $g : X \to R^+$ such that for every $u \geq 0$

$$\phi(x, 2u) \leq K\phi(x, u) + g(x) \quad \rho\text{-}a.e.$$

A generalization of Musielak-Orlicz spaces.

$\mathcal{M}$ family of measures on (X, Σ).

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Orlicz Δ_2 condition $+$ the set function

$$\sup_{\mu \in \mathcal{M}} \int_{(\cdot)} g \ d\mu$$

is order continuous.

Admissibility

The notion of admissibility, introduced by Klee, allows one to approximate the identity on compact sets by finite dimensional maps. Locally convex spaces are admissible. Not all nonlocally convex spaces are admissible [Cauty has provided an example of a metric linear space in which the admissibility fails].

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Definition

Let E be a Haudorff topological vector space. The space E is said to be *admissible* if for every compact subset K of E and for every neighborhood V of zero in E there exists a continuous mapping $H : K \to E$ such that $\dim(\text{span }[H(K)]) < +\infty$ and $f - Hf \in V$ for every $f \in K$.

Main result

Theorem

The space E_ρ is admissible.

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K a compact set in E_ρ and $\varepsilon > 0$. We consider $H : K \longrightarrow E_\rho$

$$H = H_\varepsilon \circ P_k \circ T_a \circ F_n.$$

1.
Set $F_n f = f \chi_{X_n}$, where $X = \bigcup_{n=1}^{\infty} X_n$. Find $n \in N$ such that

$$\sup\{\|F_n f - f\|_\rho : f \in K\} \leq \varepsilon.$$

2.
Find $a > 0$ satisfying

$$\sup\{\|T_a F_n f - F_n f\|_\rho : f \in K\} \leq \varepsilon.$$

3. Let $\Pi_k = \{E_1, ...E_l\}$, $\Pi_n = \{F_1, ..., F_m\}$ and $\Pi_k \leq \Pi_n$. Define $P_{k,n} : S_{\Pi_n} \to S_{\Pi_k}$

$$P_{k,n}s = \sum_{i=1}^{l} \frac{\sum_{j=1}^{m_i} w_j}{m_i} \chi_{E_i}.$$

where $s = \sum_{j=1}^{m} w_j \chi_{F_j}$, and $E_i = \sum_{j=1}^{m_i} F_{ij}$, with $F_{ij} \in \Pi_n$, for $j = 1, ..., m_i$.

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$Q = \{\Pi_n\}$ sequence of partitions of X with $\Pi_1 = \{X\}$ and $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ....$ Then for any $k \in N$, define $P_k : S(Q) = \bigcup_{j=1}^{\infty} S_{\Pi_j} \to S_{\Pi_k}$

$$P_k s = \begin{cases} s & \text{if } \; s \in \bigcup_{j=1}^{k} S_{\Pi_j} \\ P_{k,n}s & \text{if } \; s \in \bigcup_{j=k+1}^{\infty} S_{\Pi_j}. \end{cases}$$

For $f \in cl(S(Q))$ such that $\sup\{\|f(x)\| : x \in X\} \leq a < \infty$, for any $k \in N$, define

$$P_k f = \lim_n P_k s_n,$$

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Lemma

Let $\{f_n\}$ be a sequence in $cl(S(Q))$ and $f \in cl(S(Q))$. If $\sup\{\|f_n(x)\| : x \in X, n \in N\} \leq a < \infty$, $\sup\{\|f(x)\| : x \in X\} \leq a < \infty$ and $\|f - f_n\|_\rho \to 0$, then

$$\lim_n(\sup\{\|P_k f_n - P_k f\|_\rho : k \in N\}) = 0.$$

Theorem

Let K_a be a compact subset of E_ρ such that

$$\sup\{\|f(x)\| : x \in X, f \in K_a\} \leq a < \infty.$$

Then there exists a sequence $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ...$, which will be denoted by Q, of partitions of X such that $\Pi_1 = \{X\}$ and $K_a \subset cl(S(Q))$.
Moreover, for any $\varepsilon > 0$ there exists $k \in N$ such that

$$\sup\{\|P_k f - f\|_\rho : f \in K_a\}) \leq \varepsilon,$$

being $\{P_k\}$ be the sequence of operators corresponding to the sequence of partitions Q.

4. The space of $\mathcal{P}$-simple functions generated by a given partition of X is admissible.

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Proposition

Let $\Pi = \{A_1, \cdots, A_n\}$ be a partition of X. Then the subspace

$$S_\Pi = \left\{ s \in E_\rho : s = \sum_{i=1}^{n} w_i \chi_{A_i}, \quad w_i \in W \right\}$$

of E_ρ is admissible.

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Proof. K a compact set in E_ρ and $\varepsilon > 0$.

1. $$\sup\{\|F_n f - f\|_\rho : f \in K\} \leq \varepsilon.$$

2. $$\sup\{\|T_a F_n f - F_n f\|_\rho : f \in K\} \leq \varepsilon.$$

3. Considering $(T_a \circ F_n)(K)$ we have

$$\sup\{\|P_k T_a F_n f - T_a F_n f\|_\rho : f \in K\} \leq \varepsilon.$$

4. $V = (P_k \circ T_a \circ F_n)(K)$ is a compact subset of S_{Π_k}. Let $H_\varepsilon : V \to E_\rho$ such that span$[H_\varepsilon(V)]$ is finite-dimensional and

$$\sup\{\|H_\varepsilon P_k T_a F_n f - P_k T_a F_n f\|_\rho : f \in K\} \leq \varepsilon.$$

Now we consider

$$H = H_\varepsilon \circ P_k \circ T_a \circ F_n.$$

We have dim[span[$H(K)$] $< \infty$. Moreover, by the above facts, for any $f \in K$

$$\|f - Hf\|_\rho \leq \varepsilon.$$

and the admissibility of E_ρ is proved.

Corollary

Let $T : L_\rho \to E_\rho$ *be a compact and continuous mapping. Then there exists* $f \in E_\rho$ *such that* $Tf = f$.

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