Absolute continuity of non-additive measures and applications to function spaces

Paola Cavaliere

University of Salerno

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Quasi-triangular functions

- $\mathcal{R}$ is a **Boolean ring**,
- $\mathcal{S} = (S, \tau)$ is a **Hausdorff topological space**

Definition

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η is quasi-triangular **IFF**
$$\begin{cases} \forall U \in \tau[\eta(0)] \;\; \exists V = V(U) \in \tau[\eta(0)] \;\; s.t. \\ \quad \forall a, b \in \mathcal{R}, \quad a \wedge b = 0 \; : \\ \qquad \eta(a), \eta(b) \in V \implies \eta(a \vee b) \in U; \\ \eta(a), \eta(a \vee b) \in V \implies \eta(b) \in U. \end{cases}$$

[Klimkin-Sribnaya, 1997]

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Clearly

- classical measures on σ-algebras are quasi-triangular

The class of quasi-triangular functions also encompasses

❶ ***k*-triangular functions,** $k \geq 1$, *i.e.* each $\eta : \mathcal{R} \to [0, \infty]$ fulfilling $\eta(0) = 0$ and

$$(T_k) \quad \eta(a) - k\eta(b) \leq \eta(a \vee b) \leq \eta(a) + k\eta(b)$$

for all $a, b \in \mathcal{R}$, $a \wedge b = 0$.

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1. **k-triangular functions, $k \geq 1$**, *i.e.* each $\eta : \mathcal{R} \to [0, \infty]$ fulfilling $\eta(0) = 0$ and

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2. **$\oplus$-decomposable functions**, *i.e.* each $\eta : \mathcal{R} \to [0, 1]$ fulfilling $\eta(0) = 0$ and

$$(wA) \quad \eta(a \vee b) = \eta(a) \oplus \eta(b) \text{ for all } a, b \in \mathcal{R},\ a \wedge b = 0,$$

where $\oplus$ is a **t-conorm** on $[0, 1]$ **continuous at 0**.

[S.Weber, Pap ,.....]

A **t-conorm** $\oplus$ on $[0, 1]$ is an internal composition law on $[0, 1]$, commutative, associative and increasing at each place with 0 as neutral element.

T-conorms continuous at 0 are, for instance, $\oplus_\infty(x, y) := \max\{x, y\}$, $\oplus_p(x, y) := \{x^p + y^p\}^{\frac{1}{p}}$ for $p > 0$ and $\oplus_*(x, y) := x + y - xy$.

3. **quasi-sub-measures**, *i.e.* each $\eta : \mathcal{R} \to [0, \infty]$ fulfilling $\eta(0) = 0$ and for some $C_1, C_2 \geq 1$

$$(qM) \quad \eta(a) \leq C_1\, \eta(b) \text{ for all } a, b \in \mathcal{R},\ a \leq b,$$

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- $\eta = \lambda^p$ and $\eta = (1+\lambda)^p \big(\log(1+\lambda)\big)^\alpha$ for all $p > 0$, $\alpha \geq 0$

Absolute Continuity and Well-known classical result

- $\mathcal{A}$ is a **σ-complete** Boolean algebra,
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Notions of Absolute Continuity of ν with respect to η

(*I*) $\mathbf{0}$-*continuity* of $\boldsymbol{\nu}$ with respect to $\boldsymbol{\eta}$

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(II) (ε, δ)-*continuity* of ν with respect to η

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \eta(a) < \delta \ \text{ for } a \in \mathcal{A} \quad \Longrightarrow \quad \nu(a) < \varepsilon.$$

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Theorem (Relationship between $\ll$ and [AC])

- $\nu\ [AC]\ \eta \implies \nu \ll \eta$.
- For **finite** ν : $\nu\ [AC]\ \eta \impliedby \nu \ll \eta$.

For notions

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($\star$) $\nu, \eta : \mathcal{A} \to [0, +\infty]$ are **non-decreasing**, and **σ-completeness** for $\mathcal{A}$ is **unimportant.**

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On defining the kernel of ν as

$$\mathcal{N}(\nu) := \left\{ a \in \mathcal{A} :\ \nu(a) = 0 \right\},$$

clearly

$$(I) \iff \mathcal{N}(\eta) \subseteq \mathcal{N}(\nu)$$

Finitely additive functions having values in topological groups

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For **finitely additive** $\eta : \mathcal{R} \to \mathcal{G}'$, where $\mathcal{G}' = (G', \tau')$ is a **top. group**

$$(I) \quad \overset{def}{\Longleftrightarrow} \quad \mathcal{N}(\eta) \subseteq \mathcal{N}(\nu)$$

Absolute Continuity in quasi-triangular setting

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Clearly

$$\nu \text{ [AC] } \eta \quad \Longrightarrow \quad \nu \ll \eta$$

Conditions for $\nu \ll \eta \implies \nu$ [AC] η

An improvement of the quoted classical result

Theorem (D.Mitrea-I.Mitrea-M.Mitrea-Ziade, *J. Funct. Anal.* **2012)**

$$\nu \ll \eta$$

for

- *dom*$(\nu) =$ *dom*$(\eta) =: \mathcal{A}$ σ*-algebra*
- ν: $\mathcal{A} \to [0, +\infty[$, *s.t.* $\nu(\emptyset) = 0$, σ*-additive*

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Main ingredients of 3M+Z' s proof

Capacitary estimates on 'semigroupoids' in order to prove

- "some $C \geq 1$ and $\beta \in \left]0, (\log_2 C)^{-1}\right[$ exist such that

$$\eta(\vee_{i=1}^{n} d_i) \leq C^2 \left(\sum_{i=1}^{n} \eta^{\beta}(d_i) \right)^{1/\beta} \tag{1}$$

for every disjoint $\{d_i : i = 1, \cdots, n\} \subset \mathcal{A}$"

Note that in

Theorem (3M+Z)

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for η: $\mathcal{A} \to [0, +\infty]$ finitely additive, the crucial estimate

$$\eta(\vee_{i=1}^n d_i) \ \le \ C^2 \left(\sum_{i=1}^n \eta^\beta(d_i) \right)^{1/\beta} \quad \text{for disjoint } \{d_i : i = 1, \cdots, n\} \subset \mathcal{A}$$

does hold with $C = \beta = 1$.

3M+Z's Theorem for group-valued f.a. functions

For

- $\mathcal{A}$ **σ-complete** Boolean algebra,
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the implication "$\nu \ll \eta \implies \nu$ **[AC]** η" **FAILS**

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"$\nu \ll \eta \implies \nu$ **[AC]** η" **HOLDS** when

- $\mathcal{G}'$ is **pseudometrizable** [Traynor, *Canad. Math. Bull.*, 1973]

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or

- **Each disjoint family in $\mathcal{A} \setminus \mathcal{N}(\eta)$ is countable** [Lipecki, *Colloquium Math.*, 1974]

REMARK

An analysis of Traynor's and Lipecki's arguments displays that, for group-valued functions, **the validity of** the implication

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$$\mathcal{N}(\eta) = \left\{a \in \mathcal{A} : \eta([0,a]) \subseteq \bigcap_{n\in\mathbb{N}} U_n\right\} \tag{2}$$

for some sequence $(U_n)_{n\in\mathbb{N}}$ in $\tau'[0']$.

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3. $\eta \colon \mathcal{R} \to \mathcal{G}'$ *is* **finitely additive** *&* **fulfils** (2)

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A finitely additive function $\mu : \mathcal{R} \to [0, +\infty[$, where $\mathcal{R}$ is a ring, induces a pseudometric on $\mathcal{R}$, that is

$$d_\mu : (a, b) \in \mathcal{R} \times \mathcal{R} \longmapsto \mu(a \triangle b) \in [0, +\infty[.$$

Denoted as Γ_μ the topology induced by d_μ on $\mathcal{R}$, then

i) $(\boldsymbol{\mathcal{R}}, \triangle, \boldsymbol{\Gamma_\mu})$ *is a topological group*

ii) *Functions* $a \in \boldsymbol{\mathcal{R}} \longmapsto a \wedge x \in \boldsymbol{\mathcal{R}}$ *are* $\boldsymbol{\Gamma_\mu}$*-continuous, uniformly with respect to* $x \in \boldsymbol{\mathcal{R}}$.

Every topology Γ on $\mathcal{R}$ obeying i)-ii) is called an FN-topology.

Sketch of the proof of quasi-triangular 3M+Z's Th.

1. **Each quasi-triangular function η acting on a Boolean ring $\mathcal{R}$ induces an FN-topology Γ_η on $\mathcal{R}$.**

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2. **Let ν and η be quasi-triangular functions, acting on the same Boolean ring $\mathcal{R}$.**

   If $\Gamma_\nu = \Gamma_\eta$, then ν[AC]η & η[AC]ν.

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3. **If ν is a quasi-triangular [& order-continuous] function, acting on a Boolean ring $\mathcal{R}$, then there exists a finitely [σ-] additive function ν^*: $\mathcal{R} \to \mathcal{G}$, with $\mathcal{G}$ a topological group, such that**

   ν[AC]ν^* & ν^*[AC]ν ($\nu \sim \nu^*$, for short).

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The converse holds *when* $\|\cdot\|^\star$ *is order-continuous* .