Nonseparable spaceability and strong algebrability of sets of continuous singular functions

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Integration, Vector Measures and Related Topics VI

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CBV and strongly singular functions

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A singular function $f \in CBV$ is called strongly singular if its restriction to every subinterval of $[0, 1]$ is singular.

Measures μ_p and distribution functions F_p

Let $p \in (0, 1/2)$, μ_p be the distribution of the sum

$$X = \sum_{k=1}^{\infty} \left(\frac{1}{2^k} \right) X_k$$

where X_k, $k \in \mathbb{N}$, is a sequence of independent random variables with $\Pr(X_k = 0) = p$ and $\Pr(X_k = 1) = 1 - p$.

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We will denote

$$F_p(t) := \Pr(X < t)$$

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$$\mu_p\left(\left[t, t+\frac{1}{2^n}\right]\right) = F_p\left(t+\frac{1}{2^n}\right) - F_p(t) = p^{l(t)}(1-p)^{r(t)}.$$

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(3) If $x \in (0,1)$ and $F_p'(x)$ exists then $F_p'(x) = 0$.

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Hence $\frac{\mu_p(I_{n+1})}{\mu_p(I_n)} \to \frac{1}{2}$, but $\frac{\mu_p(I_{n+1})}{\mu_p(I_n)}$ is equal to p or $1-p$.

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(6) The set

$$B_p = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{x_1 + \ldots + x_n}{n} = 1 - p \right\}$$

where $x = 0.x_1x_2x_3x_4\ldots_{(2)}$, is Borel and $\mu_p\left(B_p\right) = 1$.

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(7) For any distinct $p, q \in \left(0, \frac{1}{2}\right)$, $Var_{[0,1]}(F_p - F_q) = 2$.

Sketch of the proof

Let $\varepsilon \in (0, 1/4)$. Pick closed sets $C_p \subset B_p$ and $C_q \subset B_q$ such that $\mu_p(C_p) \geq 1 - \varepsilon$ and $\mu_q(C_q) \geq 1 - \varepsilon$.

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Then $\mu_p(G_p) \geq 1 - \varepsilon$, that is $\sum_n Var_{cl(I_n)} F_p \geq 1 - \varepsilon$, and $\mu_p(G_q) \leq \varepsilon$, that is $\sum_n Var_{cl(J_n)} F_p \leq \varepsilon$.

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Hence

$$Var_{[0,1]}(F_p - F_q) \geq 2 - 4\varepsilon.$$

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(7) For any distinct $p, q \in \left(0, \frac{1}{2}\right)$, $Var_{[0,1]}\left(F_p - F_q\right) = 2$.

Corollary: The space CBV is nonseparable

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A subset A of a topological vector space V is called spaceable if $A \cup \{\theta\}$ contains an infinite dimensional closed vector subspace W of V. If W is nonseparable, we say that A is nonseparably spaceable.

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Corollary: Each nonzero function from W is strongly singular.

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If $f' = 0$ a.e., $g \in CBV$ and $|g'| > \alpha$ on the set X of positive measure $\lambda(X) = \beta$ then

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Lemma 2: If $f \in cl(W)$ in CBV then $f' = 0$ almost everywhere in $[0, 1]$.

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Lemma 3: Consider arbitrary birational numbers $t_0 = i_0/2^{n_0}$ and $t_1 = i_1/2^{n_1}$ such that $n_1 \geq n_0$, $\ell(t_1) \geq \ell(t_0)$ and $r(t_1) \geq r(t_0)$. Put $I_0 = [t_0, t_0 + 1/2^{n_0}]$ and $I_1 = [t_1, t_1 + 1/2^{n_1}]$. Then there exists a subinterval $J = [j/2^{n_1}, (j+1)/2^{n_1+1}]$ of I_0 such that $\mu_p(J) = \mu_p(I_1)$ for each $p \in (0, 1/2)$.
Moreover, for any real numbers $\alpha, \beta \in I_1$, $\alpha < \beta$, there exists a subinterval $[\alpha_1, \beta_1]$ of I_0 such that $\mu_p([\alpha_1, \beta_1]) = \mu_p([\alpha, \beta])$ for each $p \in (0, 1/2)$.

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Lemma 4: If $f \in cl(W)$ is constant in some interval of $[0,1]$ then f is equal to 0 in $[0,1]$.

Algebrability

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E is called strongly κ-algebrable if there exists a free algebra $A \subset E \cup \{\theta\}$ such that

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$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x}, \quad x \in \mathbb{R},$$

for some distinct nonzero real numbers $\beta_1, \ldots \beta_m$ and some nonzero real numbers $a_1, \ldots a_m$.

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METHOD. Given a family $\mathcal{F} \subset \mathbb{R}^{[0,1]}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential like function $f\colon \mathbb{R} \to \mathbb{R}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More exactly, if $H \subset \mathbb{R}$ is a set of cardinality $\mathfrak{c}$, linearly independent over $\mathbb{Q}$, then $\exp \circ (rF)$, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

Lemma 5. For any exponential like function $f\colon [0,1] \to \mathbb{R}$ of range m, and each $c \in \mathbb{R}$, the preimage $f^{-1}[\{c\}]$ has at most m elements. Consequently, f is not constant in every subinterval of $[0,1]$.

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Proof of Theorem 2: Let $F = F_{1/4}$ and $f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x}$. Since $F' = 0$ almost everywhere in $[0,1]$

$$(f \circ F)'(x) = F'(x) \sum_{i=1}^{m} a_i \beta_i e^{\beta_i F(x)} = 0 \text{ for almost all } x \in [0,1].$$

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Suppose that $f \circ F$ is constant in some subinterval $[c,d]$ of $[0,1]$ with

$c < d$. Since F^{-1} is a continuous increasing bijection from $[0,1]$ onto $[0,1]$, the function $f = (f \circ F) \circ F^{-1}$ is constant in the interval $[F(c), F(d)]$ which gives a contradiction.

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