Absolute continuity of non-additive measures and applications to function spaces

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Definition

Let \( \eta : \mathcal{R} \to S \).
Quasi-triangular functions

- $\mathcal{R}$ is a Boolean ring,
- $\mathcal{S} = (\mathcal{S}, \tau)$ is a Hausdorff topological space

**Definition**

Let $\eta : \mathcal{R} \to \mathcal{S}$.

$\eta$ is quasi-triangular IFF

$$\forall U \in \tau[\eta(0)] \; \exists V = V(U) \in \tau[\eta(0)] \; \text{s.t.}$$

$$\forall a, b \in \mathcal{R}, \quad a \land b = 0 :$$

$$\eta(a), \eta(b) \in V \implies \eta(a \lor b) \in U;$$

$$\eta(a), \eta(a \lor b) \in V \implies \eta(b) \in U.$$

[Klimkin-Sribnaya, 1997]
Quasi-triangular functions

- \( \mathcal{R} \) is a Boolean ring,
- \( \mathcal{S} = (S, \tau) \) is a Hausdorff topological space

Definition

Let \( \eta : \mathcal{R} \rightarrow \mathcal{S} \).

\[ \eta \text{ is quasi-triangular } \iff \begin{cases} \forall U \in \tau[\eta(0)] \exists V = V(U) \in \tau[\eta(0)] \text{ s.t.} \\ \forall a, b \in \mathcal{R}, \ a \land b = 0 : \\ \eta(a), \eta(b) \in V \implies \eta(a \lor b) \in U; \\ \eta(a), \eta(a \lor b) \in V \implies \eta(b) \in U. \end{cases} \]

[Klimkin-Sribnaya, 1997]

Clearly

- classical measures on \( \sigma \)-algebras are quasi-triangular
The class of quasi-triangular functions also encompasses

1. **\( k \)-triangular functions**, \( k \geq 1 \), i.e. each \( \eta : \mathcal{R} \to [0, \infty] \) fulfilling \( \eta(0) = 0 \) and

\[
(T_k) \quad \eta(a) - k \eta(b) \leq \eta(a \vee b) \leq \eta(a) + k \eta(b)
\]

for all \( a, b \in \mathcal{R}, a \wedge b = 0 \).

[Gusel’nikov, H.Weber, Saeki, Pap ,.....]
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[Gusel’nikov, H. Weber, Saeki, Pap ,.....]

2. \( \oplus \)-decomposable functions, i.e. each \( \eta : \mathcal{R} \to [0, 1] \) fulfilling 
\[
\eta(0) = 0 \text{ and }
\]
\[
(wA) \quad \eta(a \vee b) = \eta(a) \oplus \eta(b) \text{ for all } a, b \in \mathcal{R}, a \wedge b = 0,
\]
where \( \oplus \) is a t-conorm on \([0,1]\) continuous at \( 0 \).

[S. Weber, Pap ,.....]

A t-conorm \( \oplus \) on \([0,1]\) is an internal composition law on \([0,1]\), commutative, associative and increasing at each place with 0 as neutral element.

T-conoms continuous at 0 are, for instance, \( \oplus_{\infty}(x, y) := \max \{x, y\} \), \( \oplus_p(x, y) := \left\{x^p + y^p\right\}^{\frac{1}{p}} \text{ for } p > 0 \) and \( \oplus_*(x, y) := x + y - xy \).
quasi-sub-measures, i.e. each \( \eta : R \to [0, \infty] \) fulfilling \( \eta(0) = 0 \) and for some \( C_1, C_2 \geq 1 \)

\[
\begin{align*}
(qM) \quad & \eta(a) \leq C_1 \eta(b) \text{ for all } a, b \in R, a \leq b, \\
(qSA) \quad & \eta(a \lor b) \leq C_2 (\eta(a) + \eta(b)) \text{ for all } a, b \in R, a \land b = 0.
\end{align*}
\]
quasi-sub-measures, i.e. each \( \eta : \mathcal{R} \rightarrow [0, \infty] \) fulfilling \( \eta(0) = 0 \) and for some \( C_1, C_2 \geq 1 \)

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**Examples:**

\( \mathcal{L} := \sigma\)-algebra of the Lebesgue measurable subsets of \( \mathbb{R}^n \)

\( \lambda := \text{Lebesgue measure on } \mathcal{L} \)
quasi-sub-measures, i.e. each \( \eta : \mathcal{R} \rightarrow [0, \infty] \) fulfilling \( \eta(0) = 0 \) and for some \( C_1, C_2 \geq 1 \)

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\]

\[
\eta : A \in \mathcal{L} \rightarrow \begin{cases} 
\lambda(A) & \text{if } \lambda(A) \leq 1, \\
\lambda(A) - \frac{1}{2} & \text{if } \lambda(A) > 1.
\end{cases}
\]
quasi-sub-measures, i.e. each $\eta : \mathcal{R} \to [0, \infty]$ fulfilling $\eta(0) = 0$ and for some $C_1, C_2 \geq 1$

(qM) $\eta(a) \leq C_1 \eta(b)$ for all $a, b \in \mathcal{R}$, $a \leq b$,

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Examples:

$\mathcal{L} := \sigma$-algebra of the Lebesgue measurable subsets of $\mathbb{R}^n$

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It fails to be monotone, sub-additive and even $k$-triangular
quasi-sub-measures, i.e. each \( \eta : \mathcal{R} \rightarrow [0, \infty] \) fulfilling \( \eta(0) = 0 \)
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It fails to be monotone, sub-additive and even \( k \)-triangular

\[\eta = \lambda^p \text{ and } \eta = (1 + \lambda)^p \left( \log(1 + \lambda) \right)^\alpha \quad \text{for all } p > 0, \alpha \geq 0\]
\( \mathcal{A} \) is a \( \sigma \)-complete Boolean algebra,

\( \nu, \eta : \mathcal{A} \to [0, +\infty] \) are \( \sigma \)-additive functions s.t. \( \nu(0) = \eta(0) = 0 \).
Absolute Continuity and Well-known classical result

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Notions of Absolute Continuity of \( \nu \) with respect to \( \eta \)

1. \( 0 \)-continuity of \( \nu \) with respect to \( \eta \)
   \[ \eta(a) = 0 \quad \text{for} \quad a \in \mathcal{A} \quad \implies \quad \nu(a) = 0; \]
Absolute Continuity and Well-known classical result

- $\mathcal{A}$ is a $\sigma$-complete Boolean algebra,
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Notions of Absolute Continuity of $\nu$ with respect to $\eta$

(I) 0-continuity of $\nu$ with respect to $\eta$

$$\eta(a) = 0 \text{ for } a \in \mathcal{A} \implies \nu(a) = 0;$$

(II) $(\varepsilon, \delta)$-continuity of $\nu$ with respect to $\eta$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \eta(a) < \delta \text{ for } a \in \mathcal{A} \implies \nu(a) < \varepsilon.$$

Convention: $\nu \ll \eta$ for (I), $\nu \ [AC] \eta$ for (II)

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**Notions of Absolute Continuity of $\nu$ with respect to $\eta$**

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**Convention:** $\nu \ll \eta$ for (I), $\nu \ [AC] \eta$ for (II)

**Theorem (Relationship between $\ll$ and $[AC]$)**

- $\nu \ [AC] \eta \implies \nu \ll \eta$.
- For finite $\nu$ : $\nu \ [AC] \eta \iff \nu \ll \eta$.

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For notions

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IT SUFFICES that

(⋆) $\nu, \eta : \mathcal{A} \to [0, +\infty]$ are non-decreasing,
and $\sigma$-completeness for $\mathcal{A}$ is unimportant.
For notions

(Ⅰ) \(\nu \ll \eta\)  
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On defining the kernel of \(\nu\) as

\[\mathcal{N}(\nu) := \{a \in A : \nu(a) = 0\},\]

clearly

(Ⅰ) \[\iff \mathcal{N}(\eta) \subseteq \mathcal{N}(\nu)\]
Finitely additive functions having values in topological groups

- $\mathcal{R}$ is a Boolean ring, $\mathcal{G} = (G, \tau)$ is a topological group
- $\nu : \mathcal{R} \to \mathcal{G}$ is finitely additive

The kernel of $\nu$ is now defined by $N(\nu) := \{a \in \mathcal{R} : \nu([0, a]) \subseteq \{0\}\}$ where $[0, a] = \{x \in \mathcal{R} : 0 \leq x \leq a\}$.

When $G$ is Hausdorff, $N(\nu) = \{a \in \mathcal{R} : \nu([0, a]) = \{0\}\}$.

For finitely additive $\eta : \mathcal{R} \to \mathcal{G}'$, where $\mathcal{G}' = (G', \tau')$ is a top. group $(I)$ def $\iff N(\eta) \subseteq N(\nu)$.
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- When $\mathcal{G}$ is Hausdorff, $\mathcal{N}(\nu) = \left\{ a \in \mathcal{R} : \nu([0, a]) = \{0\} \right\}$.

For finitely additive $\eta : \mathcal{R} \to \mathcal{G}'$, where $\mathcal{G}' = (G', \tau')$ is a top. group

$$(I) \quad \overset{\text{def}}{\iff} \quad \mathcal{N}(\eta) \subseteq \mathcal{N}(\nu)$$
Absolute Continuity in quasi-triangular setting

- $\mathcal{R}$ is a Boolean ring
- $\nu : \mathcal{R} \to \mathcal{S}$ and $\eta : \mathcal{R} \to \mathcal{S}'$ are quasi-triangular
  where $\mathcal{S} = (S, \tau)$ and $\mathcal{S}' = (S', \tau')$ are Haus. topol. spaces

The kernel of $\nu$ is defined by $N(\nu) := \{ a \in \mathcal{R} : \nu([0,a]) = \nu(0) \}$.

Notions of Absolute Continuity of $\nu$ with respect to $\eta$:

- $(I)$ $\nu \ll \eta \iff N(\eta) \subseteq N(\nu)$
- $(II)$ $\nu[AC] \eta \iff \{ \forall U \in \tau \ [\nu(0)] \exists V \in \tau' \ [\eta(0)] \ s.t. \ \eta([0,a]) \subseteq V = \Rightarrow \nu([0,a]) \subseteq U$
**Absolute Continuity in quasi-triangular setting**

1. $\mathcal{R}$ is a Boolean ring
2. $\nu: \mathcal{R} \to S$ and $\eta: \mathcal{R} \to S'$ are quasi-triangular

   where $S = (S, \tau)$ and $S' = (S', \tau')$ are Haus. topol. spaces

The kernel of $\nu$ is defined by $\mathcal{N}(\nu) := \left\{ a \in \mathcal{R} : \nu([0, a]) = \{\nu(0)\} \right\}$. 
Absolute Continuity in quasi-triangular setting

- \( R \) is a Boolean ring
- \( \nu : R \to S \) and \( \eta : R \to S' \) are quasi-triangular
  where \( S = (S, \tau) \) and \( S' = (S', \tau') \) are Haus. topol. spaces

The kernel of \( \nu \) is defined by \( \mathcal{N}(\nu) := \{ a \in R : \nu([0, a]) = \{ \nu(0) \} \} \).

Notions of Absolute Continuity of \( \nu \) with respect to \( \eta \)

(1) \( \nu \ll \eta \) if and only if \( \mathcal{N}(\eta) \subseteq \mathcal{N}(\nu) \)

(II) \( \nu \) \( [AC] \) \( \eta \) if and only if

\[
\forall U \in \tau[\nu(0)] \exists V \in \tau'[\eta(0)] \text{ s.t. } \\
\eta([0, a]) \subseteq V \implies \nu([0, a]) \subseteq U
\]
Absolute Continuity in quasi-triangular setting

- $\mathcal{R}$ is a Boolean ring
- $\nu : \mathcal{R} \to S$ and $\eta : \mathcal{R} \to S'$ are quasi-triangular
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Notions of Absolute Continuity of $\nu$ with respect to $\eta$

(I) $\nu \ll \eta$ \quad def \quad $\mathcal{N}(\eta) \subseteq \mathcal{N}(\nu)$

(II) $\nu$ [AC] $\eta$ \quad def \quad $\forall U \in \tau[\nu(0)] \exists V \in \tau'[\eta(0)]$ \quad s.t. \quad $\eta([0, a]) \subseteq V \implies \nu([0, a]) \subseteq U$

Clearly

$\nu$ [AC] $\eta$ \implies $\nu \ll \eta$
Conditions for $\nu \ll \eta \implies \nu \text{ [AC]} \eta$

An improvement of the quoted classical result


\[ \nu \ll \eta \]

for

- $\text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{A}$, $\sigma$-algebra
- $\nu: \mathcal{A} \rightarrow [0, +\infty]$, s.t. $\nu(0) = 0$, $\sigma$-additive
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- $\nu: \mathcal{A} \to [0, +\infty]$, s.t. $\nu(0) = 0$, σ-additive
- $\eta: \mathcal{A} \to [0, +\infty]$, s.t. $\eta(0) = 0$, quasi-monotone & quasi-subadditive
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$\nu \ll \eta \implies \nu \text{ [AC]} \eta$ ,

for

- $\text{dom}(\nu) = \text{dom}(\eta) = A \quad \sigma$-algebra
- $\nu : A \rightarrow [0, +\infty], \text{ s.t. } \nu(0) = 0, \sigma$-additive
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- $\eta: \mathcal{A} \to [0, +\infty]$, s.t. $\eta(0) = 0$, quasi-monotone & quasi-subadditive

**Main ingredients of 3M+Z’ s proof**

Capacitary estimates on ‘semigroupoids’ in order to prove

- “some $C \geq 1$ and $\beta \in ]0, (\log_2 C)^{-1}[$ exist such that
  \[
  \eta(\bigvee_{i=1}^{n} d_i) \leq C^2 \left( \sum_{i=1}^{n} \eta^\beta(d_i) \right)^{1/\beta} 
  \]
  for every disjoint $\{d_i : i = 1, \ldots, n\} \subset \mathcal{A}$"
Note that in

\textbf{Theorem (3M+Z)}

\[ \nu \ll \eta \quad \Longleftrightarrow \quad \nu \text{ [AC]} \eta, \]

for

- \( \text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{A} \), \( \sigma \)-algebra
- \( \nu : \mathcal{A} \to [0, +\infty[, \text{ s.t. } \nu(0) = 0, \sigma\text{-additive} \)
- \( \eta : \mathcal{A} \to [0, +\infty], \text{ s.t. } \eta(0) = 0, \text{ quasi-monotone \& quasi-subadditive} \)

for \( \eta : \mathcal{A} \to [0, +\infty] \) \text{ finitely additive}, the crucial estimate

\[ \eta(\bigvee_{i=1}^{n} d_i) \leq C^2 \left( \sum_{i=1}^{n} \eta^\beta(d_i) \right)^{1/\beta} \quad \text{for disjoint } \{ d_i : i = 1, \ldots, n \} \subset \mathcal{A} \]

does hold with \( C = \beta = 1. \)
3M+Z’s Theorem for group-valued f.a. functions

For

- $\mathcal{A}$ $\sigma$-complete Boolean algebra,
- $\nu : \mathcal{A} \to \mathcal{G}$ $\sigma$-additive, with $\mathcal{G} = (G, \tau)$ a topological group
- $\eta : \mathcal{A} \to \mathcal{G}'$ $\sigma$-additive, with $\mathcal{G}' = (G', \tau')$ a topological group

the implication $\nu \ll \eta \iff \nu \text{ [AC]} \eta$ fails unless additional assumptions on $\eta$ are made.
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Known results

“$\nu \ll \eta \implies \nu \text{ [AC]} \eta$” HOLDS when

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the implication "$\nu \ll \eta \implies \nu [AC] \eta$" \textbf{FAILS} unless additional assumptions on $\eta$ are made.

**Known results**

"$\nu \ll \eta \implies \nu [AC] \eta$" \textbf{HOLDS} when


or

- Each disjoint family in $\mathcal{A} \setminus \mathcal{N}(\eta)$ is countable
  [Lipecki, \textit{Colloquium Math.}, 1974]
REMARK

An analysis of Traynor’s and Lipecki’s arguments displays that, for group-valued functions, **the validity of** the implication

\[ \nu \ll \eta \implies \nu [AC] \eta \]

requires a ‘good’ behaviour of \( N(\eta) \),
REMARK

An analysis of Traynor’s and Lipecki’s arguments displays that, for group-valued functions, the validity of the implication

\[ \nu \ll \eta \implies \nu \text{ [AC]} \eta \]

requires a ‘good’ behaviour of \( \mathcal{N}(\eta) \), that is

\[ \mathcal{N}(\eta) = \left\{ a \in A : \eta([0, a]) \subseteq \bigcap_{n \in \mathbb{N}} U_n \right\} \quad (2) \]

for some sequence \( (U_n)_{n \in \mathbb{N}} \) in \( \tau' [0'] \).
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An analysis of Traynor’s and Lipecki’s arguments displays that, for group-valued functions, the validity of the implication

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Theorem (C.- de Lucia - De Simone, Funct. Approx., 2014)

\[
\nu \ll \eta \implies \nu \text{ [AC]} \eta
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when

1. \( \text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{R} \) is a \( \sigma \)-ring
2. \( \nu : \mathcal{R} \to \mathcal{G} \) is \( \sigma \)-additive
REMARK

An analysis of Traynor’s and Lipecki’s arguments displays that, for group-valued functions, the validity of the implication
\[ \nu \preccurlyeq \eta \implies \nu \text{ [AC]} \eta \]
requires a ‘good’ behaviour of \( \mathcal{N}(\eta) \), that is
\[ \mathcal{N}(\eta) = \left\{ a \in A : \eta([0, a]) \subseteq \bigcap_{n \in \mathbb{N}} U_n \right\} \]  
for some sequence \((U_n)_{n \in \mathbb{N}}\) in \( \tau'[0'] \).

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when

1. \( \text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{R} \) is a \( \sigma \)-ring
2. \( \nu : \mathcal{R} \to \mathcal{G} \) is \( \sigma \)-additive
3. \( \eta : \mathcal{R} \to \mathcal{G}' \) is finitely additive \& fulfils (2)
3M+Z’s Theorem for quasi-triangular functions

**Theorem (C.- de Lucia - De Simone, *Funct. Approx.*, 2014)**

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Recall that \( \nu \) is order-continuous if

\[ \forall (b_k)_{k \in \mathbb{N}} \subset \mathbb{R} \text{ decreasing to } 0: \]

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The main idea of the proof is........................
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Recall that

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**3M+Z’s Theorem for quasi-triangular functions**

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\[ \nu \ll \eta \implies \nu \text{ [AC]} \eta, \]

when

1. \( \text{dom}(\nu) = \text{dom}(\eta) =: R \) is a \( \sigma \)-ring
2. \( \nu: R \to S \) is quasi-triangular & order-continuous
3. \( \eta: R \to S' \) is quasi-triangular & fulfills (2)

Recall that

\[ \nu \text{ is order-continuous} \iff \forall (b_k)_{k \in \mathbb{N}} \subseteq R \text{ decreasing to } 0: \lim_k \nu(b_k) = \nu(0) \]

**The main idea of the proof is..................**
.... to exhibit a link between quasi-triangular functions and group-valued finitely additive functions

In Financial Mathematics this is called "additivization of non-additive measures" (see, e.g., [Gilboa, 1989])
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.... Fréchet-Nikodým topologies

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A finitely additive function $\mu : \mathcal{R} \to [0, +\infty]$, where $\mathcal{R}$ is a ring, induces a pseudometric on $\mathcal{R}$, that is

$$d_\mu : (a, b) \in \mathcal{R} \times \mathcal{R} \mapsto \mu(a \triangle b) \in [0, +\infty].$$

Denoted as $\Gamma_\mu$ the topology induced by $d_\mu$ on $\mathcal{R}$, then

i) $(\mathcal{R}, \triangle, \Gamma_\mu)$ is a topological group

ii) Functions $a \in \mathcal{R} \mapsto a \land x \in \mathcal{R}$ are $\Gamma_\mu$-continuous, uniformly with respect to $x \in \mathcal{R}$.

Every topology $\Gamma$ on $\mathcal{R}$ obeying i)-ii) is called an FN-topology.
Sketch of the proof of quasi-triangular 3M+Z’s Th.

1. Each quasi-triangular function $\eta$ acting on a Boolean ring $\mathcal{R}$ induces an FN-topology $\Gamma_\eta$ on $\mathcal{R}$.

Sketch of the proof of quasi-triangular 3M+Z’s Th.

1. Each quasi-triangular function \( \eta \) acting on a Boolean ring \( \mathcal{R} \) induces an FN-topology \( \Gamma_\eta \) on \( \mathcal{R} \).
   

2. Let \( \nu \) and \( \eta \) be quasi-triangular functions, acting on the same Boolean ring \( \mathcal{R} \).
   
   If \( \Gamma_\nu = \Gamma_\eta \), then \( \nu[AC] \eta \) & \( \eta[AC] \nu \).
   
Sketch of the proof of quasi-triangular 3M+Z’s Th.

1. Each quasi-triangular function $\eta$ acting on a Boolean ring $\mathcal{R}$ induces an FN-topology $\Gamma_\eta$ on $\mathcal{R}$.
   
   \[\text{[C. - de Lucia, Commun. Appl. Anal., 2009]}\]

2. Let $\nu$ and $\eta$ be quasi-triangular functions, acting on the same Boolean ring $\mathcal{R}$.
   
   If $\Gamma_\nu = \Gamma_\eta$, then $\nu[\text{AC}]\eta \& \eta[\text{AC}]\nu$.
   
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3. If $\nu$ is a quasi-triangular [\& order-continuous] function, acting on a Boolean ring $\mathcal{R}$, then there exists a finitely $[\sigma]$-additive function $\nu^*$: $\mathcal{R} \to \mathcal{G}$, with $\mathcal{G}$ a topological group, such that $\nu[\text{AC}]\nu^* \& \nu^*[\text{AC}]\nu$ (\(\nu \sim \nu^*, \text{ for short}\)).
   
   \[\text{[C. - de Lucia, Atti Accad. Naz. Lincei, 2009]}\]
1. $\text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{R}$ is a $\sigma$-ring

2. $\nu: \mathcal{R} \to \mathcal{S}$ is quasi-triangular & order-continuous

3. $\eta: \mathcal{R} \to \mathcal{S}'$ is quasi-triangular & fulfils (2)

4. $\nu \ll \eta$
1. \( \text{dom}(\nu) = \text{dom}(\eta) =: \mathcal{R} \) is a \( \sigma \)-ring
2. \( \nu: \mathcal{R} \to \mathcal{S} \) is quasi-triangular & order-continuous
3. \( \eta: \mathcal{R} \to \mathcal{S}' \) is quasi-triangular & fulfills (2)
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\[ \Downarrow \] [C. - de Lucia, 2009]

1. \( \text{dom}(\nu^*) = \text{dom}(\eta^*) =: \mathcal{R} \) is a \( \sigma \)-ring
2. \( \nu^*: \mathcal{R} \to \mathcal{G} \) is \( \sigma \)-additive
3. \( \eta^*: \mathcal{R} \to \mathcal{G}' \) is finitely additive & fulfills (2)
4. \( \nu^* \ll \eta^* \)
dom(\nu) = \text{dom}(\eta) =: \mathcal{R} \text{ is a } \sigma\text{-ring}
\nu: \mathcal{R} \to \mathcal{S} \text{ is quasi-triangular & order-continuous}
\eta: \mathcal{R} \to \mathcal{S}' \text{ is quasi-triangular \& fulfils (2)}
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\[ \text{[our 3M+Z’s Theorem]} \]

\[ \nu^* \succeq [\text{AC}] \eta^* \]

\[ \nu \succeq \nu^* \, \& \, \eta^* \succeq \eta \]

\[ \nu \succeq [\text{AC}] \eta. \]
Let $(E, \Sigma, \mu)$ be a measure space, and $L^0(E) := \text{Riesz space of all } (\mu\text{-equiv. classes of}) \text{ measurable } f : E \to \mathbb{R}$. 
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**Theorem (C.- de Lucia - De Simone, *in preparation*)**

\(X(E)\) solid Riesz subspace of \(L^0(E),\)
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- \(X(E)\) solid Riesz subspace of \(L^0(E)\), with \((E, \Sigma, \mu)\) \(\sigma\)-finite,
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- \(\| \cdot \|^{*} : X(E) \to [0, \infty[\) quasi triangular
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- \(\| \cdot \|^* : X(E) \to [0, \infty[\) quasi triangular obeying the following

\[\eta \quad \text{there exists a function } \omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0, \text{ s.t.} \quad \|\alpha f\|^* \leq \omega(\alpha) \|f\|^* \quad \text{for all } \alpha \in [0, \infty[, \ f \in X(E),\]

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*\(X(E)\) solid Riesz subspace of \(L^0(E)\), with \((E, \Sigma, \mu)\) \(\sigma\)-finite,*

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\[\exists \text{ there exists a function } \omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0, \text{ s.t.}\]

\[|\alpha f|^{\star} \leq \omega(\alpha) |f|^{\star} \quad \text{for all } \alpha \in [0, \infty[, f \in X(E),\]

\[|f|^{\star} > 0 \quad \text{if } f \text{ does not vanishes a.e.;}\]
Let \((E, \Sigma, \mu)\) be a measure space, and 
\(L^0(E) := \text{Riesz space of all } (\mu\text{-equiv. classes of}) \text{ measurable } f : E \to \mathbb{R}.\)

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- \(\| \cdot \|^* : X(E) \to [0, \infty]\) quasi triangular obeying the following:
  
  - \(\exists \) there exists a function \(\omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0\), s.t. 
    \[\| \alpha f \|^* \leq \omega(\alpha) \| f \|^* \quad \text{for all } \alpha \in [0, \infty[, \ f \in X(E),\]
  
  - \(\| f \|^* > 0\) if \(f\) does not vanishes a.e.;
  
  - for any \(\lambda > 0\) there is \(r_\lambda > 0\) s.t. if \(f, g \in X(E), \ |g| \leq_{a.e.} |f|\)
    \[\| f \|^* < r_\lambda \implies \| g \|^* < \lambda\]

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Let \((E, \Sigma, \mu)\) be a measure space, and 
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- \(\| \cdot \|^{\ast} : X(E) \to [0, \infty[\) *quasi triangular* obeying the following:
  - there exists a function \(\omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0\), s.t.
    \[ \| \alpha f \|^{\ast} \leq \omega(\alpha) \| f \|^{\ast} \quad \text{for all } \alpha \in [0, \infty[, \ f \in X(E), \]
  - \(\| f \|^{\ast} > 0\) if \(f\) does not vanishes a.e.;
  - for any \(\lambda > 0\) there is \(r_{\lambda} > 0\) s.t. if \(f, g \in X(E), \ |g| \leq_{a.e.} |f|\)
    \[ \| f \|^{\ast} < r_{\lambda} \implies \| g \|^{\ast} < \lambda \]

Then 
\((X(E), \tau_{\| \cdot \|^{\ast}})\)
Let \((E, \Sigma, \mu)\) be a measure space, and 
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**Theorem (C.- de Lucia - De Simone, in preparation)**

\(\nabla \quad X(E) \text{ solid Riesz subspace of } L^0(E), \text{with } (E, \Sigma, \mu) \text{ } \sigma\text{-finite}, \quad \text{s.t. } \chi_{K_j} \in X(E) \text{ for all } j, \text{ where } \mu(K_j) < \infty \text{ and } K_j \nearrow E\)

\(\nabla \quad \| \cdot \|^{*} : X(E) \to [0, \infty[ \text{ quasi triangular obeying the following}\)

\(\leadsto \text{there exists a function } \omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0, \text{ s.t.}, \quad \| \alpha f \|^{*} \leq \omega(\alpha) \| f \|^{*} \text{ for all } \alpha \in [0, \infty[, \ f \in X(E),\)

\(\leadsto \| f \|^{*} > 0 \text{ if } f \text{ does not vanishes a.e.;}\)

\(\leadsto \text{for any } \lambda > 0 \text{ there is } r_\lambda > 0 \text{ s.t. if } f, g \in X(E), \ |g| \leq_{a.e.} |f|, \quad \| f \|^{*} < r_\lambda \implies \| g \|^{*} < \lambda\)

Then

\((X(E), \tau_{\| \cdot \|^{*}}) \hookrightarrow (L^0(E), \tau_\mu)\)
Corollary (C.- de Lucia - De Simone)

- **X(E)** solid Riesz subspace of \( L^0(E) \), with \( (E, \Sigma, \mu) \) \( \sigma \)-finite,
  s.t. \( \chi_{K_j} \in X(E) \) for all \( j \), where \( \mu(K_j) < \infty \) and \( K_j \uparrow E \)

- \( \| \cdot \|^* : X(E) \to [0, \infty[ \) quasi triangular obeying the following

  \( \bowtie \) there exists a function \( \omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0 \), s.t.
  \[ \| \alpha f \|^* \leq \omega(\alpha) \| f \|^* \quad \text{for all } \alpha \in [0, \infty[, f \in X(E), \]

  \( \bowtie \) \( \| f \|^* > 0 \) if \( f \) does not vanishes a.e.;

  \( \bowtie \) for any \( \lambda > 0 \) there is \( r_\lambda > 0 \) s.t. if \( f, g \in X(E), |g| \leq \text{a.e. } |f| \)
  \[ \| f \|^* < r_\lambda \quad \Rightarrow \quad \| g \|^* < \lambda \]

If \( (f_n)_{n \in \mathbb{N}} \subset X(E) \) converges to an \( f \in X(E) \), then every \( (f_{nk})_{k \in \mathbb{N}} \)
admits a subsequence converging to \( f \) \( \mu \)-a.e. in \( E \).
Corollary (C.- de Lucia - De Simone)

- $X(E)$ solid Riesz subspace of $L^0(E)$, with $(E, \Sigma, \mu)$ $\sigma$-finite, s.t. $\chi_{K_j} \in X(E)$ for all $j$, where $\mu(K_j) < \infty$ and $K_j \nearrow E$

- $\| \cdot \|^* : X(E) \to [0, \infty[$ quasi triangular obeying the following:
  - There exists a function $\omega : [0, +\infty[ \to [0, +\infty[, \omega(0) = 0$, s.t. $\| \alpha f \|^* \leq \omega(\alpha) \| f \|^*$ for all $\alpha \in [0, \infty[, f \in X(E)$,
  - $\| f \|^* > 0$ if $f$ does not vanishes a.e.;
  - For any $\lambda > 0$ there is $r_\lambda > 0$ s.t. if $f, g \in X(E), |g| \leq_{a.e.} |f|$ then $\| f \|^* < r_\lambda \implies \| g \|^* < \lambda$

If $(f_n)_{n \in \mathbb{N}} \subset X(E)$ converges to an $f \in X(E)$, then every $(f_{n_k})_{k \in \mathbb{N}}$ admits a subsequence converging to $f$ $\mu$-a.e. in $E$.

The converse holds when $\| \cdot \|^*$ is order-continuous.