The variation and semivariations of a vector measure

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The main body of the talk falls into three parts, each of them being based on a separate paper of mine. We start with some notation and definitions to be used throughout the talk.

S – nonempty set

Σ – σ-algebra of subsets of S

$\nu\colon \Sigma \to [0, \infty]$ – (positive) measure, i.e., σ-additive and $\nu(\varnothing) = 0$

Def. $A \in \Sigma$ is a *ν-atom* if $\nu(A) > 0$ and for every $E \in \Sigma$ with $E \subset A$ we have $\nu(E) = 0$ or $\nu(A \setminus E) = 0$.

Def. ν is *nonatomic* if it has no atom.

For a measure $\mu: \Sigma \to [0, \infty]$ we write $\nu \ll \mu$ if for every $E \in \Sigma$ with $\mu(E) = 0$ we have $\nu(E) = 0$, and $\nu \equiv \mu$ if both $\nu \ll \mu$ and $\mu \ll \nu$ hold.

$(X, \|\cdot\|)$ – Banach space (real or complex)

$ca(\Sigma, X) := \{\varphi: \Sigma \to X : \varphi \text{ is } \sigma\text{-additive}\}$

$\|\varphi\| = \sup\{\|\varphi(E)\| : E \in \Sigma\}$

$$cca(\Sigma, X) := \{\varphi \in ca(\Sigma, X) : \varphi(\Sigma) \text{ is relatively compact}\}$$

$|\varphi|$ stands for the variation of φ. In particular, $|\varphi|$ is a measure on Σ.

Thm. (Bartle–Dunford–Schwartz (1955)). *For every $\varphi \in ca(\Sigma, X)$ there exists $\lambda \in ca_+(\Sigma)$ with $\lambda \equiv |\varphi|$.*

$\mathcal{E}_\nu(X) := \{\varphi \in ca(\Sigma, X) : |\varphi| = \nu\}$

Part I. Variation

Thm. 1. *For a nonatomic measure* $\nu\colon\Sigma \to [0,\infty]$ *the following conditions are equivalent:*
(i) there exists $\lambda \in ca_+(\Sigma)$ *with* $\lambda \equiv \nu$*;*
(ii) there exists a Banach space Z such that $\mathcal{E}_\nu(Z) \neq \varnothing$*;*
(iii) given an infinite-dimensional Banach space X, we have $\mathcal{E}_\nu(X) \cap cca(\Sigma, X) \neq \varnothing$*.*

Cor. 1. *Let* X *and* Z *be Banach spaces and let* X *be infinite-dimensional. For every nonatomic* $\psi \in ca(\Sigma, Z)$ *there exists* $\varphi \in cca(\Sigma, X)$ *with* $|\varphi| = |\psi|$*.*

The proof of Thm. 1, (i)$\Longrightarrow$(iii), is based on the following lemma.

Lemma 1. *Let $\lambda \in ca_+(\Sigma)$ be nonatomic and let X be an infinite-dimensional Banach space. Then*
(a) for every $\varepsilon > 0$ there exists $\varphi_\varepsilon \in ca(\Sigma, X)$ with $\varphi_\varepsilon(\Sigma)$ finite-dimensional, $\|\varphi_\varepsilon\| < \varepsilon$ and $|\varphi_\varepsilon| = \lambda$;
(b) there exists $\varphi \in cca(\Sigma, X)$ with $|\varphi| = \infty \cdot \lambda$.

(a) is a simple consequence of the Dvoretzky–Rogers lemma. It is implicit in Janicka–Kalton (1977).

(b) is due essentially to E. Thomas (1974) (for λ Lebesgue measure on $[0,1]$). A new proof, based on (a) and the closed graph theorem, was subsequently given by Janicka–Kalton. Still another proof of (b) is due to Drewnowski–ZL (1995). We define φ rather explicitly as the sum of a series of λ-simple measures, avoiding the closed graph theorem.

Another ingredient of the proof of Thm. 1, (i)$\Longrightarrow$(iii), is a Σ-partition $\{S_0, S_1, S_2, ...\}$ of S such that $\nu(S_0 \cap E) = 0$ or ∞ for $E \in \Sigma$ and $\nu(S_i) < \infty$ for $i \geq 1$.

Def. A measure $\nu: \Sigma \to [0, \infty]$ has *property* (a_0) if the following conditions hold:

(1) there exists $\lambda \in ca_+(\Sigma)$ with $\lambda \equiv \nu$;

(2) $\nu(A) < \infty$ for every ν-atom A;

(3) $(\nu(A_i)) \in c_0$ whenever (A_i) is a sequence of pairwise disjoint ν-atoms.

It is plain that $|\varphi|$, where $\varphi \in ca(\Sigma, X)$, has property (a_0).

Thm. 2. *For a measure* $\nu: \Sigma \to [0, \infty]$ *the following conditions are equivalent:*
(i) ν *has property* (a_0)*;*
(ii) there exists a Banach space Z *such that* $\mathcal{E}_\nu(Z) \neq \emptyset$*;*
(iii) $\mathcal{E}_\nu(c_0) \cap cca(\Sigma, c_0) \neq \emptyset$*.*

Clearly, (iii) implies (ii). As mentioned above, (ii) implies (i). That (i) implies (iii) is a simple consequence of the corresponding implication of Thm. 1 and the decomposition of ν into its nonatomic and atomic parts. The appearance of c_0 in (iii) is due, of course, to the atomic part.

Cor. 2. *Let* Z *be a Banach space. For every* $\psi \in ca(\Sigma, Z)$ *there exists* $\varphi \in cca(\Sigma, c_0)$ *with* $|\varphi| = |\psi|$*.*

Thus, c_0 is “variation universal” in the class of Banach spaces.

Part II. Borel complexity and denseness of $\mathcal{E}_\nu(X)$

S, Σ, ν and X are the same as in Part I.

$ca(\Sigma,\nu,X) := \{\varphi \in ca(\Sigma,X) : |\varphi| \ll \nu\}$

This is, clearly, a closed subspace of $ca(\Sigma,X)$ and we have $\mathcal{E}_\nu(X) \subset ca(\Sigma,\nu,X)$.

In a special case where $\nu = \infty\cdot\lambda$ and λ is Lebesgue measure on [0, 1], the set $\mathcal{E}_\nu(X) \cap cca(\Sigma,X)$ was first studied by Anantharaman–Garg (1983). They showed that it is a dense G_δ-set in an appropriate subspace of $ca(\Sigma,X)$ provided X is infinite-dimensional. This result was generalized by Drewnowski–ZL. Namely, we replaced Lebesgue measure by an arbitrary nonatomic finite measure and $cca(\Sigma,X)$ by a more general subspace of $ca(\Sigma,X)$. We shall now present some extensions of those results, joint with Drewnowski. We shall also answer the question of when $\mathcal{E}_\nu(X)$ is an F_σ-set in $ca(\Sigma,X)$.

Thm. 3. *$\mathcal{E}_\nu(X)$ is a G_δ-set in $ca(\Sigma, X)$.*

Thm. 4. *The following conditions are equivalent:*
(i) $\mathcal{E}_\nu(X)$ is dense in $ca(\Sigma, \nu, X)$;
(ii) $\mathcal{E}_\nu(X)$ is not nowhere dense in $ca(\Sigma, \nu, X)$;
(iii) $\mathcal{E}_\nu(X)$ is of second category in $ca(\Sigma, \nu, X)$;
(iv) $\nu = 0$ or $\nu = \infty \cdot \lambda$ for some nonatomic $\lambda \in ca_+(\Sigma)$ and X is infinite-dimensional.

About the proof of Thm. 4. (i)$\Longrightarrow$(iii) by Thm. 3 and the Baire category theorem.

The nontrivial part of (iv)$\Longrightarrow$(i) is due to Drewnowski and myself.

(i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(ii) are obvious.

(ii)$\Longrightarrow$(iv) uses BDS but is otherwise elementary. In particular, one shows as a lemma that $\mathcal{E}_\nu(X)$ is always closed in $ca(\Sigma, X)$ if X is finite-dimensional.

Thm. 5. *The following conditions are equivalent:*
(i) $\mathcal{E}_\nu(X)$ *is an* F_σ*-set in* $ca(\Sigma,\nu,X)$*;*
(ii) $\mathcal{E}_\nu(X)$ *is closed in* $ca(\Sigma,\nu,X)$*;*
(iii) $\mathcal{E}_\nu(X)$ *is empty or* ν *is atomic or* X *is finite-dimensional.*

About the proof. (ii)$\Longrightarrow$(i) is obvious; (iii)$\Longrightarrow$(ii) is elementary. (A part has been just mentioned.) The main ingredient of the proof of (i)$\Longrightarrow$(iii) is the following lemma.

Lemma 2. *Let* $\lambda \in ca_+(\Sigma)$ *be nonatomic and* $\lambda \neq 0$*, and let* X *be infinite-dimensional. Then*
(a) $\mathcal{E}_\lambda(X)$ *is not an* F_σ*-set in* $ca(\Sigma,X)$*;*
(b) $\mathcal{E}_{\infty\cdot\lambda}$ *is not an* F_σ*-set in* $ca(\Sigma,X)$*.*

Part (b) is a simple consequence of Thms. 3 and 4 and the Baire category theorem.

The proof of part (a) also applies the Baire category theorem, but in another space.

Let $\lambda \in ca_+(\Sigma)$, and set

$$ca_\sigma(\Sigma, \lambda, X) = \{\varphi \in ca(\Sigma, \lambda, X) : |\varphi| \text{ is } \sigma\text{-finite}\}.$$

Denote by τ the Janicka-Kalton topology on $ca_\sigma(\Sigma, \lambda, X)$. A base of neighbourhoods of zero for τ consists of the sets $\mathcal{U}_\varepsilon$, $\varepsilon > 0$, where $\varphi \in ca_\sigma(\Sigma, \lambda, X)$ is in $\mathcal{U}_\varepsilon$ if and only if $\|\varphi\| \leq \varepsilon$ and there exists $E \in \Sigma$ with $\lambda(E) \leq \varepsilon$ and $|\varphi|(S \setminus E) \leq \varepsilon$. This is a complete metrizable linear topology on $ca_\sigma(\Sigma, \lambda, X)$.

Lemma 3. *For every $\lambda \in ca_+(\Sigma)$ the set $\mathcal{E}_\lambda(X)$ is τ-closed in $ca_\sigma(\Sigma, \lambda, X)$.*

In the proof of Lemma 2(a), we denote by τ_0 the restriction of τ to $\mathcal{E}_\lambda(X)$. We show that every $\mathcal{G} \subset \mathcal{E}_\lambda(X)$ with nonempty τ_0-interior is non-closed in $ca(\Sigma, \lambda, X)$, which is again elementary. The Baire category theorem applied to τ_0 then yields the assertion.

Part III. Semivariations

As before, Σ stands for a σ-algebra of subsets of a set S.

Def. $\eta: \Sigma \to [0, \infty)$ is a *submeasure* if it is increasing, subadditive, and order continuous (at $\varnothing$), i.e., $\eta(E_n) \to 0$ whenever (E_n) is a decreasing sequence of sets in Σ with empty intersection. The submeasure η is *separable* if Σ equipped with the Fréchet–Nikodym semimetric d_η, defined by

$$d_\eta(E, F) = \eta(E \triangle F) \text{ for } E, F \in \Sigma,$$

is separable.

Def. (G. G. Lorentz (1952)). $\eta\colon \Sigma \to [0,\infty)$ is *multiply subadditive* (*m.s.* for short) if, given $E, E_1, \ldots, E_n \in \Sigma$ and $k \in \mathbb{N}$ with $k1_E = \sum_{i=1}^n 1_{E_i}$, we have $k\eta(E) \le \sum_{i=1}^n \eta(E_i)$.

It is a special case of a result of Lorentz that a submeasure η on Σ is m.s. if and only if there exists $\Gamma \subset ca_+(\Sigma)$ such that

$$\eta(E) = \sup_{\mu\in\Gamma} \mu(E) \text{ for all } E \in \Sigma.$$

The semivariations of $\varphi \in ca(\Sigma, X)$, where X is a Banach space, are defined in the usual way:

$$\begin{aligned}\tilde{\varphi}(E) = \sup\{\| \sum_{i=1}^n t_i\varphi(E_i)\| : |t_i| \le 1,\\ E_i \in \Sigma \text{ are pairwise disjoint}\\ \text{and } \textstyle\bigcup_{i=1}^n E_i = E\};\end{aligned}$$

$$\bar{\varphi}(E) = \sup\{\|\varphi(F)\| : F \in \Sigma \text{ and } F \subset E\}.$$

Lemma 4. *Let* $\varphi \in ca(\Sigma, X)$. *Then*
(a) $\tilde{\varphi} = \sup\{|x^*\varphi| : x^* \in X^*$ *and* $\|x^*\| \le 1\}$;
(b) $\tilde{\varphi}$ *and* $\bar{\varphi}$ *are m.s. submeasures on* Σ.

Part (a) is well known, and so is (b) up to multiple subadditivity. As for $\tilde{\varphi}$, this property is a direct consequence of (a) and Lorentz' result mentioned above. As for $\bar{\varphi}$, we can use the formula

$\bar{\varphi} = \sup\{(x^*\varphi)_+, (x^*\varphi)_- : x^* \in X^*$ and $\|x^*\| \le 1\}$ if X is real, and if X is complex, we consider it to be real with the same norm.

We note that for φ, $\psi \in ca(\Sigma, X)$ the following implications hold:

$$\tilde{\varphi} = \tilde{\psi} \implies |\varphi| = |\psi| \quad \text{and} \quad \bar{\varphi} = \bar{\psi} \implies |\varphi| = |\psi|,$$

and none of them can be reversed.

Thm. 6. *For* $\eta\colon\Sigma\to[0,\infty)$ *the following conditions are equivalent:*
(i) η *is a m.s. submeasure;*
(ii) there exists a uniformly σ*-additive* $\Gamma\subset ca_+(\Sigma)$ *such that* $\sup\Gamma=\eta$*;*
(iii) there exist a Banach space X *and* $\varphi\in ca(\Sigma,X)$ *such that* $\tilde{\varphi}=\eta$*;*
(iv) there exist a Banach space X *and* $\varphi\in ca(\Sigma,X)$ *such that* $\bar{\varphi}=\eta$*;*
(v) there exist a Banach space X *and* $\varphi\in ca(\Sigma,X)$ *such that* $\tilde{\varphi}=\bar{\varphi}=\eta$*.*

Sketch of proof. (i)$\Longleftrightarrow$(ii) is a simple consequence of the result of Lorentz mentioned above. Clearly, (v)$\Longrightarrow$(iii), (iv). By Lemma 4(b), (iii)$\Longrightarrow$(i) and (iv)$\Longrightarrow$(i).

Finally, let (ii) hold, and define $\varphi\colon\Sigma\to l_\infty(\Gamma)$ by

$\varphi(E)(\gamma)=\gamma(E)$ for $E\in\Sigma$ and $\gamma\in\Gamma$.

The uniform σ-additivity of Γ implies that φ is in $ca(\Sigma, l_\infty(\Gamma))$, while the equalities of (v) follow from $\sup \Gamma = \eta$.

Thm. 7. *For $\eta\colon \Sigma \to [0, \infty)$ the following conditions are equivalent:*
(i) there exists a relatively compact $\Gamma \subset ca_+(\Sigma)$ such that $\sup \Gamma = \eta$*;*
(ii) there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\tilde{\varphi} = \eta$;
(iii) there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\bar{\varphi} = \eta$;
(iv) there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\tilde{\varphi} = \bar{\varphi} = \eta$.

The proof is similar to that of Thm. 6. A new element is the following lemma: *for $\varphi \in ca(\Sigma, X)$ we have φ is in $cca(\Sigma, X)$ if and only if $\{x^*\varphi : x^* \in X^* \text{ and } \|x^*\| \le 1\}$ is relatively compact in $ca(\Sigma)$.*

Remark. In general, it is not possible to decide whether $\varphi \in ca(\Sigma, X)$ has relatively compact range knowing only its semivariations $\tilde{\varphi}$ and $\bar{\varphi}$. Indeed, if Σ admits a nonatomic probability measure λ, then, setting $\varphi(E) = 1_E$ for $E \in \Sigma$, we obtain $\varphi \in ca(\Sigma, L_1(\lambda))$ such that

$\tilde{\varphi} = \bar{\varphi} = \lambda$ and $\varphi(\Sigma)$ is *not* relatively compact.

The first part of the following result also follows from a theorem of Curbera (1994).

Thm. 8. *If X is a Banach space and $\varphi \in ca(\Sigma, X)$ [resp., $\varphi \in cca(\Sigma, X)$] is separable and nonatomic, then there exists $\psi \in ca(\Sigma, c_0)$ [resp., $\psi \in cca(\Sigma, c_0)$] such that $\tilde{\psi} = \tilde{\varphi}$.*

Both separability and nonatomicity of φ are essential for the validity of Thm. 8.

Problems. 1. Can we dispense with the separability assumption in the first part of Thm. 8 at the cost of replacing c_0 by $c_0(\Gamma)$ for Γ large enough?

2. Does Thm. 8 hold for the bar semivariation of a vector measure?