Non-absolute gage integrals for multifunctions with values in an arbitrary Banach space

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Integration, Vector Measures and Related Topics VI

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Joint results with Kazimierz Musial

The papers

- L. Di Piazza and K. Musial, *Relations among Henstock, McShane and Pettis integrals for multifunctions with compact convex values*, Monatshefte fr Mathematik, Vol. 173, Issue 4 (2014), pp. 459-470.

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- L. Di Piazza and K. Musial, *Relations among Henstock, McShane and Pettis integrals for multifunctions with compact convex values*, Monatshefte fr Mathematik, Vol. 173, Issue 4 (2014), pp. 459-470.
- L. Di Piazza and K. Musial, *Henstock-Kurzweil-Pettis integrability of compact valued multifunctions with values in an arbitrary Banach space*, Jour. Math. Anal. Applic. Vol. 408 (2013), pp. 452-464, ISSN: 0022-247X

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- A **gauge** on $[0, 1]$ is a positive function on $[0, 1]$
- Given a gauge δ, a partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is said to be **δ-fine** if $I_j \subset (t_j - \delta(t_j), t_j + \delta(t_j))$, $j = 1, \ldots, p$.

Definition

A function $h : [0,1] \to \mathbb{R}$ is said to be **Henstock-Kurzweil-integrable**, or simply **HK-integrable**, on $[0,1]$ if there exists $a \in \mathbb{R}$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\left| \sum_{j=1}^{p} h(t_j)|I_j| - a \right| < \varepsilon \,, \tag{1}$$

for each δ–fine **Perron partition** $\{(I_j, t_j) : \; j = 1, .., p\}$ of $[0,1]$.
We set $(HK)\int_0^1 h dt := a$.

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- We consider on $cb(X)$ the **Minkowski addition** ($A \bigoplus B := \overline{\{a + b : a \in A,\ b \in B\}}$) and the standard multiplication by scalars
- d_H is the Hausdorff metric on $cb(X)$

- For each $C \in cb(X)$ the **support function of C** is denoted by $s(\cdot, C)$ and defined on X^* by

$$s(x^*, C) = \sup\{< x^*, x >: \; x \in C\},$$

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- A **multifunction** is map $\Gamma\colon [0,1] \to cb(X)$
- A function $f\colon [0,1] \to X$ is called a **selection of** Γ if $f(t) \in \Gamma(t)$, for every $t \in [0,1]$.

Multifunctions

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Definition

A scalarly HK-integrable multifunction $\Gamma \colon [0,1] \to cb(X)$ is said to be **Henstock-Kurzweil-Pettis integrable** (or simply **HKP-integrable**) in $cb(X)$, $[ck(X)$, $cwk(X)]$ if for each $I \in \mathcal{I}$ there exists a set $\Phi_{\Gamma}(I) \in cb(X)$ $[ck(X)$, $cwk(X)$, respectively$]$ such that

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$$\mathbf{s(x^*, \Phi_\Gamma(I)) = (HK)\int_I s(x^*, \Gamma(t))\, dt \quad for\ every\ x^* \in X^*.}$$

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We write $(HKP)\int_I \Gamma(t)\, dt := \Phi_\Gamma(I)$ and call $\Phi_\Gamma(I)$ the **Henstock-Kurzweil-Pettis integral of Γ** over I.

Definition

A scalarly integrable multifunction $\Gamma\colon [0,1] \to cb(X)$ is said to be **Pettis integrable** (or simply **P-integrable**) in $cb(X)$, $[ck(X)$, $cwk(X)]$ if for each $E \in \mathcal{L}$ there exists a set $\Phi_\Gamma(E) \in cb(X)$ $[ck(X)$, $cwk(X)$, respectively$]$ such that

$$\mathbf{s(x^*, \Phi_\Gamma(E)) = \int_E s(x^*, \Gamma(t))\, dt \quad \text{for every } x^* \in X^*.} \tag{2}$$

We write $(P)\int_E \Gamma(t)\,dt := \Phi_\Gamma(E)$ and call $\Phi_\Gamma(E)$ the **Pettis integral of Γ** over E.

Proposition

(*i*) Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly HK-integrable multifunction. Then Γ is HKP-integrable in $cwk(X)$ **if and only if** for each $I \in \mathcal{I}$ the mapping

$$\mathbf{x}^* \longrightarrow \mathbf{(HK)} \int_{\mathbf{I}} \mathbf{s}(\mathbf{x}^*, \Gamma(\mathbf{t}))\, \mathbf{dt}$$

is $\tau(X^*, X)$-continuous (where $\tau(X^*, X)$ is the Mackey topology on X^*).

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(*ii*) Let $\Gamma : [0,1] \to ck(X)$ be a scalarly HK-integrable multifunction. Then Γ is HKP-integrable in $ck(X)$ **if and only if** for each $I \in \mathcal{I}$ the mapping

$$\mathbf{x}^* \longrightarrow (\mathbf{HK})\int_{\mathbf{I}} \mathbf{s}(\mathbf{x}^*, \Gamma(\mathbf{t}))\, \mathbf{dt}$$

is $\tau_c(X^*, X)$-continuous (where $\tau_c(X^*, X)$ is the topology on X^* of uniform convergence on elements of $ck(X)$).

Selections of HKP-integrable multifunctions

Proposition

Let $\Gamma : [0,1] \to cwk(X)$ be a multifunction HKP-integrable in $cwk(X)$. Then there exists an HKP-integrable selection f of Γ. Moreover each scalarly measurable selection f of Γ is HKP-integrable.

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Sketch of the proof.

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Then, for each $x^* \in X^*$ we have

$$-s(-x^*, \Gamma(t)) \leq x^* f(t) \leq s(x^*, \Gamma(t)) .$$

$$0 \leq x^* f(t) + s(-x^*, \Gamma(t)) \leq s(x^*, \Gamma(t)) + s(-x^*, \Gamma(t)) .$$

and the HK-integrability of the function $x^* f$ follows.

Moreover for each $I \in \mathcal{I}$

$$-(HK)\int_I s(-x^*, \Gamma(t))\,dt \le (HK)\int_I x^* f(t)\,dt \le (HK)\int_I s(x^*, \Gamma(t))\,dt\,.$$

So by previous characterization f is HKP-integrable. □

By the symbol $\mathcal{S}_{\mathbf{HKP}}(\Gamma)$ we denote the family of all selections of Γ that are HKP-integrable.

HKP-integrable selections

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If $\varGamma : [0,1] \to ck(X)$ is scalarly HK-integrable, then TFAE:

1. $\varGamma$ is HKP-integrable in $ck(X)$ and $\mathbf{\Phi}_{\varGamma}(\mathcal{I}) := \bigcup_{\mathbf{I}\in\mathcal{I}} \mathbf{\Phi}_{\varGamma}(\mathbf{I})$ is relatively compact

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3. Each scalarly measurable selection of Γ is *HKP*-integrable and has continuous primitive.

Theorem

A scalarly HK-integrable multifunction $\Gamma : [0,1] \to ck(X)[cwk(X)]$ is **HKP-integrable** in $ck(X)[cwk(X)]$ **if and only if** $\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_{HKP}(\Gamma)$ the multifunction **G** : $[0,1] \to ck(X)[cwk(X)]$ defined by

$$\Gamma(t) = G(t) + f(t)$$

is **Pettis integrable** in $ck(X)[cwk(X)]$.

We know that for Pettis integrable functions the space c_0 is that space which makes problems: if $c_0 \subset X$ isomorphically, then there are X-valued scalarly integrable functions that are not Pettis integrable.

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Let us recall that X is called **weakly sequentially complete** if each weakly Cauchy sequence in X is weakly convergent. It is known that no weakly sequentially complete Banach space can contain an isomorphic copy of c_0.

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(Gordon 1989): **A separable Banach space X is weakly sequentially complete if and only if each X-valued scalarly HK-integrable function $f : [0, 1] \to X$ is HKP integrable.**

Integration in weakly sequentially complete Banach spaces

We recall that a space Y **determines a function** $f : [0,1] \to X$ (resp. **a multifunction** $\Gamma : [0,1] \to cb(X)$) if $x^*f = 0$ (resp. $s(x^*, \Gamma) = 0$) a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on x^*).

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For an arbitrary Banach space X TFAE:

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2. Each scalarly HK-integrable function $f : [0,1] \to X$ that is determined by a weakly compactly generated (WCG) space is HKP-integrable

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3. Each scalarly HK-integrable multifunction $\Gamma : [0,1] \to cwk(X)[ck(X)]$ that is determined by a WCG space, is HKP-integrable in $cwk(X)$.

We recall that a Banach space X has the **Schur property** is each sequence weakly convergent to 0 is also norm convergent. It well known that each space with the Schur property is weakly sequentially complete.

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Proof. (1) $\Rightarrow$ (2) According to previous theorem, if $\Gamma : [0,1] \to ck(X)$ is scalarly HK-integrable and determined by a WCG space, then it is HKP-integrable in $cwk(X)$. The Schur property of X forces the integrability in $ck(X)$. □

Definition

A multifunction $\Gamma : [0,1] \to cb(X)$ is said to be **Henstock** (resp. **McShane**) **integrable**, if there exists a set $\Phi_\Gamma[0,1] \in cb(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that for each δ–fine Perron partition (resp. partition) $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0,1]$, we have

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$$d_H\left(\Phi_\Gamma[0,1], \bigoplus_{i=1}^{p} \Gamma(t_i)|I_i| \right) < \varepsilon \, .$$

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We write then $(H) \int_0^1 \Gamma(t)\,dt := \Phi_\Gamma[0,1]$ (resp. $(MS) \int_0^1 \Gamma(t)\,dt := \Phi_\Gamma[0,1]$).

Remarks:

- From the definition and the completeness of the Hausdorff metric in $cwk(X)[ck(X)]$, it is easy to see that if a $cwk(X)[ck(X)]$-valued multifunction is Henstock integrable, than also $\Phi_{\Gamma}[0,1] \in cwk(X)[ck(X)]$.

Remarks:

- From the definition and the completeness of the Hausdorff metric in $cwk(X)[ck(X)]$, it is easy to see that if a $cwk(X)[ck(X)]$-valued multifunction is Henstock integrable, than also $\Phi_{\Gamma}[0,1] \in cwk(X)[ck(X)]$.
- Each McShane integrable multifunction, is also Henstock integrable (with the same values of the integrals)

- According to Hörmander's equality

$$\mathbf{d_H}\left(\mathbf{K}, \bigoplus_{\mathbf{i}=1}^{\mathbf{p}} \Gamma(\mathbf{t_i})|\mathbf{I_i}|\right) = \sup_{\|\mathbf{x}^*\|\leq 1} \left|\mathbf{s}(\mathbf{x}^*, \mathbf{K}) - \sum_{\mathbf{i}=1}^{\mathbf{p}} \mathbf{s}(\mathbf{x}^*, \Gamma(\mathbf{t_i}))|\mathbf{I_i}|\right|.$$

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Let us consider the embedding $j : cb(X) \to l_\infty(B(X^*))$ defined by

$$\mathbf{j}(\mathbf{K})(\mathbf{x}^*) = \mathbf{s}(\mathbf{x}^*, \mathbf{K}).$$

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  The images $j(cb(X))$, $j(ck(X))$ and $j(cwk(X))$ are closed cones of $l_\infty(B(X^*))$. So, if $z \in l_\infty(B(X^*))$ is the value of the Henstock integral of $j \circ \Gamma$, then there exists a set $K \in cb(X)\,[ck(X),\, cwk(X)]$ with $j(K) = z$.

- According to Hörmander's equality

$$\mathbf{d_H}\left(\mathbf{K}, \bigoplus_{\mathbf{i=1}}^{\mathbf{p}} \Gamma(\mathbf{t_i})|\mathbf{I_i}|\right) = \sup_{\|\mathbf{x}^*\|\leq 1} \left|\mathbf{s}(\mathbf{x}^*, \mathbf{K}) - \sum_{\mathbf{i=1}}^{\mathbf{p}} \mathbf{s}(\mathbf{x}^*, \Gamma(\mathbf{t_i}))|\mathbf{I_i}|\right|.$$

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Therefore: **a multifunction $\Gamma : [0,1] \to cb(X)$ is Henstock (or McShane) integrable if and only if the single valued function $j \circ \Gamma : [0,1] \to l_\infty(B(X^*))$ is Henstock (or McShane) integrable in the usual sense**.

- If $\Gamma : [0,1] \to cb(X)[cwk(X), ck(X)]$ is **Henstock integrable**, then it is also **Henstock-Kurzweil-Pettis integrable** in $cb(X)[cwk(X), ck(X)]$.

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- If $\Gamma : [0,1] \to cb(X)[cwk(X), ck(X)]$ is **McShane integrable**, then it is also **Pettis integrable** in $cb(X)[cwk(X), ck(X)]$.

Equi-integrability.

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Definition

We recall that a family of real valued HK-integrable (or McShane integrable) functions $\{g_\alpha : \alpha \in \mathbb{A}\}$ is **Henstock** (resp. **McShane**) **equi-integrable** on $[0,1]$ whenever for every $\varepsilon > 0$ there is a gauge δ such that

$$\sup \left\{ \left| \sum_{j=1}^{p} g_\alpha(t_j)|I_j| - (HK)\int_0^1 g_\alpha \, dt \right| : \alpha \in \mathbb{A} \right\} < \varepsilon \,,$$

for each δ–fine Perron partition (resp. partition) $\{(I_j, t_j) : \; j = 1, .., p\}$ of $[0,1]$.

Given a multifunction $\Gamma : [0,1] \to cb(X)$ we set

$$\mathcal{Z}_\Gamma := \{\mathbf{s}(\mathbf{x}^*, \Gamma(\cdot)) : \|\mathbf{x}^*\| \leq \mathbf{1}\},$$

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$$\mathcal{Z}_{\mathit{\Gamma}} := \{\mathbf{s}(\mathbf{x}^*, \mathit{\Gamma}(\cdot)) : \|\mathbf{x}^*\| \leq \mathbf{1}\},$$

Proposition

A scalarly HK–integrable (resp. integrable) multifunction $\mathit{\Gamma} : [0,1] \to cb(X)$, is Henstock (resp. McShane) integrable **iff** the family $\mathcal{Z}_{\mathit{\Gamma}}$ is Henstock (resp. McShane) equi-integrable.

Selections of Henstock or McShane integrable multifunctions.

By $\mathcal{S}_{\mathbf{H}}(\Gamma)$ $[\mathcal{S}_{\mathbf{MS}}(\Gamma)\,,\ \mathcal{S}_{\mathbf{P}}(\Gamma)]$ we denote the family of all scalarly measurable selections of Γ that are **Henstock [McShane, Pettis] integrable**.

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Theorem

If $\Gamma : [0,1] \to cwk(X)$ is Henstock integrable, then $\mathcal{S}_H(\Gamma) \neq \emptyset$.

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Theorem

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Sketch of the proof. In the first part we proceed in a way similar to that of Cascales-Kadets-Rodriguez (2009) for Pettis integrable multifunctions.

Continuation of the proof

- Let $\Gamma : [0,1] \to cwk(X)$ be Henstock integrable. Since $H := (H)\int_0^1 \Gamma(t)\,dt \in cwk(X)$, there exists a **strongly exposed point** $x_0 \in H$. Assume that $x_0^* \in B(X^*)$ is such that $x_0^*(x_0) > x_0^*(x)$ for every $x \in H \setminus \{x_0\}$ and the sets $\{x \in H : x_0^*(x) > x_0^*(x_0) - \alpha\}$, $\alpha \in \mathbb{R}$, form a neighborhood basis of x_0 in the norm topology on H.

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- We define $G : [0,1] \to cwk(X)$ by

$$G(t) := \{x \in \Gamma(t) : x_0^*(x) = s(x_0^*, \Gamma(t))\}.$$

  Since Γ is Henstock integrable, then Γ is also HKP-integrable in $cwk(X)$ and also G is HKP-integrable in $cwk(X)$. Let $g : [0,1] \to X$ be any selection of G. Then g is scalarly measurable (and of course HKP-integrable). Moreover $x_0^*(x_0) = (HK)\int_0^1 x_0^* g(t)\,dt$.

Continuation of the proof

- Let $\varepsilon > 0$ and $0 < \varepsilon' < \varepsilon/2$ be arbitrary. Then, let $0 < \eta < \varepsilon'$ be such that

$$\forall\, x \in H \quad [|x_0^*(x) - x_0^*(x_0)| < \eta \Rightarrow \|x - x_0\| < \varepsilon']. \qquad (3)$$

Since Γ is Henstock integrable and $x_0^* g$ is HK-integrable we can find a gauge δ on $[0,1]$ such that for each δ-fine Perron partition $\mathcal{P} := \{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0,1]$

$$d_H\left(H, \bigoplus_{i=1}^{p} \Gamma(t_i)|I_i|\right) < \eta/2$$

and

$$\left| \int_0^1 x_0^* g(t)\, dt - \sum_{i=1}^{p} x_0^* g(t_i)|I_i| \right| < \eta/2.$$

So there exists a point $x_{\mathcal{P}} \in H$ with

$$\left\| \sum_{i=1}^{p} g(t_i)|I_i| - x_{\mathcal{P}} \right\| < \eta/2.$$

and so

$$|x_0^*(x_{\mathcal{P}}) - x_0^*(x_0)| \leq \left| x_0^*(x_{\mathcal{P}}) - x_0^*\left(\sum_{i=1}^{p} g(t_i)|I_i| \right) \right| +$$

$$+ \left| \sum_{i=1}^{p} x_0^* g(t_i)|I_i| - x_0^*(x_0) \right| < \eta.$$

- Now, previous inequality yields $\|x_{\mathcal{P}} - x_0\| < \varepsilon'$

- Finally

$$\left\|\sum_{i=1}^{p} g(t_i)|I_i| - x_0\right\| \leq \left\|\sum_{i=1}^{p} g(t_i)|I_i| - x_{\mathcal{P}}\right\| + \|x_{\mathcal{P}} - x_0\| < \varepsilon\,.$$

□

Fremlin (1994) proved that a Banach space valued function is McShane integrable if and only if it is Henstock and Pettis integrable.

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We will see that answer is positive for multifunctions with compact convex values being subsets of an arbitrary Banach space.

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2. a Decomposition Theorem;
3. a technical (but useful) Lemma.

A Decomposition Theorem

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$$\Gamma(t) = G(t) + f(t)$$

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$$\mathcal{Z}_\Gamma := \{\mathbf{s}(\mathbf{x}^*, \Gamma(\cdot)) = \mathbf{s}(\mathbf{x}^*, \mathbf{G}(\cdot)) + \mathbf{x}^*\mathbf{f}(\cdot) : \|\mathbf{x}^*\| \leq \mathbf{1}\},$$

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Let $\mathcal{A} = \{g_\alpha \colon [0,1] \to [0,\infty) : \alpha \in S\}$ be a family of functions satisfying the following conditions:

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Then the family $\mathcal{A}$ is also McShane equi-integrable.

Theorem

Let $\Gamma : [0, 1] \to ck(X)$ be a multifunction. Then TFAE:

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Relations among the integrals

Theorem

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Proof.

(2) $\Rightarrow$ (1) Since $\Gamma : [0,1] \to ck(X)$ is Henstock and Pettis integrable in $ck(X)$, then $\mathcal{S}_{MS}(\Gamma) \neq \emptyset$. Let f be a McShane integrable selection Γ. It follows from the Decomposition Theorem that there exists a multifunction $G : [0,1] \to ck(X)$ that is McShane integrable such that $\Gamma = G + f$. It follows that Γ is also McShane integrable. $\square$

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Theorem

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1. $\varGamma$ is McShane integrable;
2. $\varGamma$ is Henstock integrable and $\mathcal{S}_H(\varGamma) \subset \mathcal{S}_{MS}(\varGamma)$;
3. $\varGamma$ is Henstock integrable and $\mathcal{S}_H(\varGamma) \subset \mathcal{S}_P(\varGamma)$;

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Theorem

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1. Γ is McShane integrable;
2. Γ is Henstock integrable and $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$;
3. Γ is Henstock integrable and $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_P(\Gamma)$;
4. Γ is Henstock integrable and $\mathcal{S}_P(\Gamma) \neq \emptyset$.

THANK YOU!

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