A class of hereditarily separably Sobczyk spaces

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Notation

- K is always compact, Hausdorff topological space,
- $C(K)$ denotes the Banach space of continuous real-valued functions on set K with supremum norm $\|\cdot\|$,
- any linear subspace F of a Banach space E is closed (it is denoted by $F \leqslant E$).

Preliminaries

Weak* topology

$M(K)$ denotes the space of all signed measures on K and $P(K)$ denotes the space of probability measures. Obviously $P(K) \subset M(K)$. Any measure $\mu \in M(K)$ can be seen as a function

$$\mu\colon C(K) \to \mathbb{R} \quad \text{given by the formula} \quad \mu(f) = \int f \, \mathrm{d}\mu.$$

These functions are linear, so in fact $M(K)$ forms a closed subset of the product $\mathbb{R}^{C(K)}$. It is natural to equip the space $M(K) \subseteq \mathbb{R}^{C(K)}$ in the product topology (pointwise convergence). This topology is called the weak* topology.

Definition

A subspace $F \leqslant E$ is complemented in E if there exists a subspace $G \leqslant E$ such that $E = F \oplus G$. Equivalently there exists a projection $P\colon E \to F$ onto F, i.e. a bounded linear operator P such that $P^2 = Id$.

Theorem (Sobczyk)

Any isomorphic copy of c_0 in separable superspace is complemented.

Example

- c_0 is not complemented in l_∞,
- if K contains a non-trivial convergent sequence then there exists an embedding T such that $T[c_0]$ is complemented in $C(K)$,
- there exists an uncomplemented subspace X of l_1.

Definition

A space such that any subspace isomorphic to c_0 is complemented is called separably Sobczyk (s.S.) or space with Sobczyk property (S.p.).

The natural question which arises is the following:

Question

Which Banach space have the Sobczyk property?

1. $C(K)$ if K is metric,
2. any isometric copy of c_0 is complemented in $C(DA)$ (W. M. Patterson, 1993),
3. more general, $C(K)$ if K is compact line (C. Correa, D. V. Tausk, 2014),
4. space with certain topological property of its dual space (Moltó, 1991).

Hereditarily separably Sobczyk spaces

Definition

A sapce E is heredirarily separably Sobczyk if any isomorphic copy of c_0 in E contains further copy of c_0 which is complemented in E.

To express the theorem we need another definition:

Definition

A measure $\mu \in P(K)$ is of Maharam type ω if there exists countable family $\mathcal{C} \subseteq \mathrm{Bor}(K)$ such that for any borel set $B \subseteq K$ and any $\varepsilon > 0$ there is $C \in \mathcal{C}$ such that $\mu(B \triangle C) < \varepsilon$. Equivalently, $L_1(\mu)$ is separable.

Theorem

If K be a compact space such that any measure $\mu \in P(K)$ is of Maharam type ω then the space $C(K)$ is hereditarily separably Sobczyk.

How to prove the theorem?

We need one lemma, one definition and another theorem.

Lemma

Let $T\colon c_0 \to E$ be an embedding and suppose that there exists a convergent to 0 sequence $\langle x_n^* \rangle \subseteq E^*$ such that $x_n^*(Te_k) = \delta_{nk}$. Then $T[c_0]$ is complemented in E.

Proof.

The proof is quite straightforward. Simply define an operator $P\colon E \to T[c_0]$ by the formula

$$Px = \sum_{i \in \omega} x_i^*(x) Te_i. \qquad \square$$

To express a necessary theorem we need one more definition.

Definition

Let X denote a vector space. $\langle y_n \rangle$ is a convex block subsequence of $\langle x_n \rangle \subseteq X$ if there exist finite sets $I_n \subset \omega$ and non-negative real numbers t_j such that for all $n \in \omega$

$$\max I_n < \min I_{n+1},$$
$$y_n = \sum_{j \in I_n} t_j x_j \quad \text{and} \quad \sum_{j \in I_n} t_j = 1.$$

Theorem (Haydon — Levy — Odell)

If K is a compact space such that all measures $\mu \in M(K)$ are of Maharam type ω then every bounded sequence $\langle \lambda_n : n \in \omega \rangle \subseteq M(K)$ has a convex block subsequence $\langle \nu_n : n \in \omega \rangle$ convergent to some measure $\nu \in M(K)$.

Theorem

If K be a compact space such that any measure $\mu \in P(K)$ is of Maharam type ω then the space $C(K)$ is hereditarily separably Sobczyk.

The proof is simply application of the previous theorem and lemma.

Let $T\colon c_0 \to C(K)$ be an embedding.

We want to find another copy of c_0 in $T[c_0]$ which is complemented, i.e. we want to have an embedding $S\colon c_0 \to T[c_0]$ such that its image is complemented.

How to do it?

Let $\langle e_n \rangle_n$ be a standard base in c_0. Recall that $c_0^* = l_1$.

Define $g_n = Te_n$. Since T is an embedding hence a dual operator

$$T^*\colon C(K)^* = M(K) \to c_0^* = l_1$$

is onto.

Therefore for any $n \in \omega$ there is a bounded sequence of measures $\mu_n \in M(K)$ such that $T^* \mu_n = e_n^*$.

By theorem HLO there exists convex block subsequence ν_n convergent to some measure $\nu \in M(K)$:

$$\nu_n = \sum_{k \in I_n} t_k \mu_k \to \nu,$$

Denote $h_n = \sum_{k \in I_n} g_k$ and define an operator $S : c_0 \to C(K)$ by the following formula:

$$Se_n = h_n$$

The functions $\langle h_n \rangle_n$ span the space $S[c_0]$ which is isomorphic to c_0. It is easy to see that $S[c_0] \leqslant T[c_0]$. By application of previous lemma to the sequence $\langle \nu_n - \nu \rangle_n$, we can see that $S[c_0]$ is complemented in $C(K)$.

Compacta with all measures of Maharam type ω

1. Eberlein compacta,
2. Rosenthal compacta,
3. (linear ordered compacta),
4. Radon-Nikodym compacta,
5. Stone spaces of minimally generated algebras,

Moreover this class is closed under subspaces, continuous images and countable Cartesian products.

It is worth to mention that not every hereditarily separably Sobczyk space is separably Sobczyk. The example of such space was given by E.M. Galego and A. Plichko (2003).