Factorization theorems for multilinear operators

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Domination theorems have proved to be relevant tools in the theory of operators between Banach spaces. For instance, they play a fundamental role in the theory of summability in Banach spaces thanks to:

- The Pietsch Domination Theorem An operator $T\colon X \to Y$ between Banach spaces is p-summing ($1 \leqslant p < \infty$), i.e., there exists $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|Tx_j\|_Y^p\Big)^{1/p} \leqslant C \sup_{x^* \in B_{X^*}} \Big(\sum_{j=1}^{n} |x^*(x_j)|^p\Big)^{1/p}$$

for every finite set $\{x_1, \ldots, x_n\} \subset X$, if and only if there exists a regular Borel probability measure μ on (B_{X^*}, w^*) such that

$$\|Tx\|_Y \leqslant C \Big(\int_{B_{X^*}} |x^*(x)|^p \, d\mu\Big)^{1/p}, \quad x \in X.$$

- The Grothendieck Theorem Let K_1 and K_2 be compact Hausdorff spaces, and let T be a scalar-valued bounded bilinear form on $C(K_1) \times C(K_2)$. Then there exist probability measures μ_1 and μ_2, on K_1, K_2, respectively, such that

$$|T(f,g)| \leqslant K\,\|T\|\Big(\int_{K_1} |f|^2\,d\mu_1\Big)^{1/2}\Big(\int_{K_2} |g|^2\,d\mu_2\Big)^{1/2}$$

for every $f \in C(K_1)$ and $g \in C(K_2)$, where K is an absolute constant.

Remark The smallest value constant K is called the Grothendieck constant and is denoted by K_G.

- Corollary Let T be as before. Let $(f_j)_{j=1}^n$, $(g_j)_{j=1}^n$ be finite sequences in $C(K_1)$ and $C(K_2)$, respectively. We then have

$$\Big|\sum_{j=1}^n T(f_j, g_j)\Big| \leqslant K_G \, \|T\| \Big\| \Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2} \Big\|_{C(K_1)} \Big\| \Big(\sum_{j=1}^n |g_j|^2\Big)^{1/2} \Big\|_{C(K_2)}.$$

- Corollary Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $[a_{ij}]$ be an $n \times n$ matrix such that for all $(s_1, \dots, s_n) \in \mathbb{K}^n$, $(t_1, \dots, t_n) \in \mathbb{K}^n$

$$\Big|\sum_{i,j=1}^n s_i t_j a_{ij}\Big| \leqslant \sup_i |s_i| \, \sup_j |t_j|.$$

We then have, for all sequences $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in an arbitrary Hilbert space H,

$$\Big|\sum_{i,j=1}^n a_{ij} \langle x_i, y_j\rangle\Big| \leqslant K_G \sup_i \|x_i\|_H \, \sup_j \|y_j\|_H.$$

Let $X := (X, \|\cdot\|)$ be a quasi-normed space.

- Definitions The Mackey semi-norm $\|\cdot\|^c$ on X is the Minkowski functional of the convex hull $\mathrm{conv}(B_X)$ of the unit ball $B_X := \{x \in X;\ \|x\| \leqslant 1\}$,

$$\|x\|^c = \inf\left\{\lambda > 0;\ x \in \lambda\,\mathrm{conv}(B_X)\right\}, \quad x \in X.$$

- If the topological dual X^* of X separates the points of X, then it is a Banach space under the norm

$$\|x^*\|_{X^*} = \sup_{x \in B_X} |x^*(x)|$$

and it is easy to check that $X^* = (X, \|\cdot\|^c)^*$ with equality of norms; the completion of $(X, \|\cdot\|^c)$ is called the Banach envelope of X and is denoted by $\widehat{X}$.

- Remark If $\kappa\colon X \to X^{**}$ is the canonical embedding defined by $\kappa x(x^*) = x^*(x)$ for all $x \in X$, $x^* \in X^*$, then

$$\|\kappa x\|_{X^{**}} = \sup_{\|x^*\|_{X^*} \leqslant 1} |x^*(x)| = \|x\|^c.$$

Given $f \in L^0(\mu) := L^0(\Omega, \Sigma, \mu)$. The distribution function $\mu_f \colon (0, \infty) \to [0, \infty]$ is given by

$$\mu_f\{x \in \Omega;\, |f(x)| > \lambda\}, \quad \lambda > 0.$$

- The decreasing rearrangement f^* of f is defined by

$$f^*(t) = \inf\{\lambda > 0;\, \mu_f(\lambda) \leqslant t\}, \quad t > 0.$$

- A quasi-Banach space $(X, \|\cdot\|) \subset L^0(\mu)$ is said to be rearrangement invariant (r.i.) on (Ω, Σ, μ) provided

$$\big(f \in L^0(\mu),\, g \in X \text{ and } \mu_f \leqslant \mu_g\big) \;\Rightarrow\; \big(f \in X \text{ and } \|f\| \leqslant \|g\|\big).$$

- If (Ω, Σ, μ) is a nonatomic measure space. An r.i. space X on (Ω, Σ, μ) is said to be generated by a quasi-Banach lattice E on $I = (0, \mu(\Omega))$ equipped with the Lebesgue measure provided that the following condition is satisfied:

$$f \in X \text{ if and only if } f^* \in E \text{ and } \|f\|_X = \|f^*\|_E.$$

Theorem (A. Kamińska & M. M.)

If X is an r.i. quasi-Banach space on a non-atomic measure space (Ω, Σ, μ) generated by an r.i. quasi-Banach space E on $I = (0, \mu(\Omega))$, which is 1-concave on the cone Q of decreasing simple functions, i.e., there exists $C > 0$ such that for $f_1, \ldots, f_n \in Q$,

$$C \, \|f_1 + \cdots + f_n\|_E \geqslant \|f_1\|_E + \cdots + \|f_n\|_E,$$

and $\psi(t) := \|\chi_{(0,t)}\|_E$ for all $t \in I$ denotes the fundamental function of E, then

(i) $X' = M_\psi$ and $X' \neq \{0\}$ if and only if $\widehat{\psi}(t) > 0$ for all $t \in I$, where

$$\widehat{\psi}(t) = t \inf\{\psi(s)/s;\, 0 < s \leqslant t\}, \qquad t \in I.$$

(ii) If E is order continuous and $\widehat{\psi}(t) > 0$ for all $t \in I$, then the Banach envelope $\widehat{X}$ of X coincides up to equivalence of norms with the Lorentz space $\Lambda_{\widehat{\psi}}$.

- Definitions Φ denote the set of all increasing, continuous functions $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$. A function $\varphi \in \Phi$ is said to satisfy the Δ_2-condition ($\varphi \in (\Delta_2)$ for short) provided there exists $C > 0$ such that $\varphi(2t) \leqslant C\varphi(t)$ for all $t > 0$.

- For any quasi-normed function lattice X on a measure space (Ω, Σ, μ) and $\varphi \in \Phi$, we define an order ideal in $L^0(\mu)$,

$$X_\varphi = \big\{f \in L^0(\mu);\ \exists\, \lambda > 0,\ \varphi(\lambda|f|) \in X\big\},$$

and the functional $\|\cdot\|_{X_\varphi}\colon X_\varphi \to [0,\infty)$ by

$$\|f\|_{X_\varphi} = \inf\big\{\lambda > 0;\ \|\varphi(|f|/\lambda)\|_X \leqslant 1\big\}, \quad f \in X_\varphi.$$

Clearly, $\|f\|_{X_\varphi} = 0$ if and only if $f = 0$ and $\|\lambda f\|_{X_\varphi} = |\lambda|\,\|f\|_{X_\varphi}$ for every $\lambda \in \mathbb{R}$, $f \in X_\varphi$.

- Remarks If φ is a convex function and X is a Banach function lattice, X_φ is a Banach function lattice. In the case when $X = L_1(\mu)$ and $\varphi \in \Phi$, X_φ is the Orlicz space denoted as usual by $L_\varphi(\mu)$. This fact motivates to call X_φ the generalized Orlicz space provided X_φ is a quasi-Banach space.

- If $0 < p < \infty$ and $\varphi(t) = t^p$ for all $t \geqslant 0$, the space X_φ is known as the p-convexification X_p of X whose quasi-norm is given by

$$\|f\|_{X_p} = \||f|^p\|_X^{1/p}, \qquad f \in X_p.$$

- Definition For a given function $\varphi \in \Phi$ and a quasi-Banach function lattice X, X is said to be φ-admissible provided that $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ. If in addition the topological dual $(X_\varphi)^*$ separates the points of X_φ, then X is called strongly φ-admissible. For the case $\varphi(t) = t^p$, we simply say that the space X is strongly p-admissible.

- Let $\phi, \varphi_1, \varphi_2 \in \Phi$ and let X, Y be quasi-Banach function lattices on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively, such that X is strongly φ_1^{-1}-admissible and Y is strongly φ_2^{-1}-admissible. Suppose that T is a bilinear operator from $X \times Y$ into a quasi-Banach space E. Assume $0 < C_1, C_2 < \infty$ and that $A \subset X$, $B \subset Y$ are non-empty sets.

Theorem (M. M. & E. A. Sánchez Pérez)

Consider the following statements:

(i) For any finite set of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and any finite sets $\{f_k\}_{k=1}^n$ in A and $\{g_k\}_{k=1}^n$ in B, $n \in \mathbb{N}$, the following inequality holds:

$$\sum_{k=1}^n \alpha_k \phi\big(\|T(f_k, g_k)\|_E\big) \leqslant C_1 \Big\| \sum_{k=1}^n \alpha_k \varphi_1(|f_k|)\Big\|^c_{X_{\varphi_1^{-1}}} + C_2 \Big\| \sum_{k=1}^n \alpha_k \varphi_2(|g_k|)\Big\|^c_{Y_{\varphi_2^{-1}}}.$$

(ii) There exist positive functionals $x^* \in (X_{\varphi_1^{-1}})^*$ and $y^* \in (Y_{\varphi_2^{-1}})^*$ such that

$$\phi\big(\|T(f,g)\|_E\big) \leqslant C_1\, x^*(\varphi_1(|f|)) + C_2\, y^*(\varphi_2(|g|)), \quad (f,g) \in A \times B.$$

(iii) There exist $0 \leqslant u \in B_{(X_{\varphi_1^{-1}})'}$ and $0 \leqslant v \in B_{(Y_{\varphi_2^{-1}})'}$ such that

$$\phi(\|T(f,g)\|_E) \leqslant C_1 \int_{\Omega_1} \varphi_1(|f|)u\,d\mu_1 + C_2 \int_{\Omega_2} \varphi_2(|g|)v\,d\mu_2, \quad (f,g) \in A \times B.$$

Then (i) is equivalent to (ii). If $X_{\varphi_1^{-1}}$ and $Y_{\varphi_2^{-1}}$ are order continuous, then all three statements are equivalent.

- Remark Under the adequate requirements, order continuity of X_φ is often inherited from X; if an order continuous Banach function lattice X is φ-admissible and $\varphi \in (\Delta_2)$, then $\|f_n\|_{X_\varphi} \to 0$ if and only if $\|\varphi(|f_n|)\|_X \to 0$. This implies that X_φ is order continuous if and only if X is order continuous (i.e., $0 \leqslant x_n \downarrow 0 \implies \|x_n\|_X \to 0$).

- If $\varphi_1, \varphi_2 \in \Phi$, then $\varphi_1 \oplus \varphi_2 \colon [0,\infty) \to [0,\infty)$ is defined by

$$(\varphi_1 \oplus \varphi_2)(t) := \inf_{s>0} \big(\varphi_1(s) + \varphi_2(t/s)\big), \quad t \geqslant 0.$$

Theorem (M. M. & E. A. Sánchez Pérez)

For $j = 0, 1$ let $L_{\varphi_j} = (L_{\varphi_j}(\mu_j), \|\cdot\|_{\varphi_j})$ be Orlicz spaces on a measure space $(\Omega_j, \Sigma_j, \mu_j)$ generated by $\varphi_j \in \Phi$ satisfying $\varphi_j(s)\varphi_j(t) \leqslant C_j\varphi_j(st)$ for some $C_j > 0$ and all $s, t > 0$. If T is a bilinear operator from $L_{\varphi_1} \times L_{\varphi_2}$ into a quasi-Banach space E with $\|T\| \leqslant 1$, then for $\phi = \varphi_1 \oplus \varphi_2$ we have

$$\phi(\|T(f,g)\|_E) \leqslant C_1 \int_{\Omega_1} \varphi_1(|f|)\, d\mu_1 + C_2 \int_{\Omega_2} \varphi_2(|g|)\, d\mu_2$$

for all $(f,g) \in L_{\varphi_1} \times L_{\varphi_2}$.

Theorem (M. M. & E. A. Sánchez Pérez)

Let $0 < p, q < \infty$ and let X and Y be quasi-Banach lattices such that both duals $(X_{1/p})^*$ and $(Y_{1/q})^*$ separates the points of $X_{1/p}$ and $Y_{1/q}$, respectively. Assume $\phi \in \Phi$, $0 < C_1, C_2 < \infty$ and that $A \subset X$, $B \subset Y$ are non-empty sets. The following are equivalent statements about a bilinear operator T from $X \times Y$ to a quasi-Banach space E.

(i) For any set of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and any sets $\{f_k\}_{k=1}^n$ in A and $\{g_k\}_{k=1}^n$ in B, $n \in \mathbb{N}$,

$$\sum_{k=1}^n \alpha_k \phi(\|T(f_k, g_k)\|_E) \leqslant C_1 \Big\| \sum_{k=1}^n \alpha_k |f_k|^p \Big\|_{X_{1/p}}^c + C_2 \Big\| \sum_{k=1}^n \alpha_k |g_k|^q \Big\|_{Y_{1/q}}^c .$$

(ii) There exist positive functionals $x^* \in B_{(X_{1/p})^*}$ and $y^* \in B_{(Y_{1/q})^*}$ such that

$$\phi\big(\|T(f,g)\|_E\big) \leqslant C_1\, x^*(|f|^p) + C_2\, y^*(|g|^q), \qquad (f,g) \in A \times B.$$

- Definition A quasi-Banach lattice X is p-convex, where $0 < p < \infty$, if there is a constant $C > 0$ so that

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p} \Big\|_X \leqslant C \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}, \quad x_1, \dots, x_n \in X.$$

The smallest value of the constant C is denoted by $M^{(p)}(X)$.

- Remark We notice the well-known easily verified fact that X is p-convex if and only if $X_{1/p}$ is normable.

Theorem (M. M. & E. A. Sánchez Pérez)

Let $0 < p, q < \infty$ and let $1/r = 1/p + 1/q$. Assume that X and Y are quasi-Banach lattices such that X is strongly $1/p$-admissible and Y is strongly $1/q$-admissible. Assume $0 < C < \infty$. The following are equivalent statements about a bilinear operator T from $X \times Y$ to a quasi-Banach space E.

(i) For each couple of finite sets $\{f_k\}_{k=1}^n$ and $\{g_k\}_{k=1}^n$ of elements of X and Y, respectively,

$$\Big(\sum_{k=1}^n \|T(x_k, y_k)\|_E^r\Big)^{1/r} \leqslant C\Big(\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)\Big\|_{X_{1/p}}^c\Big)^{1/p} \Big(\Big\|\Big(\sum_{k=1}^n |y_k|^q\Big)\Big\|_{Y_{1/q}}^c\Big)^{1/q}.$$

(ii) There are positive functionals $x^* \in B_{(X_{1/p})^*}$ and $y^* \in B_{(Y_{1/q})^*}$ such that

$$\|T(x, y)\|_E \leqslant C\,\big(x^*(|x|^p)\big)^{1/p}\big(x^*(|y|^q)\big)^{1/q}, \quad (x, y) \in X \times Y.$$

- If X is p-convex, Y is q-convex with $M^{(p)}(X) = M^{(q)}(Y) = 1$, then both $X_{1/p}$ and $Y_{1/q}$ are Banach spaces and so

$$(X_{1/p})^c = X_{1/p}, \quad (Y_{1/q})^c = Y_{1/q}$$

  with equality of norms. As a consequence we obtain a remarkable result for bilinear forms due to A. Defant (2001).

- The following variant of Grothendieck's theorem was proved by R. Blei. It is a consequence of the multilinear version of the above Theorem in combination with the Riesz Representation Theorem and the fact that $C(K)$-spaces are p-convex for every $0 < p < \infty$.

Theorem (R. Blei, 1988)

Suppose that K_1,...,K_n are compact Hausdorff spaces and let $1 \leqslant p_j < \infty$, for $j = 1, ..., n$. Let $0 < C < \infty$ and $\sum_{j=1}^{n} 1/p_j = 1$. The following are equivalent statements about an n-linear functional U on $C(K_1) \times ... \times C(K_n)$:

(i) For any set $\{f_j^{(k)}\}_{j=1}^{m}$ in $C(K_k)$ with $k = 1, ..., n$

$$\sum_{j=1}^{m} \left|U(f_j^{(1)}, ..., f_j^{(n)})\right| \leqslant C\left\|\Big(\sum_{j=1}^{m} |f_j^{(1)}|^{p_1}\Big)^{1/p_1}\right\|_{C(K_1)} \cdots \left\|\Big(\sum_{j=1}^{m} |f_j^{(n)}|^{p_n}\Big)^{1/p_n}\right\|_{C(K_n)}$$

(ii) There exist probability Borel measures $\mu_1, ..., \mu_n$ on $K_1, ..., K_n$, respectively, so that

$$|U(f_1, ..., f_n)| \leqslant C\left(\int_{K_1} |f_1|^{p_1}\, d\mu_1\right)^{1/p_1} \cdots \left(\int_{K_n} |f_n|^{p_n}\, d\mu_n\right)^{1/p_n}$$

for all $f_1 \in C(K_1)$,...,$f_n \in C(K_n)$.

- Definition A Banach lattice E if p-concave for $1 < p < \infty$ if there is a constant $C > 0$ so that

$$\Big(\sum_{k=1}^{n} \|x_k\|_E^p\Big)^{1/p} \leqslant C\Big\|\Big(\sum_{k=1}^{n} |x_k|^p\Big)^{1/p}\Big\|_E, \quad x_1, ..., x_n \in E.$$

Theorem (M. M. & E. A. Sánchez Pérez) Let $1 < p < \infty$, $1 < p_i < \infty$ for $i = 1, ..., n$ be such that $1/p = 1/p_1 + \ldots + 1/p_n$. Assume that X_k is a p_k-convex Banach lattice for each $k = 1, ..., n$, and let E be a p-concave Banach lattice. For any n-linear positive operator $T\colon X_1 \times \ldots \times X_n \to E$ there exist a constant $C > 0$ and positive functionals $x_k^* \in B_{(X_{1/p_k})^*}$ so that

$$\big\|T(x_1, ..., x_n)\big\|_E \leqslant C\big(x_1^*(|x_1|^{p_1})\big)^{1/p_1} \cdots \big(x_n^*(|x_n|^{1/p_n})\big)^{1/p_n}$$

for every $(x_1, ..., x_n) \in X_1 \times \cdots \times X_n$.

Theorem (M. M. & E. A. Sánchez Pérez)

Let $\varphi_1, \varphi_2 \in \Phi$ with $\varphi_1, \varphi_2 \in (\Delta_2)$, and let X, Y be order continuous Banach function lattices on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively, such that X is strongly φ_1^{-1}-admissible and Y is strongly φ_2^{-1}-admissible. Suppose $0 < C_1, C_2 < \infty$ and consider an operator $S\colon X \to Y'$, where Y' is the Köthe dual of Y. The following statements about the bilinear form $T\colon X \times Y \to \mathbb{R}$ defined by

$$T(f,g) = \int_{\Omega_2} g\, S(f)\, d\mu_2, \quad (f,g) \in X \times Y$$

are equivalent:

(i) For every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequences $\{f_k\}_{k=1}^n$ in X and $\{g_k\}_{k=1}^n$ in Y,

$$\sum_{k=1}^n \alpha_k |T(f_k, g_k)| \leqslant C_1 \Big\| \sum_{k=1}^n \alpha_k \varphi_1(|f_k|) \Big\|_{X_{\varphi_1^{-1}}}^c + C_2 \Big\| \sum_{k=1}^n \alpha_k \varphi_2(|g_k|) \Big\|_{Y_{\varphi_2^{-1}}}^c .$$

(ii) There exist functions $0 \leqslant u \in B_{(X_{\varphi_1^{-1}})'}$ and $0 \leqslant v \in B_{(X_{\varphi_2^{-1}})'}$ such that

$$|T(f,g)| \leqslant C_1 \Big(\int_{\Omega_1} \varphi_1(|f|) u d\mu_1 \Big) + C_2 \Big(\int_{\Omega_2} \varphi_2(|g|) v d\mu_2 \Big), \quad (f,g) \in X \times Y.$$

Pisier's factorization Theorem

Theorem (Pisier, 1986) An operator T from a $C(K)$-space to a Banach space Y is (q,p)-summing with $1 \leqslant p < q < \infty$ if and only if there is a probability Borel measure μ on K such that T factors as follows:

$$T\colon C(K) \xrightarrow{j} L_{q,1}(\mu) \xrightarrow{S} Y,$$

where j is the inclusion map. Here $L_{q,1}(\mu)$ is the Lorentz space on the measure space $(K, \mathcal{B}(K), \mu)$ equipped with the norm

$$\|f\| := \int_0^1 f^*(t)t^{1/q-1}\,dt,$$

where $\mathcal{B}(K)$ is the σ-algebra of the Borel sets in K and

$$f^*(t) = \inf\{s > 0; \mu(\{|f| > s\}) \leqslant t\}, \quad t \in [0,1]$$

is the decreasing rearrangement of $|f|$.

- Remark. Notice that Pisier's theorem is a cornerstone in the theory of (q,p)-concave operators, which is deeply connected with the linear theory of (q,p)-summing operators.

- Definition Let $X_1,\ldots,X_n$, Y be Banach spaces, and let $1 \leqslant q,p < \infty$. An n-linear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be factorable (q,p)-summing if for every each positive integers M, N and all $M \times N$ matrices $\big(x_{jk}^{(1)}\big),\ldots,\big(x_{jk}^{(n)}\big)$ in $X_1,\ldots,X_n$, respectively, we have

$$\Big(\sum_{j=1}^{M}\Big\|\sum_{k=1}^{N} T\big(x_{jk}^{(1)},\ldots,x_{jk}^{(n)}\big)\Big\|_Y^q\Big)^{1/q}$$
$$\leqslant C \sup_{x_1^*\in B_{X_1^*},\ldots,x_n^*\in B_{X_n^*}} \Big(\sum_{j=1}^{M}\Big|\sum_{k=1}^{N}\langle x_{jk}^{(1)},x_1^*\rangle\ldots\langle x_{jk}^{(n)},x_n^*\rangle\Big|^p\Big)^{1/p}.$$

- Definition Let L be a space of real or complex-valued functions defined on the product $K_1 \times \cdots \times K_n$ of compact Hausdorff spaces $K_1, \ldots, K_n$ $(n \geqslant 2)$. We denote by $\odot$ the map from $C(K_1 \times \cdots \times K_n)$ into L given by

$$\odot(f_1, \ldots, f_n)(t_1, \ldots, t_n) := f_1(t_1) \cdots f_n(t_n)$$

for all $f_i \in C(K_i)$, $t_i \in K_i$ and each $1 \leqslant i \leqslant n$.

The bilinear version of Pisier's Theorem (M. M. & E. A. Sánchez Pérez)

Let $1 \leqslant p < q < \infty$. The following assertions are equivalent for a Banach space valued bilinear map $T\colon C(K_1) \times C(K_2) \to Y$.

(i) T is factorable (q,p)-summing.

(ii) For each positive integers M, N and all $M \times N$ matrices (f_{jk}) and (g_{jk}) in $C(K_1)$ and $C(K_2)$, respectively, the following inequality holds

$$\Big(\sum_{j=1}^{M}\Big\|\sum_{k=1}^{N} T(f_{jk}, g_{jk})\Big\|_Y^q\Big)^{1/q} \leqslant C\Big\|\Big(\sum_{j=1}^{M}\Big|\sum_{k=1}^{N} \odot(f_{jk}, g_{jk})\Big|^p\Big)^{1/p}\Big\|_{C(K_1\times K_2)}.$$

(iii) There is a probability Borel measure μ on $K_1 \times K_2$ such that T admits a factorization:

$$T\colon C(K_1) \times C(K_2) \xrightarrow{\odot} L_{q,1}(\mu) \xrightarrow{\widetilde{T}} Y$$

Theorem (M. M. & E. A. Sánchez Pérez)

Let $1 \leqslant p, q, r < \infty$ satisfy $1/r \leqslant 1/p + 1/q$. The following are equivalent statements about a Banach space valued bilinear operator

$$T\colon C(K_1) \times C(K_2) \to Y.$$

(i) There are probability Borel measures μ_1 and μ_2 on K_1 and K_2 and a constant $C > 0$ such that for every $f \in B_{C(K_1)}$ and $g \in B_{C(K_2)}$,

$$\|T(f,g)\|^r \leqslant C\Big(\int_{K_1} |f|^p \, d\mu_1 + \int_{K_2} |g|^q d\, \mu_2\Big).$$

(ii) For every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and for any finite sequences $\{f_k\}_{k=1}^n$ in $B_{C(K_1)}$ and $\{g_k\}_{k=1}^n$ in $B_{C(K_2)}$, the following inequality holds

$$\sum_{k=1}^n \alpha_k \|T(f_k, g_k)\|^r \leqslant C_1 \Big\| \sum_{k=1}^n \alpha_k |f_k|^p \Big\|_{C(K_1)} + C_2 \Big\| \sum_{k=1}^n \alpha_k |g_k|^q \Big\|_{C(K_2)}.$$

Moreover, each of the above conditions implies

(iii) T is $(r; p, q)$-summing operator.

All conditions are equivalent whenever T is a positive operator.

- Ky-Fan's Lemma Let E be a Hausdorff topological vector space, and let K be a compact convex subset of E. Let Ψ be a set of functions on K with values in $(-\infty, \infty]$ having the following properties:
  (a) each $f \in \Psi$ is convex and lower semicontinuous,
  (b) Ψ is concave, i.e., if $g \in \text{conv}(\Psi)$, there is an $f \in \Psi$ with $g(x) \leqslant f(x)$, for every $x \in K$,
  (c) there is an $r \in \mathbb{R}$ such that each $f \in \Psi$ has a value not greater than r.

  Then there is an $x_0 \in K$ such that $f(x_0) \leqslant r$ for all $f \in \Psi$.