Quantitative Schur and Dunford-Pettis properties

(and some other properties of a similar nature)

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Properties defined or characterized by comparing convergence

- Schur, Grothendieck and Dunford-Pettis properties
- Quantitative versions
- More on the quantitative Schur property
- Relationship to quantitative Dunford-Pettis properties

Properties defined or characterized by possibility to extract a subsequence

- Norm compactness, weak compactness, not containing ℓ_1, Banach-Saks property, weak Banach-Saks property, weak* sequential compactness
- Quantitative versions
- Dichotomies on the unit ball

References

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Comparing convergence

- X has the Schur property if
  (x_k) weakly convergent $\Rightarrow$ (x_k) norm convergent
- X is Grothendieck if
  (x_k^*) weak* convergent $\Rightarrow$ (x_k^*) weakly convergent
- X has the Dunford-Pettis property
  iff (x_k) weakly convergent $\Rightarrow$ (x_k) $\mu(X^{**}, X^*)$-convergent
  iff (x_k^*) weakly convergent $\Rightarrow$ (x_k^*) $\mu(X^*, X)$-convergent

Comparing convergence

- X has the Schur property if
  $$(x_k) \text{ weakly Cauchy} \Rightarrow (x_k) \text{ norm Cauchy}$$
- X is Grothendieck if
  $$(x_k^*) \text{ weak}^* \text{ Cauchy} \Rightarrow (x_k^*) \text{ weakly Cauchy}$$
- X has the Dunford-Pettis property
  iff (x_k) weakly Cauchy $\Rightarrow (x_k)$ $\mu(X^{**}, X^*)$-Cauchy
  iff (x_k^*) weakly Cauchy $\Rightarrow (x_k^*)$ $\mu(X^*, X)$-Cauchy

Quantification of non-cauchyness

Let $(x_k) \subset X$ be a bounded sequence

- $\operatorname{ca}(x_k) = \inf_{n\in\mathbb{N}} \operatorname{diam}\{x_k : k \geq n\}$
- $\delta_w(x_k) = \sup_{x^*\in B_{X^*}} \inf_{n\in\mathbb{N}} \operatorname{diam}\{x^*(x_k) : k \geq n\}$
- $\delta_\rho(x_k) = \sup_{\substack{L\subset B_{X^*} \\ \text{weak compact}}} \inf_{n\in\mathbb{N}} \operatorname{diam}_{\|\cdot\|_L}\{x_k : k \geq n\}$

$$\|x\|_L = \sup_{x^*\in L} |x^*(x)|$$

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Let $(x_k^*) \subset X^*$ be a bounded sequence

- $\delta_{w^*}(x_k^*) = \sup_{x\in B_X} \inf_{n\in\mathbb{N}} \operatorname{diam}\{x_k^*(x) : k \geq n\}$
- $\delta_{\rho^*}(x_k^*) = \sup_{\substack{L\subset B_X\\ \text{weak compact}}} \inf_{n\in\mathbb{N}} \operatorname{diam}_{\|\cdot\|_L}\{x_k^* : k \geq n\}$

Quantitative versions of the properties

- X has the C-Schur property if
$$\text{ca}(x_k) \le C\delta_w(x_k) \quad \text{for } (x_k) \subset X \text{ bounded.}$$
- X is C-Grothendieck if
$$\delta_w(x_k^*) \le C\delta_{w^*}(x_k^*) \quad \text{for } (x_k^*) \subset X^* \text{ bounded.}$$
- X has the direct quantitative DPP if there is $C > 0$ with
$$\delta_{\rho^*}(x_k^*) \le C\delta_w(x_k^*) \quad \text{for } (x_k^*) \subset X^* \text{ bounded.}$$
- X has the dual quantitative DPP if there is $C > 0$ with
$$\delta_\rho(x_k) \le C\delta_w(x_k) \quad \text{for } (x_k) \subset X \text{ bounded.}$$

Facts on the quantitative DPP

The following are proved in [KKS 2013]:

- X has the direct or dual qDPP $\underset{\not\Leftarrow}{\Rightarrow}$ X has DPP.
- $\mathcal{L}_\infty$ spaces and $\mathcal{L}_1$ spaces have both direct and dual qDPP.
- X^* has the dual qDPP $\Rightarrow$ X has the direct qDPP.
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- X has the direct qDPP $\underset{\not\Leftarrow}{\not\Rightarrow}$ X has the dual qDPP.

Facts on the quantitative Grothendieck property

The following are proved in [Bendová, JMAA 2014]:

- The space ℓ_∞ is 1-Groghendieck.
- X is C-Grothedieck for some C $\underset{\not\Leftarrow}{\Rightarrow}$ X is Grothendieck.

Facts on the quantitative Schur property

The following are proved in [KS 2012]:

- The space $\ell_1(\Gamma)$ enjoys the 1-Schur property for any set Γ.
- X has the C-Schur property for some C

$$\overset{\Rightarrow}{\not\Leftarrow}$$ X has the Schur property.

Quantitative Schur property – a new positive result

Theorem [KS 2013]

Let X be a subspace of $c_0(\Gamma)$ for a set Γ. Then X^* enjoys the 1-Schur property.

Key lemma [KS 2012]

Let $(y_n) \subset \ell_1(\Gamma) = c_0(\Gamma)^*$ such that

- $y_n \xrightarrow{w^*} 0$ in $\ell_1(\Gamma)$,
- $\|y_n\| > c$ for $n \in \mathbb{N}$.

Then for each $\eta > 0$ there is (y_{n_k}) such that each weak* cluster point of (y_{n_k}) in $\ell_1(\Gamma)^{**}$ has norm at least $c - \eta$.

Quantitative Schur property – a new positive result

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Let X be a subspace of $c_0(\Gamma)$ for a set Γ. Then X^* enjoys the 1-Schur property.

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Let X be a subspace of $c_0(\Gamma)$ and $(y_n) \subset \ell_1(\Gamma) = c_0(\Gamma)^*$ such that

- $y_n \xrightarrow{w^*} 0$ in $\ell_1(\Gamma)$,
- $\|y_n|_X\| > c$ for $n \in \mathbb{N}$.

Then for each $\eta > 0$ there is (y_{n_k}) such that each weak* cluster point of $(y_{n_k}|_X)$ in X^{***} has norm at least $c - \eta$.

Key lemma $\Rightarrow$ Theorem

- $(x_k^*) \subset X^*$ bounded, $\mathrm{ca}(x_k^*) > c$
- There are (a_k), (b_k) subsequences of (x_k^*) with $\|a_k - b_k\| > c$.
- Set $\varphi_k = a_k - b_k$.
- Let $A_k, z_k \in \ell_1(\Gamma) = c_0(\Gamma)^*$ be H-B extensions of a_k, φ_k.
- Let $B_k = A_k - z_k$.
- WLOG $A_k \xrightarrow{w^*} A$, $B_k \xrightarrow{w^*} B$ in $\ell_1(\Gamma)$.
- Set $y_k = z_k - A + B$. Then $y_k \xrightarrow{w^*} 0$ and $\|y_k|_X\| > c - \|(A-B)|_X\|$.
- Apply Key lemma to (y_k) and $\varepsilon > 0$ to get (y_{n_k}).
- Let (a, b) be a weak*-cluster point of (a_{n_k}, b_{n_k}) in X^{***}.
- Then $\|a - b\| > c - \varepsilon$, hence $\delta_w(x_k^*) > c - \varepsilon$.

Ingredients to prove the Key lemma

Lemma (essentially [Brown 1995])

Let X be a Banach space and (x_k^*) a sequence in X^* weak* converging to some $x^* \in X^*$. Then for each finite-dimensional $F \subset X^*$ we have

$$\liminf \operatorname{dist}(x_k^*, F) \geq \frac{1}{2}(\liminf \|x_k^*\| - \|x^*\|).$$

Lemma [KS 2013]

Let $X \subset c_0(\Gamma)$ and (x_k^*) a sequence in X^* weak* converging to some $x^* \in X^*$. Then for each finite-dimensional $F \subset X^*$ we have

$$\liminf \operatorname{dist}(x_k^*, F) \geq \liminf \|x_k^*\| - \|x^*\|.$$

Lemma (follows from [Kalton and Werner, 1995])

Let $X \subset c_0(\Gamma)$, $x^* \in X^*$ and $x_n^* \xrightarrow{w^*} 0$ in X^*. Then

$$\limsup \|x^* + x_n^*\| = \|x^*\| + \limsup \|x_n^*\|.$$

Theorem

- X has the Schur property $\Rightarrow$ X has direct qDPP. [easy]
- X has the Schur property $\not\Rightarrow$ X has dual qDPP. [KKS 2013]
- X has the quantitative Schur property $\Rightarrow$ X has both direct and dual qDPP. [KS 2013]

Theorem [KKS 2013]

- X^* has the Schur property $\Rightarrow$ X has dual qDPP.
- X^* has the Schur property $\not\Rightarrow$ X has direct qDPP.

Theorem

X^* has the Schur property $\Leftrightarrow$

X has the DPP and $\ell_1 \not\subset X$.

Theorem

- X has the Schur property $\Rightarrow$ X has direct qDPP. [easy]
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- X has the quantitative Schur property $\Rightarrow$ X has both direct and dual qDPP. [KS 2013]

Theorem [KKS 2013]

- X^* has the Schur property $\Rightarrow$ X has dual qDPP.
- X^* has the Schur property $\not\Rightarrow$ X has direct qDPP.

Theorem [KS 2013]

X^* has the quantitative Schur property $\Leftrightarrow$

X has the direct qDPP and $\ell_1 \not\subset X$.

Hereditary DPP

Fact

- DPP is not inherited by subspaces.
- Each subpaces of $c_0(\Gamma)$ has DPP.

Corollary [KS 2013]

Let $n \in \mathbb{N}$, K be compact with $K^{(n)} = \emptyset$ and $X \subset C(K)$. Then:

- X^* has the C_n-Schur property where C_n is the Banach-Mazur distance of c_0 and $C[0, \omega^n]$.
- X has the direct qDPP (with a constant dependent just on n).

Hereditary DPP - continued

Theorem [KS 2013]

There are Banach spaces X_n for $n \in \mathbb{N}$ such that:

- X_n is isomorphic to c_0.
- X_n is isometric to a subspace of $C[0, \omega^n]$.
- X_n^* does not have the C-Schur property for $C < \frac{n}{4}$.

Theorem [Pełczyński and Szlenk 1965] [KS 2013]

There are Y_1 and Y_2, subspaces of $C[0, \omega^\omega]$ such that:

- Y_1 does not have DPP (hence Y_1^* does not have the Schur property).
- Y_2^* has the Schur property (hence Y_2 has the dual qDPP) but Y_2 does not have the direct qDPP.

Construction of the examples

For $x \in c_0$ and $p \in \mathbb{N}$ set

$$A_p(x) = \sup\{|x_p + x_{i_1} + \cdots + x_{i_p}| : p < i_1 < \cdots < i_p\}.$$

Then:

- $Y_1 = \{x \in c_0 : A_n(x) \to 0\}$ with the norm $\|x\| = \sup_n A_n(x)$. [Pełczyński and Szlenk 1965]
- $X_n = (c_0, \|\cdot\|_n)$, where $\|x\|_n = \max\{A_1(x), \dots, A_n(x)\}$. [KS 2013]
- $Y_2 = (\bigoplus_n X_n)_{c_0}$. [KS 2013]

DPP in the non-commutative case

Observation

The space $K(\ell^2)$ does not have DPP.

$(e_1 \otimes e_n \to 0$ weakly but not in $\mu(B(\ell^2), N(\ell^2))$.)

DPP in the non-commutative case

Observation

The space $K(\ell^2)$ does not have DPP.

Theorem [Saksman and Tylli 1999]

Let $X \subset K(\ell^p)$ for some $p \in (1, \infty)$. Then X has DPP iff X is isomorphic to a subspace of c_0 iff X^* has the Schur property.

Lemma [KS 2013]

Let $X \subset K(\ell_p)$ have DPP. Then for each $\varepsilon > 0$ there is an isomorphic embedding $T : X \to c_0$ with $\|T\| \cdot \|T^{-1}\| < 4 + \varepsilon$.

Corollary [KS 2013]

Let $X \subset K(\ell_p)$. TFAE:

- X has DPP.
- X^* has the 4-Schur property.
- X has both direct qDPP and dual qDPP.

Some questions

Let X be a subspace of $C(K)$, K scattered. Suppose that X has DPP.

1. Does X^* have the Schur property?
2. Does X have the dual qDPP?
3. Suppose that each subspace of X has DPP. Is then X isomoprhic to a subspace of $C(L)$, where L has finite height?

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Quantitative versions

Dichotomies on the unit ball

Extracting a subsequence

Let X be a Banach space and $A \subset X$

- A is relatively norm-compact iff any $(x_k) \subset A$ admits a norm-convergent subsequence.
- A is relatively weakly compact iff any $(x_k) \subset A$ admits a weakly convergent subsequence.
- A contains no ℓ_1-sequence iff any $(x_k) \subset A$ admits a weakly Cauchy subsequence.
- A is Banach-Saks if any $(x_k) \subset A$ admits a subsequence Cesàro convergent in the norm.
- A is weakly Banach-Saks if any weakly convergent $(x_k) \subset A$ admits a subsequence Cesàro convergent in the norm.
- If $X = Y^*$, A is relatively weak*-sequentially compact if any $(x_k) \subset X$ admits a weak*-convergent subsequence.

Quantities measuring extraction of a subsequence

Let (x_k) be a bounded sequence in a Banach space.

- $\widetilde{\text{ca}}(x_k) = \inf\{\text{ca}(x_{n_k}) : (x_{n_k}) \text{ is a subsequence of } (x_k)\}$
- Similarly we define $\widetilde{\delta}_w$ and $\widetilde{\delta}_{w^*}$.
- $\text{cca}(x_k) = \text{ca}\left(\frac{x_1 + \cdots + x_k}{k}\right)$
- $\widetilde{\text{cca}}$ is defined by the above pattern.

Quantification of the properties

Let $A \subset X$ be bounded.

- $\beta(A) = \sup\{\widetilde{\operatorname{ca}}(x_k) : (x_k) \subset A\}$
  (measure of norm non-compactness, equivalent to the Hausdorff and Kuratowski measures)
- $\operatorname{Ros}(A) = \sup\{\widetilde{\delta}_w(x_k) : (x_k) \subset A\}$
  Quantification of the Rosenthal ℓ^1-theorem proved in [E.Behrends, 1995]:
  - X real $\Rightarrow$
    $\sup\{c > 0 : (\exists (x_k) \subset A)(\|\sum \alpha_i x_i\| \geq c \sum |\alpha_i|)\} = \frac{1}{2} \operatorname{Ros}(A)$
  - X complex $\Rightarrow \frac{1}{4\sqrt{2}} \operatorname{Ros}(A) \leq \sup\{c > 0 : \dots\} \leq \frac{1}{2} \operatorname{Ros}(A)$

Quantification of the properties – continued

Let $A \subset X$ be bounded.

- $\mathrm{wck}(A) = \sup\{\mathrm{dist}(\mathrm{clust}_{w^*}(x_k), X) : (x_k) \subset A\}$
(measure of weak non-compactness equivalent to some other ones)
- If $X = Y^*$, let $\mathrm{Seq}(A) = \sup\{\widetilde{\delta}_{w^*}(x_k) : (x_k) \subset A\}$
(measure of weak* sequential non-compactness, not yet studied)

Quantification of the properties – continued once more

Let $A \subset X$ be bounded.

- $\mathrm{bs}(A) = \sup\{\widetilde{\mathrm{cca}}(x_k) : (x_k) \subset A\}$
  (measure of the failure of the Banach-Saks property)
- $\mathrm{wbs}(A) = \sup\{\widetilde{\mathrm{cca}}(x_k) : (x_k) \subset A \text{ weakly convergent}\}$
  (measure of the failure of the weak Banach-Saks property)

Quantitative relationships of the properties

Let $A \subset X$ be bounded.

A is relatively norm compact.

$$\Downarrow$$

A is a Banach-Saks set.

$$\Updownarrow$$

A is relatively weakly compact and a weak Banach-Saks set.

Theorem [BKS 2014]

$$\max\{\operatorname{wck}(A), \operatorname{wbs}(A)\} \leq \operatorname{bs}(A) \leq \beta(A).$$

The remaining implication is purely qualitative.

Quantities applied to the unit balls

Let X be a Banach space.

- $\dim X < \infty \Rightarrow \beta(B_X) = 0$ [Bolzano-Weierstrass theorem]
- $\dim X = \infty \Rightarrow \beta(B_X) \in [1, 2]$ [Riesz lemma]
- $\beta(B_{\ell^p}) = 2^{1/p}$ for $p \in [1, \infty)$ [exercise]

- X reflexive $\Rightarrow \mathrm{wck}(B_X) = 0$ [Eberlein-Šmulyan theorem]
- X non-reflexive $\Rightarrow \mathrm{wck}(B_X) = 1$
  [A.S.Granero, J.M.Hernández and H.Pfitzner 2011]

- $\ell_1 \not\subset X \Rightarrow \mathrm{Ros}(B_X) = 0$ [Rosenthal ℓ^1-theorem]
- $\ell_1 \subset X \Rightarrow \mathrm{Ros}(B_X) = 2$ [James distortion theorem]

Quantities applied to the unit balls – continued

Let X be a Banach space.

- X has the weak Banach-Saks property $\Rightarrow \mathrm{wbs}(B_X) = 0$ [trivial]
- X fails the weak Banach-Saks property $\Rightarrow \mathrm{bs}(B_X) = \mathrm{wbs}(B_X) = 2$ [BKS 2014]
  - $\mathrm{bs}(B_{C[0,1]}) = \mathrm{wbs}(B_{C[0,1]}) = 2$ [Schreier 1930]
  - $\exists X$ reflexive with $\mathrm{bs}(B_X) = \mathrm{wbs}(B_X) = 2$ [Baernstein II 1972]

- X has the Banach-Saks property $\Rightarrow \mathrm{bs}(B_X) = 0$ [trivial]
- [BKS 2014] X fails the Banach-Saks property $\Rightarrow \mathrm{bs}(B_X) \in [1, 2]$
  In particular:
  - $\mathrm{wbs}(B_{\ell^1}) = 0$, $\mathrm{bs}(B_{\ell^1}) = 2$
  - $\mathrm{wbs}(B_{L^1}) = 0$ [Szlenk 1965], $\mathrm{bs}(B_{L^1}) = 2$
  - $\mathrm{wbs}(B_{c_0}) = 0$ [Farnum 1974], $\mathrm{bs}(B_{c_0}) = 1$
  - $\mathrm{wbs}(B_c) = 0$, $\mathrm{bs}\, B_c = 2$

On the dichotomy for the quantity wbs

- Let X fail the weak Banach-Saks property.
- There is a weakly null sequence $(x_k) \subset B_X$ and $\delta > 0$ such that

$$\operatorname{card} F \leq \min F \Rightarrow \left\| \sum_{i \in F} \alpha_i x_i \right\| \geq \delta \sum_{i \in F} |\alpha_i|$$

  (x_k generates an ℓ_1-spreading model with constant δ.)
- For each $\varepsilon > 0$ there is $(y_k) \subset B_X$ weakly null generating an ℓ_1-spreading model with constant $1 - \varepsilon$.
- $\widetilde{\operatorname{cca}}(y_{k^3}) \geq 2(1 - \varepsilon)$.
- Hence, $\operatorname{wbs}(B_X) \geq 2$.

A question

Let (x_k) be a weakly null sequence and $c > d > 0$. Let

$$\mathcal{F} = \{\{k \in \mathbb{N} : |x^*(x_k)| > c\} : x^* \in B_{X^*}\}$$

Suppose that for each $M \subset \mathbb{N}$ infinite the cardinalities of $F \cap M$, $F \in \mathcal{F}$ are not bounded.
Is there a subsequence generating an ℓ_1-spreading model with the constant d?

Remark: YES, if $d < \frac{c}{2}$.

Thank you for your attention.