Which algebras of operators on Banach spaces are Grothendieck spaces?

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Theorem (Grothendieck). *Let $(f_n)_{n=1}^{\infty}$ be a weak*-convergent sequence in $\ell_\infty(\Gamma)^*$. Then $(f_n)_{n=1}^{\infty}$ converges weakly.*

Definition: Let us call a Banach space X for which weak*-convergent sequences in X^* converge weakly *a Grothendieck space*.

Examples: reflexive spaces, quotients (hence complemented subspaces) of Grothendieck spaces, injective spaces *etc.*

> According to Corollary 2.12 and the remarks following it, $B(\Sigma)$ is a Grothendieck space if Σ is a σ-field. Grothendieck spaces are close to the heart of every vector measure theorist. A close relationship exists between $B(\mathscr{F})$ spaces that are Grothendieck spaces, the validity of the Vitali-Hahn-Saks theorem for measures on the field $\mathscr{F}$ and the validity of the Nikodým Boundedness Theorem for measures defined on $\mathscr{F}$ (see Faires **[1976]** and Seever **[1968]**). This fact further accentuates the importance of the study of the interrelationships between Grothendieck spaces and vector measure theory. Here is a list of reformulations of the definition of Grothendieck spaces. For proofs, see Diestel **[1973c]**, Faires **[1974b]**, Grothendieck **[1953]**.

Figure: J. Diestel, J. R. Uhl, Vector measures, p. 179

An easy exercise on the Eberlein–Šmulian theorem: A separable space is Grothendieck if and only if it is reflexive.

A non-example: c_0. If follows from the above but it can be seen directly: take the canonical basis $(e_k)_{k=1}^{\infty}$ of $\ell_1 \cong c_0^*$. It is w*-null, however it does *not* converge weakly as $\langle ((-1)^n)_{n=1}^{\infty}, e_k \rangle$ fails to converge whenever $k \to \infty$.

Grothendieck spaces — why bother? *A typical application: to show that, for instance, c_0 is not a quotient of ℓ_∞ or that separable quotients of ℓ_∞ are separable.*

Theorem. *Let X be a Banach space. Then the following conditions are equivalent:*

1. X is a Grothendieck space
2. every operator $T\colon X \to c_0$ is weakly compact;
3. for every separable space X, each operator $T\colon X \to Y$ is weakly compact.

Some properties:

— Duals of Grothendieck spaces are weakly sequentially complete (easy).

— Every Grothendieck space X has *Pełczyński's property* (V): for each Banach space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact[1].

— Spaces with property (V) without complemented copies of c_0 are Grothendieck (Räbiger 1984)

Theorem. (Pełczyński 1962, Cembranos 1988). $C(K)$-spaces have property (V). Consequently, a $C(K)$-space is Grothendieck if and only if it does not contain complemented copies of c_0.

[1] weakly compact operators are always unconditionally converging

Diestel and Uhl wrote in their famous monograph [p. 180]:

> *Finally, there is some evidence (Akemann [1967], [1968]) that the space $\mathscr{L}(H; H)$ of bounded linear operators on a Hilbert space is a Grothendieck space and that more generally the space $\mathscr{L}(X; X)$ is a Grothendieck space for any reflexive Banach space X.*

Akemann was interested in conditional expectations (norm-one projections) from $\mathscr{B}(H)$, the algebra of operators on a Hilbert space, onto separable sub-C*-algebras; he proved that separable C*-algebras are never complemented in $\mathscr{B}(H)$.

Theorem (Pfitzner 1994). C*-algebras have property (V). In particular, a C*-algebra is a Grothendieck space if and only if it does not contain a complemented copy of c_0; von Neumann algebras (dual C*-algebras) are thus Grothendieck.

Corollary. For any Hilbert space H, $\mathscr{B}(H)$ is a Grothendieck space.

Question. For what E, the space $\mathscr{B}(E)$ is Grothendieck?

Let us note that the question of the Grothendieck property is interesting only for E reflexive; in the non-reflexive case $\mathscr{B}(E)$ is never Grothendieck.

Easy fact. Let E be a Banach space. Then both E and E^* are isomorphic to complemented subspaces of $\mathscr{B}(E)$.

Proof. Choose a norm-one elements $x_0 \in E$ and $\lambda_0 \in E^*$ such that $\langle x_0, \lambda_0 \rangle = 1$. Consider the map $x \ni E \mapsto x \otimes \lambda_0$, which is an isomorphic embedding of E into $\mathscr{B}(E)$. It has a left inverse given by $\mathscr{B}(E) \ni T \mapsto Tx_0 \in E$. A similar proof works for E^*. $\square$

Corollary. If $\mathscr{B}(E)$ is Grothendieck, then so are E and E^*.

Proposition. A Banach space E is reflexive if and only if E and E^* are Grothendieck.

Proof. Suppose that E is not reflexive. It follows from the Eberlein–Šmulian theorem that there is a bounded sequence $(x_n)_{n=1}^{\infty}$ in E such that $\{x_n \colon n \in \mathbb{N}\}$ is not relatively weakly compact. By a result of Godefroy and Saab (1986); since E is a subspace of a dual of Grothendieck space, we may find an infinite set $A \subset \mathbb{N}$ such that the closed linear span of $\{x_n \colon n \in A\}$ is complemented in E and isomorphic to ℓ_1. $\square$

We shall need the following fact (probably first observed by W. B. Johnson):

Lemma. The Banach space $F = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{\ell_\infty}$ contains an (isometric) complemented copy of ℓ_1.

A reflexive space E for which $\mathscr{B}(E)$ is not Grothendieck. (K. 2013)

Let $p \in (1,\infty)$ and consider

$$E = \Big(\bigoplus_{n=1}^\infty \ell_1^n\Big)_{\ell_p}.$$

(We identify $T \in \mathscr{B}(E)$ with a matrix $(T_{ij})_{i,j=1}^\infty$ where $T_{ij} \in \mathscr{B}(\ell_1^j, \ell_1^i)$.) To complete the proof it is enough to embed F as a complemented subspace of $\mathscr{B}(E)$.
Let us identify ℓ_1^n with a 1-complemented subspace of $\mathscr{B}(\ell_1^n)$ via the mapping

$$e_k \mapsto e_k \otimes e_1^* \quad (k \leqslant n,\ n \in \mathbb{N}),$$

where e_1^* stands for the coordinate functional associated with e_1.
The space $D = (\bigoplus_{n=1}^\infty \mathscr{B}(\ell_1^n))_{\ell_\infty}$ contains a complemented copy of F. Let $\Delta\colon D \to \mathscr{B}(E)$ be the *diagonal embedding*, that is,

$$\Delta((T_n)_{n=1}^\infty) = \operatorname{diag}(T_1, T_2, \ldots) \quad ((T_n)_{n=1}^\infty \in D).$$

The map Δ is well-defined since the decomposition of E into the subspaces $\ell_1^1, \ell_1^2, \ldots$ is unconditional.

It is enough to notice that Δ has a left-inverse $\Xi\colon \mathscr{B}(E) \to D$ given by

$$\Xi(T_{ij})_{i,j\in\mathbb{N}} = (T_{ii})_{i=1}^{\infty} \quad ((T_{ij})_{i,j\in\mathbb{N}} \in \mathscr{B}(E)),$$

which is bounded.

With each operator $T = (T_{ij})_{i,j\in\mathbb{N}} \in \mathscr{B}(E)$ we shall associate a sequence $(S^{(n)})_{n=1}^{\infty}$ of operators such that for each $n \in \mathbb{N}$ we have $\|S^{(n)}\| \leqslant \|T\|$ and the matrix of $S^{(n)}$ agrees with the matrix of the diagonal operator $\operatorname{diag}(-T_{11}, \ldots, -T_{nn}, 0, 0, \ldots)$ at entries (i,j) with $i \leqslant n$ or $j \leqslant n$.

This will immediately yield that

$$\|\Xi(T)\| = \sup_{n\in\mathbb{N}} \|T_{nn}\| = \sup_{n\in\mathbb{N}} \|-S^{(n)}_{nn}\| \leqslant \sup_{n\in\mathbb{N}} \|S^{(n)}\| \leqslant \|T\|.$$

Define operators $T_{\mathrm{k},n}$, $T_{\mathrm{r},n}$ $(n \geqslant 1)$ which have the same columns and rows as T respectively, except the first n ones, where we instead set $(T_{\mathrm{k},n})_{ij} = -T_{ij}$ and $(T_{\mathrm{r}})_{ji} = -T_{ji}$ for $j \in \{1, \ldots, n\}$ and $i \in \mathbb{N}$. Certainly, $\|T\| = \|T_{\mathrm{k},n}\| = \|T_{\mathrm{r},n}\|$ for all $n \in \mathbb{N}$ and the norm of $S^{(n)} := (T_{\mathrm{k},n} + T_{\mathrm{r},n})/2$ does not exceed the norm of T. Consequently, $\big(S^{(n)}\big)_{n=1}^{\infty}$ is the desired sequence. $\square$

Observation. There is nothing really specific about this space; all we need are the following ingredients:

- X is reflexive;
- X contains ℓ_1^n's uniformly complemented;
- X has an unconditional basis (or more generally, $\mathscr{B}(X) \cong \ell_\infty(\mathscr{B}(X))$ as Banach spaces)

Theorem (Johnson 1974). Suppose that X is a Banach space with local unconditional structure. Then either X is super-reflexive or X contains ℓ_∞^n's uniformly or X contains ℓ_1^n's uniformly complemented.

Example. Tsierlson's space T (actually the dual of Tsirelson's original construction) – it has an unconditional basis and is easily seen that it is *not* super-reflexive.

Observation. For E reflexive, $\mathscr{B}(E)$ and $\mathscr{B}(E^*)$ are isomorphic, so $\mathscr{B}(E)$ is Grothendieck if and only $\mathscr{B}(E^*)$ is.

Another observation. Suppose that E is a reflexive space with a basis. Then $\mathscr{K}(E)^{**} \cong \mathscr{B}(E)$. It cannot happen that $\mathscr{B}(E)$ is Grothendieck and $\mathscr{K}(E)$ is of finite codimension in $\mathscr{B}(E)$.

Indeed, $\mathscr{K}(E)$ is separable so would be $\mathscr{B}(E)$. The Grothendieck property of $\mathscr{B}(E)$ would force to be reflexive, however $\mathscr{B}(E)$ is reflexive if and only if E is finite-dimensional.

Question. Is it true that if E is super-reflexive, then $\mathscr{B}(E)$ is Grothendieck? What if $E = \ell_p$ for $p \neq 1, 2, \infty$?

If yes, then there would be no super-reflexive analogues of the Argyros–Haydon space.

A weaker question: Is $\mathscr{B}(\ell_p)^*$ weakly sequentially complete for $p \neq 1, 2, \infty$?

Not sure if it helps, but by a result of Daws and Read, if E is super-reflexive then $\mathscr{B}(E)^{**}$ is a 1-complemented subalgebra of $\mathscr{B}(F)$ for some super-reflexive F; this however is another story...

Theorem (Argyros–Haydon 2011). There is a Banach space X_{AH} which has the following three remarkable properties:

(i) X_{AH} has very few operators, in the sense that each operator on X_{AH} is a compact perturbation of a scalar multiple of the identity;

(ii) X_{AH} has a Schauder basis;

(iii) the dual space of X_{AH} is isomorphic to ℓ_1.