A martingale inequality

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Theorem Suppose that (Ω, Σ, μ) is a probability space, $\Sigma_0 \subseteq \ldots \subseteq \Sigma_n$ are σ-subalgebras of Σ, $(Y_0, \ldots, Y_n)$ is a martingale adapted to $(\Sigma_0, \ldots, \Sigma_n)$, and $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for each $i < n$. Set

$$Z = \sum_{i=0}^{n-1} X_i \times (Y_{i+1} - Y_i).$$

Then $\Pr(|Z| \geq M) \leq \frac{1}{M^{2/3}}(1 + \mathbb{E}(|Y_n|))$ for every $M > 0$.

Theorem If $(Y_0, \dots, Y_n)$ is a martingale adapted to $(\Sigma_0, \dots, \Sigma_n)$, $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for $i < n$, and

$$Z = \sum_{i=0}^{n-1} X_i \times (Y_{i+1} - Y_i),$$

then $\Pr(|Z| \geq M) \leq \frac{1}{M^{2/3}}(1 + \mathbb{E}(|Y_n|))$ for every $M > 0$.

Doob's maximal inequality If $(Y_0, \dots, Y_n)$ is a martingale, and

$$Z = \max_{i \leq n} |Y_i|,$$

then $\Pr(|Z| \geq M) \leq \frac{1}{M}\mathbb{E}(|Y_n|)$ for every $M > 0$.

Theorem If $(Y_0, \ldots, Y_n)$ is a martingale adapted to $(\Sigma_0, \ldots, \Sigma_n)$, $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for $i < n$, and

$$Z = \sum_{i=0}^{n-1} X_i \times (Y_{i+1} - Y_i),$$

then $\Pr(|Z| \geq M) \leq \frac{1}{M^{2/3}}(1 + \mathbb{E}(|Y_n|))$ for every $M > 0$.

A fractionally sharper theorem If $(Y_0, \ldots, Y_n)$ is a martingale adapted to $(\Sigma_0, \ldots, \Sigma_n)$, $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for $i < n$, and

$$Z = \sum_{i=0}^{n-1} X_i \times (Y_{i+1} - Y_i),$$

then $\Pr(|Z| \geq M) \leq \frac{K^2}{M^2} + \frac{1}{K}\mathbb{E}(|Y_n|)$ for all K, $M > 0$.

(Set $K = M^{2/3}$ to get the original version.)

Case 1 Suppose that $|Y_n| \leq_{\text{a.e.}} K$. Then $\Pr(|Z| \geq M) \leq \frac{K^2}{M^2}$.

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proof We have

$$\mathbb{E}(Z^2) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}(X_i \times X_j \times (Y_{i+1} - Y_i) \times (Y_{j+1} - Y_j))$$
$$= \sum_{i=0}^{n-1} \mathbb{E}(X_i^2 \times (Y_{i+1} - Y_i)^2)$$

(because if $i < j$, $X_i \times X_j \times (Y_{i+1} - Y_i)$ is Σ_j-measurable, while 0 is a conditional expectation of $Y_{j+1} - Y_j$ on Σ_j)

$$\leq \sum_{i=0}^{n-1} \mathbb{E}((Y_{i+1} - Y_i)^2)$$
$$= \sum_{i=0}^{n-1} \mathbb{E}(Y_{i+1}^2 - Y_i^2) - 2\mathbb{E}(Y_i \times (Y_{i+1} - Y_i))$$
$$= \sum_{i=0}^{n-1} \mathbb{E}(Y_{i+1}^2 - Y_i^2) = \mathbb{E}(Y_n^2) - \mathbb{E}(Y_0^2) \leq K^2$$

and the result follows at once.

Case 2 Suppose that whenever $i < n$ and $\max_{j \le i} |Y_j(\omega)| < K \le |Y_{i+1}(\omega)|$ then $|Y_{i+1}(\omega)| = K$. Then $\Pr(|Z| \ge M) \le \frac{K^2}{M^2} + \frac{1}{K}\mathbb{E}(|Y_n|)$.

proof Set

$$\begin{aligned} Y_i'(\omega) &= 0 \text{ if } |Y_0(\omega)| \ge K, \\ &= Y_i(\omega) \text{ if } |Y_j(\omega)| < K \text{ for every } j \le i, \\ &= Y_k(\omega) \text{ if } 0 < k \le i,\ |Y_j(\omega)| < K \text{ for every } j < k,\ |Y_k(\omega)| = K \end{aligned}$$

and set

$$Z' = \sum_{i=0}^{n-1} X_i \times (Y_{i+1}' - Y_i').$$

By Case 1, $\Pr(|Z'| \ge M) \le \frac{K^2}{M^2}$, so

$$\begin{aligned} \Pr(|Z| \ge M) \le \Pr(|Z'| \ge M) + \Pr(Z' \ne Z) &\le \frac{K^2}{M^2} + \Pr(\exists\, i,\ Y_i' \ne Y_i) \\ \le \frac{K^2}{M^2} + \Pr(\exists\, i,\ |Y_i| \ge K) &\le \frac{K^2}{M^2} + \frac{1}{K}\mathbb{E}(|Y_n|) \end{aligned}$$

by Doob's inequality.

Lemma Suppose that (Ω, Σ, μ) is a probability space, $\Sigma_0 \subseteq \ldots \subseteq \Sigma_n$ are σ-subalgebras of Σ, $(Y_0, \ldots, Y_n)$ is a martingale adapted to $(\Sigma_0, \ldots, \Sigma_n)$, and $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for each $i < n$. Then there are a probability space (Ω', Σ', μ'), σ-subalgebras $\Sigma'_0 \subseteq \ldots \subseteq \Sigma'_{2n}$ of Σ', a martingale $(Y'_0, \ldots, Y'_{2n})$ adapted to $(\Sigma'_0, \ldots, \Sigma'_{2n})$, and a Σ'_{2i}-measurable random variable X'_{2i} for each $i < n$, such that

(i) whenever $i < n$ and $|Y'_{2i}(\omega')| < K$ then either $|Y'_{2i+1}(\omega')| = K$ or $|Y'_{2i+1}(\omega')| < K$ and $|Y'_{2i+2}(\omega')| < K$,

(ii) $Y'_0, Y'_2, \ldots, Y'_{2n}, X'_0, X'_2, \ldots, X'_{2n-2}$ have the same joint distribution as $Y_0, Y_1, \ldots, Y_n, X_0, X_1, \ldots, X_{n-1}$.

proof of theorem Set $X'_{2i+1} = X'_{2i}$ for $i < n$,

$$Z' = \sum_{j=0}^{2n-1} X'_j \times (Y'_{j+1} - Y'_j) = \sum_{i=0}^{n-1} X'_{2i} \times (Y'_{2i+2} - Y'_{2i}).$$

Then Z and Z' have the same distribution so

$$\Pr(|Z| \geq M) = \Pr(|Z'| \geq M) \leq \frac{K^2}{M^2} + \frac{1}{K}\mathbb{E}(|Y'_{2n}|)$$

(by Case 2)

$$= \frac{K^2}{M^2} + \frac{1}{K}\mathbb{E}(|Y_n|).$$

Lemma Suppose that $(Y_0, \dots, Y_n)$ is a martingale adapted to $(\Sigma_0, \dots, \Sigma_n)$, and $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for each $i < n$. Then there are a probability space (Ω', Σ', μ'), a martingale $(Y'_0, \dots, Y'_{2n})$ adapted to $(\Sigma'_0, \dots, \Sigma'_{2n})$, and a Σ'_{2i}-measurable random variable X'_{2i} for each $i < n$, such that

(i) whenever $i < n$ and $|Y'_{2i}(\omega')| < K$ then either $|Y'_{2i+1}(\omega')| = K$ or $|Y'_{2i+1}(\omega')| < K$ and $|Y'_{2i+2}(\omega')| < K$,

(ii) $Y'_0, Y'_2, \dots, Y'_{2n}, X'_0, X'_2, \dots, X'_{2n-2}$ have the same joint distribution as $Y_0, \dots, Y_n, X_0, \dots, X_{n-1}$.

Proving the lemma: basic case Take $n = 1$, $\Sigma_1 = \Sigma$, $\Sigma_0 = \{\emptyset, \Omega\}$, $Y_0 = \gamma = \mathbb{E}(Y_1)$ where $|\gamma| < K$. Set $\Omega' = \Omega \times [0, 1]$ with product measure μ', domain Σ'_2; set $\Sigma'_0 = \{\emptyset, \Omega'\}$, $Y'_0(\omega, t) = Y_0(\omega) = \gamma$, $X'_0(\omega, t) = X_0(\omega)$, $Y'_2(\omega, t) = Y_1(\omega)$. Seek a partition (G^+, G^-, H) of Ω' such that

$$\int_{G^+} Y'_2 d\mu' = K\mu' G^+, \quad \int_{G^-} Y'_2 d\mu' = -K\mu' G^-$$

and $H \subseteq F \times [0, 1]$ where $F = \{\omega : |Y_1(\omega)| < K\}$. Then we can take Σ'_1 to have atoms G^+, G^- and H.

Construction when $n = 1$, $\Sigma_0 = \{\emptyset, \Omega\}$, $\Omega' = \Omega \times [0,1]$, $|\mathbb{E}(Y_1)| < K$, $Y_2'(\omega, t) = Y_1(\omega)$. Seek a partition (G^+, G^-, H) of Ω' such that

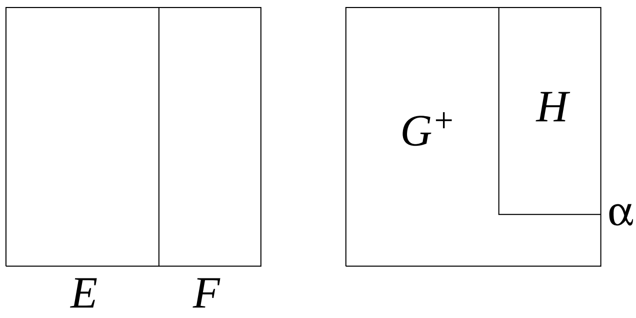
$$\int_{G^+} Y_2' d\mu' = K\mu' G^+, \quad \int_{G^-} Y_2' d\mu' = -K\mu' G^+$$

and $H \subseteq F \times [0,1]$ where $F = \{\omega : |Y_1(\omega)| < K\}$.

(α) If $\int_E Y_1 \geq K\mu E$ where $E = \Omega \setminus F$, try

$$G_\alpha = (E \times [0,1]) \cup (F \times [0, \alpha]) \text{ for } \alpha \in [0,1].$$

If $\alpha = 0$, $\int_{G_\alpha} Y_2' d\mu' \geq K\mu' G_\alpha$; if $\alpha = 1$, $\int_{G_\alpha} Y_2' d\mu' < K\mu' G_\alpha$; so for a suitable α can take $G^+ = G_\alpha$, $G^- = \emptyset$.



(β) Similarly if $\int_E Y_1 \leq -K\mu E$.

Seek a partition (G^+, G^-, H) of Ω' such that

$$\int_{G^+} Y_2' d\mu' = K\mu' G^+, \quad \int_{G^-} Y_2' d\mu' = -K\mu' G^+$$

and $H \subseteq F \times [0,1]$ where $F = \{\omega : |Y_1(\omega)| < K\}$.

(γ) If $-K\mu E < \int_E Y_1 < K\mu E$; set

$$E^+ = \{\omega : Y_1(\omega) \geq K\}, \quad E^- = \{\omega : Y_1(\omega) \leq -K\},$$

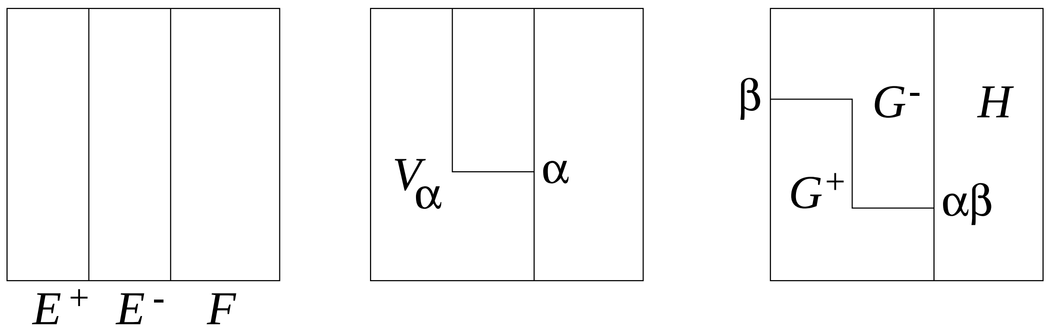
$$V_\alpha = (E^+ \times [0,1]) \cup (E^- \times [0,\alpha]) \text{ for } \alpha \in [0,1].$$

Then there is an α such that $\int_{V_\alpha} Y_2' d\mu' = K\mu' V_\alpha$. Now set

$$W_\beta = (E^+ \times [\beta,1]) \cup (E^- \times [\alpha\beta,1]) \text{ for } \beta \in [0,1].$$

Then there is a β such that $\int_{W_\beta} Y_2' = -K\mu' W_\beta$. Take

$$G^- = W_\beta, \quad G^+ = (E^+ \times [0,\beta[) \cup (E^- \times [0,\alpha\beta[).$$



Vector-valued extensions? Lemma Let U be a Banach space. Suppose that (Ω, Σ, μ) is a probability space, $\Sigma_0 \subseteq \ldots \subseteq \Sigma_n$ are σ-subalgebras of Σ, $(Y_0, \ldots, Y_n)$ is a martingale of Bochner integrable U-valued functions adapted to $(\Sigma_0, \ldots, \Sigma_n)$, and $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for each $i < n$. Then there are a probability space (Ω', Σ', μ'), σ-subalgebras $\Sigma'_0 \subseteq \ldots \subseteq \Sigma'_{2n}$ of Σ', a U-valued Bochner martingale $(Y'_0, \ldots, Y'_{2n})$ adapted to $(\Sigma'_0, \ldots, \Sigma'_{2n})$, and a Σ'_{2i}-measurable random variable X'_{2i} for each $i < n$, such that

(i) whenever $i < n$ and $\|Y'_{2i}(\omega')\| < K$ then either $\|Y'_{2i+1}(\omega')\| = K$ or $\|Y'_{2i+1}(\omega')\| < K$ and $\|Y'_{2i+2}(\omega')\| < K$,

(ii) $Y'_0, Y'_2, \ldots, Y'_{2n}, X'_0, X'_2, \ldots, X'_{2n-2}$ have the same joint distribution as $Y_0, Y_1, \ldots, Y_n, X_0, X_1, \ldots, X_{n-1}$.

Theorem Let U be a Hilbert space. Suppose that $(Y_0, \ldots, Y_n)$ is a U-valued Bochner martingale adapted to $(\Sigma_0, \ldots, \Sigma_n)$, and $X_i : \Omega \to [-1, 1]$ is Σ_i-measurable for each $i < n$. Set

$$Z = \sum_{i=0}^{n-1} X_i \times (Y_{i+1} - Y_i).$$

Then $\Pr(\|Z\| \geq M) \leq \frac{1}{M^{2/3}}(1 + \mathbb{E}(\|Y_n\|))$ for every $M > 0$.