Stability of vector measures and non-trivial twisted sums of c_0

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- to report on several results describing vector analogues of the Kalton–Roberts theorem on nearly additive set functions (for this we will need some background on twisted sums of Banach spaces);

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T.K., *Stability of vector measures and twisted sums of Banach spaces*, J. Funct. Anal. **264** (2013), 2416–2456.

T.K., *Stability of vector measures and non-trivial extensions of c_0*, in preparation.

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If the above condition is valid, then we say that X has the SVM **property**.

N.J. Kalton, J.W. Roberts, *Uniformly exhaustive submeasures and nearly additive set functions*, Trans. Amer. Math. Soc. **278** (1983), 803–816.

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SVM character

Let κ be a cardinal number. We say that a Banach space X has the **κ-SVM property** if and only if there exists a constant $v(\kappa, X) < \infty$ (depending only on κ and X) such that given any algebra $\mathcal{F} \subset 2^{\Omega}$ **of cardinality less than κ**, and any map $\nu\colon \mathcal{F} \to X$ satisfying

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If X is a Banach space which does not have the SVM property, then by the **SVM character** of X we mean the minimal cardinal number κ such that X does not have the κ-SVM property, and we denote it by $\tau(X)$.

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If X is a Banach space which does not have the SVM property, then by the **SVM character** of X we mean the minimal cardinal number κ such that X does not have the κ-SVM property, and we denote it by $\tau(X)$.

Remark. Note that $\tau(X)$ is properly defined for every Banach space not enjoying the SVM property.

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Remark. Note that $\tau(X)$ is properly defined for every Banach space not enjoying the SVM property. (That is, if X has the κ-SVM property for each κ, then X has the SVM property.)

Basic observations concerning the SVM character

Let us recall our basic assumption on a given function $\nu \colon \mathcal{F} \to X$:

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- **$\tau(X) \geq \omega$ for every Banach space X.** [Proof] Let $\mathcal{F} \subset 2^\Omega$ be a finite algebra of sets and $\nu\colon \mathcal{F} \to X$ satisfy $(*)$. We may assume that $\mathcal{F} = 2^\Omega$ and let $n = |\Omega|$. By a simple induction we get the inequality

$$\left\| \nu(A) - \sum_{a \in A} \nu\{a\} \right\| \leq |A| - 1 \quad \text{for } A \in \mathcal{F},$$

thus the measure $\mu\colon \mathcal{F} \to X$, defined by $\mu\{a\} = \nu\{a\}$ for $a \in \Omega$, does the job. Consequently, for every Banach space X we have $\tau(X) \geq \omega$ and $v(2^n, X) \leq n - 1$ for each $n \in \mathbb{N}$.

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- **If $\tau(X) > \omega$ and X is complemented in its bidual, then X has the SVM property. Moreover, if there is a projection of X^{**} onto X with norm not exceeding λ, then $v(X) \leq \lambda v(\omega, X)$.**

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- **$\tau(X) \geq \omega$ for every Banach space X.**
- **If $\tau(X) > \omega$ and X is complemented in its bidual, then X has the** SVM **property. Moreover, if there is a projection of X^{**} onto X with norm not exceeding λ, then $v(X) \leq \lambda v(\omega, X)$.** [Proof] Let $\mathcal{F}$ be an arbitrary algebra of sets and let Γ be the set of all finite subalgebras of $\mathcal{F}$, directed by the inclusion. We use the assumption $\tau(X) > \omega$ and the compactness of the unit ball of X^{**} with respect to the w^*-topology to produce an approximating measure with values in X^{**}. Next we just have to project it onto X.

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- **$\tau(c_0) > \omega$, i.e. c_0 satisfies the ω-**SVM **property.** [Proof] Let $\mathcal{F}$ be a finite algebra. Choose any $\varepsilon \in (0,1)$ and pick an $n \in \mathbb{N}$ such that $|e_j^*(\nu(A))| < \varepsilon$ for each $j > n$ and $A \in \mathcal{F}$. For each jth coordinate $(1 \leq j \leq n)$ there is an additive set function $\mu_j\colon \mathcal{F} \to \mathbb{R}$ satisfying $|e_j^*(\nu(A)) - \mu_j(A)| \leq K$ for $A \in \mathcal{F}$. Then the measure $\mu\colon \mathcal{F} \to c_0$ defined by $\mu(A) = (\mu_1(A), \ldots, \mu_n(A), 0, 0, \ldots)$ satisfies $\|\nu(A) - \mu(A)\| \leq K$ for $A \in \mathcal{F}$. We get $v(\omega, c_0) = K$.

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- **If** $\tau(X)>\omega$ **and** X **is complemented in its bidual, then** X **has the** SVM **property. Moreover, if there is a projection of** X^{**} **onto** X **with norm not exceeding** λ**, then** $v(X)\leq\lambda v(\omega,X)$**.**
- $\tau(c_0)>\omega$**, i.e.** c_0 **satisfies the** ω-SVM **property.**
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- $\tau(c_0) > \omega$**, i.e.** c_0 **satisfies the** ω-SVM **property.**
- $\tau(C[0,1]) > \omega$**, i.e.** $C[0,1]$ **satisfies the** ω-SVM **property.** [Proof] We use the uniform continuity of $\nu(A) \in C[0,1]$ (for $A \in \mathcal{F}$).

Let X, Y, Z be F-spaces. A short **exact sequence** is a diagram

$$(*) \qquad 0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{q} X \longrightarrow 0,$$

where $i\colon Y \to Z$ is a one-to-one operator with a closed range (embedding) and $q\colon Z \to X$ is a surjective operator such that $\mathrm{im}(i) = \ker(q)$.

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In other words, Z contains a closed subspace $Y_1 \simeq Y$ (isomorphically) such that the quotient space $Z/Y_1 \simeq X$. We then say that Z is a **twisted sum** of Y and X (in this order!), or that Z is an **extension** of X by Y.

In fact, twisted sums are identified via the following natural equivalence relation:

Twisted sums machinery

Equivalence between exact sequences

In fact, twisted sums are identified via the following natural equivalence relation:

We say that two exact sequences of F-spaces $0 \to Y \to Z_1 \to X \to 0$ and $0 \to Y \to Z_2 \to X \to 0$ are **equivalent**, if there exists an operator $T \colon Z_1 \to Z_2$ such that the diagram

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is commutative.

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We write $\mathrm{Ext}(X, Y) = 0$ if every exact sequence $(*)$, where Z is a Banach space, splits. We say that the pair (X, Y) splits if $(*)$ splits for every locally bounded F-space Z (so, this is something stronger).

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- c_0 and ℓ_∞, as well as all $\mathscr{L}_\infty$-spaces, **are** $\mathcal{K}$-spaces (Kalton, Roberts, 1983); this is why Kalton and Roberts proved their theorem on stability of nearly additive real-valued set functions!

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Some classical results

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Similarly, $\tau(C[0, \omega^\omega]) = \omega_1$.

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By a result of Cabello Sánchez and Castillo, we have $\mathrm{Ext}(c_0, \ell_1) \neq 0$, thus Theorem 2 implies that $\tau(\ell_1) \leq \omega_1$ and so $\tau(\ell_1) = \omega$ because ℓ_1 is certainly complemented in its bidual.

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Next, following the construction of the Johnson-Lindenstrauss space, we build a non-splitting exact sequence of the form

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Therefore, by our necessary condition, we infer that $\tau(m_0(\Gamma)) \leq \Gamma^{++}$.

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Let Γ be an infinite cardinal number. Then the space $m_0(\Gamma)$ has the $\mathrm{cf}(\Gamma)^+$-SVM property with $v(\mathrm{cf}(\Gamma)^+, m_0(\Gamma)) \leq 16K < 720$.

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Corollary

For every infinite cardinal Γ we have

$$\tau(c_0(\Gamma)) = \omega_2.$$

κ-injectivity and κ-SVM property

Let κ be a cardinal. A Banach space X is called **κ-injective** if for every Banach space E, with density character less than κ, and every subspace $F \subset E$, every operator $t \colon F \to X$ admits an extension to an operator $T \colon E \to X$. If for some $\lambda \geq 1$ there is always such an extension with $\|T\| \leq \lambda \|t\|$, then we say that X is **(λ, κ)-injective**.

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In the case $\kappa = \omega_1$ we say that X is separably injective.

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Theorem 4

If X is a (λ, κ)-injective Banach space, then X has the κ-SVM property and $v(\kappa, X) \leq 24\lambda K$.

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Theorem 4

If X is a (λ, κ)-injective Banach space, then X has the κ-SVM property and $v(\kappa, X) \leq 24\lambda K$. For instance, if Ω is compact Hausdorff and of finite height n, then $v(\omega_1, C(\Omega)) \leq 24(2n-1)K$.

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[Proof] The vector spaces $\mathrm{Ext}(Z, Y^*)$ and $\mathrm{Ext}(Y, Z^*)$ are isomorphic (Jebreen, Jamjoom, Yost), so $\mathrm{Ext}(X^*, \ell_1) = 0$ is equivalent to $\mathrm{Ext}(c_0, X^{**}) = 0$ which in turn is equivalent to X^{**} having the SVM property.

Non-trivial extensions of c_0

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Now, we shall explain how their construction may be strengthened by using the following assertion:

Theorem 6

Every infinite-dimensional Banach space having the ω_1-SVM property contains an isomorphic copy of c_0.

Sketch of the proof of Theorem 6

- Consider the pull-back diagram, where the first row is the Kalton–Peck twisted sum and $j^*\colon L_\infty \to \ell_2$ is the adjoint to the embedding $j\colon \ell_2 \hookrightarrow L_1$ given by Rademacher functions:

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\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_2 & \longrightarrow & \mathcal{Z}_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\
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  ($j^*f = (r_j^*f)_{j=1}^{\infty} \in \ell_2$, where $r_j^*f = \int_0^1 r_j(t)f(t)\,\mathrm{d}t$)
- The space $\mathcal{Z}_2$ is the completion of the direct sum $\ell_2 \oplus \ell_2$ under the quasi-norm given by $\|(y,x)\| = \|x\| + \|y - \varphi(x)\|$, where $\varphi\colon \ell_2 \to \ell_2$ is a *quasi-linear map*, that is, it is homogeneous and satisfies

$$\|\varphi(x+y) - \varphi(x) - \varphi(y)\| \leq c(\|x\| + \|y\|) \quad \text{for all } x, y \in \ell_2,$$

  where $c < \infty$ is a constant.

Sketch of the proof of Theorem 6 continued

- φ may be given as a quasi-linear extension of a map $\varphi_0\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $\varphi_0(x) = \|x\| F(x/\|x\|)$ (0 for $x = 0$), where $F\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is a quasi-additive map of the form:

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$$F(x)(k) = x(k) \cdot \theta(-\log|x(k)|) \quad \text{(convention: } 0 \cdot \infty = 0\text{)},$$

  where $\theta \colon \mathbb{R} \to \mathbb{R}$ is an (unbounded) Lipschitz function with $\theta(t) = 0$ for $t \leq 0$.

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where $L = \mathrm{Lip}(\theta)$, and hence it may be proved that the constant of quasi-linearity of φ is at most $16L \cdot \log 2$.

Sketch of the proof of Theorem 6 continued

- Let $\mathcal{F} = \{A \subset \mathbb{N} \colon |A| < \omega \text{ or } |\mathbb{N} \setminus A| < \omega\}$. We look for a suitable sequence $(f_n)_{n=1}^{\infty} \subset L_\infty$, equivalent to the canonical basis of c_0, for which we will define an 'almost' additive measure $\nu \colon \mathcal{F} \to \ell_2$ by the formula

$$\nu(A) = \begin{cases} \varphi\left(j^* \sum_{n \in A} f_n\right) & \text{for } A \text{ finite },\\ -\nu(\mathbb{N} \setminus A) & \text{for } A \text{ cofinite.} \end{cases}$$

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- So, for finite $A \subset \mathbb{N}$ we have

$$\nu(A)(k) = (r_k^* f_A) \cdot \theta\left(-\log \frac{|r_k^* f_A|}{\|(r_j^* f_A)_{j=1}^{\infty}\|_2}\right),$$

where $f_A := \sum_{n \in A} f_n$.

Sketch of the proof of Theorem 6 continued

- We choose f_n's so that they are disjointly supported characteristic functions of some unions of 'Rademacher's subintervals' of $[0, 1]$, and so that for each $n \in \mathbb{N}$ we have:
  - $\|(r_j^* f_n)_{j=1}^{\infty}\|_2 \sim \frac{1}{n}$;

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- In this way we define a function $\nu \colon \mathcal{F} \to \ell_2$ satisfying the following three conditions:

Sketch of the proof of Theorem 6 continued

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  Then $\|\nu\{n\}\| \sim \frac{\theta(\frac{1}{2}n)}{n^{1/2}}$ for each $n \in \mathbb{N}$.
- In this way we define a function $\nu \colon \mathcal{F} \to \ell_2$ satisfying the following three conditions:
  (i) ν is 1-additive, that is, for every $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, we have $\|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1$;
  (ii) for each $n \in \mathbb{N}$ we have $\|\nu\{n\}\| \geq M$, where M is arbitrarily fixed positive number;
  (iii) ν is bounded.

Sketch of the proof of Theorem 6 continued

- Now, suppose that an infinite-dimensional Banach space has the ω_1-SVM property and take M (from the previous slide) to be larger than $v(\omega_1, X)$.

Sketch of the proof of Theorem 6 continued

- Now, suppose that an infinite-dimensional Banach space has the ω_1-SVM property and take M (from the previous slide) to be larger than $v(\omega_1, X)$. Use the Dvoretzky theorem (ℓ_2 is finitely representable in X) to 'pack' the function ν into X.

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- There must exist a finitely additive vector measure $\mu\colon \mathcal{F} \to X$ which approximates ν to within $v(\omega_1, X)$.

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- There must exist a finitely additive vector measure $\mu\colon \mathcal{F} \to X$ which approximates ν to within $v(\omega_1, X)$. But then the series $\sum_{n=1}^{\infty} \mu\{n\}$ certainly diverges.

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Sketch of the proof of Theorem 6 continued

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- There must exist a finitely additive vector measure $\mu\colon \mathcal{F} \to X$ which approximates ν to within $v(\omega_1, X)$. But then the series $\sum_{n=1}^{\infty} \mu\{n\}$ certainly diverges.
- Consequently, we have produced a bounded and non-strongly additive measure with values in X, and hence the Diestel–Faires theorem tells us that X contains an isomorphic copy of c_0.

Non-trivial extensions of c_0

Corollary

For every infinite-dimensional Banach space X that is complemented in its bidual and does not contain isomorphically c_0 (e.g. for any infinite-dimensional reflexive space) we have $\mathrm{Ext}(c_0, X) \neq 0$.

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Proof. Combine Theorems 5 and 6.