Abstract versions of the Radon-Nikodym theorem

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Integration, Vector Measures and Related Topics
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Classical Radon-Nikodym theorem

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$$\int f \, d\nu = \int (f \cdot g) \, d\mu$$

for all $f \in L^1(\nu)$.

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We say that a Banach space Y has the Radon-Nikodym property if there exist a μ-integrable function $g\colon X \to Y$ such that:

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Every reflexive Banach space has the Radon-Nikodym property. There are spaces which do not have the Radon-Nikodym property, e.g. c_0, $L^1(\Omega)$, $C(\Omega)$, $L^\infty(\Omega)$.

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can be rewritten as follows: $T = V \circ \pi$.

Results of Maharam and the Luxemburg-Schep theorem

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- W.A.J. Luxemburg, A.R. Schep, *A Radon-Nikodym type theorem for positive operators and a dual*, Nederl. Akad. Wet., Proc. Ser. A, 81 (1978), 357–375.

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Then V is said to have *Maharam property* if for all $f \in F$ and for all $g \in G$ such that $f \geq 0$ and $0 \leq g \leq Vf$ there exists some $f_1 \in F$ such that $0 \leq f_1 \leq f$ and $Vf_1 = g$.

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In other words, for every positive $f \in F$, the interval $[0, Vf]$ is contained in the set $V([0, f])$.

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For every operator $T\colon F \to G$ such that $0 \leq T \leq V$ there exists an orthomorphism π of F such that $0 \leq \pi \leq I$ and $T = V \circ \pi$.

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The dual theorem: conditions for factorization $T = \pi \circ V$.

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with some function g (for example, if the domain of π is $C(X)$, then $g \in C(X)$; if it is L^1, then $g \in L^\infty$).
To derive the Radon-Nikodym theorem from the Luxemburg-Schep theorem we need that every orthomorphism of $L^1(\mu)$ is a multiplication operator (so g is the Radon-Nikodym derivative).

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Factorization theorems of Arendt

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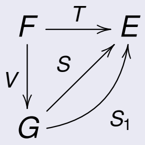
Theorem (Arendt)

Let E be a Dedekind complete Riesz space, F, G be Riesz spaces and $V: F \to G$ be a Riesz homomorphism. Then, given a positive linear mapping $S: G \to E$, every positive linear mapping $T: F \to E$ which satisfies $T \leq S \circ V$ admits a factorization

$$T = S_1 \circ V,$$

where $S_1: G \to E$ is a linear mapping such that $0 \leq S_1 \leq S$.



Factorization theorems of Arendt

Theorem (Arendt)

Let E, F and G be Banach lattices with G having an order-continuous norm and let $U\colon G \to F$ be an interval preserving positive linear mapping. Then, given a positive linear mapping $S\colon E \to G$, every positive linear mapping $T\colon E \to F$ which satisfies $T \leq U \circ S$ admits a factorization

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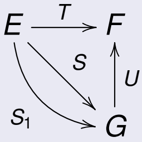

Some definitions

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A group with a lattice order compatible with its algebraic structure is called ℓ-group.
A map $f \colon G \to F$ between ℓ-groups is called *monotone* if

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in G$ and f is called Φ-*invariant* if $f \circ \Phi_s = f$ for all $s \in X$.

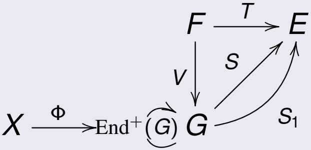
Result 1

Assume that E is a Dedekind complete Riesz space and F and G are Abelian ℓ-groups. Further, denote by $\mathrm{End}^+(G)$ the semigroup of all monotone endomorphisms of G. Moreover, let X be a right-amenable semigroup and let $\Phi\colon X \to \mathrm{End}^+(G)$ be a representation of X.

Result 1

Theorem

Let $V\colon F \to G$ be an ℓ-group homomorphism such that $\Phi_s \circ V = V$ for all $s \in X$. Given an additive monotone and Φ-invariant mapping $S\colon G \to E$, every additive monotone mapping $T\colon F \to E$ such that $T \le S \circ V$ admits a factorization $T = S_1 \circ V$,



where $S_1\colon G \to E$ is an additive and Φ-invariant mapping such that $0 \le S_1 \le S$.

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$$\begin{aligned} 0 &\le |T(x_1) - T(x_2)| = |T(x_1 - x_2)| \le T(|x_1 - x_2|) \\ &\le S(V(|x_1 - x_2|)) = S(|Vx_1 - Vx_2|) = 0. \end{aligned}$$

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Therefore, $T(x_1) = T(x_2)$.

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$S_0(\Phi_s(Vx)) = S_0(Vx)$ for all $x \in F$.

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Introduce $p\colon G \to E$ by:

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$$\begin{aligned} p(y_1 + y_2) &= S((y_1 + y_2)^+) \leq S(y_1^+ + y_2^+) \\ &= S(y_1^+) + S(y_2^+) = p(y_1) + p(y_2). \end{aligned}$$

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Proof (continued)

Introduce $p\colon G \to E$ by:

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$$p(\Phi_s(y)) = S((\Phi_s y)^+) \leq S(\Phi_s(y^+)) = S(y^+) = p(y).$$

Proof (continued)

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$$\begin{aligned} S_0(y) &= S_0(Vx) = T(x) \leq T(x^+) \leq S(V(x^+)) \\ &= S((Vx)^+) = S(y^+) = p(y). \end{aligned}$$

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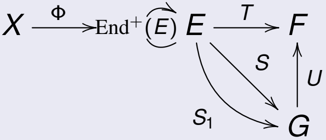
Result 2

Assume that G is a Dedekind complete Riesz space and E and F are Abelian ℓ-groups. Further, assume that X is a right-amenable group and $\Phi\colon X \to \mathrm{End}^+(E)$ is a representation of X in the set of of all monotone endomorphisms of E.

Result 2

Theorem

Let $U\colon G \to F$ be an injective ℓ-group homomorphism. Given an additive monotone and Φ-invariant mapping $S\colon E \to G$, every additive monotone and Φ-invariant mapping $T\colon E \to F$ such that $T \leq U \circ S$ admits a factorization $T = U \circ S_1$,



where $S_1\colon E \to G$ is an additive and Φ-invariant map such that $0 \leq S_1 \leq S$.

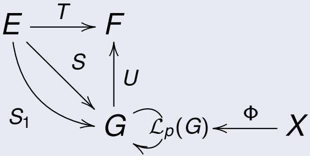
Result 3

Assume that E, F and G are Banach lattices with G having an order-continuous norm. Further, assume that X is a right-amenable semigroup and $\Phi\colon X \to \mathcal{L}_p(G)$ is a representation of X in the set $\mathcal{L}_p(G)$ of of all positive linear self-mappings of G.

Result 3

Theorem

Let $U\colon G \to F$ be an interval preserving and Φ-invariant positive linear mapping. Given a positive linear mapping $S\colon E \to G$ such that $\Phi_s \circ S = S$ for all $s \in X$, every positive linear mapping $T\colon E \to F$ such that $T \leq U \circ S$ admits a factorization $T = U \circ S_1$,



where $S_1\colon E \to G$ is a linear map such that $0 \leq S_1 \leq S$ and $\Phi_s \circ S_1 = S_1$ for all $s \in X$.

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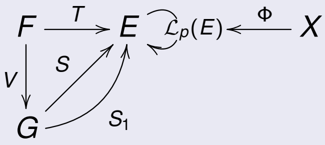
Theorem

Let $V: F \to G$ be an interval preserving positive and injective linear mapping. Given a positive linear mapping $S: G \to E$ such that $\Phi_s \circ S = S$ for all $s \in X$, every positive linear mapping $T: F \to E$ such that $T \leq S \circ V$ and $\Phi_s \circ T = T$ for all $s \in X$ admits a factorization $T = S_1 \circ V$,



where $S_1: G \to E$ is a linear mapping such that $0 \leq S_1 \leq S$ and $\Phi_s \circ S_1 = S_1$ for all $s \in X$.

Theorem of Z. Gajda

Assume that X is a right-amenable semigroup, G a partially ordered Abelian group, $\Phi\colon X \to \mathrm{End}(G)$ is a representation of X, E is a Dedekind complete Riesz space and G_1 is a subgroup of G such that $\Phi_s(G_1) \subseteq G_1$ for every $s \in X$.

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Theorem (Gajda)

Assume that $p\colon G \to E$ is a monotone, subadditive and Φ-subinvariant function and $a_0\colon G_1 \to E$ is an additive monotone and Φ-invariant function such that $a_0 \le p$ on G_1. Then a_0 has an extension to an additive monotone and Φ-invariant function $a\colon G \to E$ such that $a \le p$ on G.

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Zbigniew Gajda, *Sandwich theorems and amenable semigroups of transformations*, Grazer Math. Ber., 316 (1992), 43–58.

Thank you for your kind attention!!!