Fuzzy Integration

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$[a, b]$ – a bounded closed interval of the real line equipped with Lebesgue measure λ.
$\mathcal{L}$ – the family of all Lebesgue measurable subsets of $[a, b]$.
$\mathcal{I}$ – the family of all closed subintervals of $[a, b]$.
If $I \in \mathcal{I}$, then $|I|$ denotes its length.
A *partition in $[a, b]$* **is a collection of pairs $\mathcal{P} = \{(I_i, t_i) : i = 1, ..., p\}$, where I_i, are non-overlapping subintervals of $[a, b]$ and t_i are points of $[a, b]$, $i = 1, \ldots, p$.**
If $\cup_{i=1}^{p} I_i = [a, b]$ we say that $\mathcal{P}$ is a *partition of $[a, b]$*.
If $t_i \in I_i$, $i = 1, \ldots, p$, we say that $\mathcal{P}$ is a *Perron partition in (of) $[a, b]$*.
A *gauge* **on $[a, b]$ is a positive function on $[a, b]$.**
We say that a partition $\mathcal{P} = \{(I_i, t_i) : i = 1, ..., p\}$ is *δ-fine* **if**

$$I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)), \qquad i = 1, \ldots, p.$$

Given $f : [a, b] \to \mathbb{R}^n$ and a partition $\mathcal{P} = \{(I_i, t_i) : \ i = 1, ..., p\}$ in $[a, b]$ we set

$$\boldsymbol{\sigma(\mathbf{f}, \mathcal{P}) = \sum_{\mathbf{i=1}}^{\mathbf{p}} |\mathbf{I_i}|\mathbf{f(t_i)}}.$$

Definition

A function $g\colon [a,b] \to \mathbb{R}^n$ is said to be McShane (resp. Henstock) integrable on $[a,b]$ if there exists a vector $w \in \mathbb{R}^n$ with the following property: **for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that**

$$||\boldsymbol{\sigma}(\mathbf{g}, \mathcal{P}) - \mathbf{w}|| < \varepsilon\,.$$

for each δ-fine partition (resp. Perron partition) $\mathcal{P}$ of $[a,b]$.
We set $(Mc)\int_a^b g(t)dt := w$ (resp. $(H)\int_a^b g(t)dt := w$).

If $n = 1$ instead of Henstock, rather the name *Henstock-Kurzweil* is used. We denote by $\mathcal{M}c[a,b]$ (resp. $\mathcal{HK}[a,b]$) the set of all real valued McShane (resp. Henstock-Kurzweil) integrable functions on $[a,b]$.

Definition

A function $g\colon [a,b] \to \mathbb{R}^n$ *is said to be Pettis integrable if*

1. $\forall\, y \in \mathbb{R}^n$ $\langle y, g\rangle$ *is Lebesgue integrable, and*
2. $\forall\, A \in \mathcal{L}\ \exists x_A \in \mathbb{R}^n\ \forall\, y \in \mathbb{R}^n\ \langle y, x_A\rangle = \int_A \langle y, g(t)\rangle\, dt.$

Then $(P)\int_A g\, dt := x_A$.

McShane, Pettis and Bochner integrability coincide for functions taking values in a finite dimensional space.

$\mathbf{ck(\mathbb{R}^n)}$ denotes the family of all non-empty compact and convex subsets of $\mathbb{R}^n$.
If $A, B \in ck(\mathbb{R}^n)$ and $k \in \mathbb{R}$, then

$$A + B := \{x + y : \; x \in A, y \in B\} \;, \quad kA := \{kx : \; x \in A\}.$$

For every $A \in ck(\mathbb{R}^n)$ the **support function** of A is denoted by $s(\cdot, A)$ and defined by

$$\mathbf{s(x, A) = \sup\{\langle x, y\rangle : \; y \in A\}},$$

for each $x \in \mathbb{R}^n$.
The map $x \longmapsto s(x, A)$ is sublinear on $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$.
Each mapping $\Gamma \colon [a, b] \to ck(\mathbb{R}^n)$ is called a multifunction.

$\mathbf{S^{n-1}}$ – the closed unit sphere in $\mathbb{R}^n$.
$\mathbf{d_H}$ – the **Hausdorff distance** on $ck(\mathbb{R}^n)$.

$$\mathbf{d_H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}\|x-y\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}}.$$

The space $ck(\mathbb{R}^n)$ endowed with the Hausdorff distance is a complete metric space.
According to Hörmander's equality (cf. [9], p. 9), for A and B non empty members of $ck(\mathbb{R}^n)$, we have the equality

$$\mathbf{d_H(A,B) = \sup_{x\in S^{n-1}} \left|s(x,A) - s(x,B)\right|}.$$

Definition

- A multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ is said to be *measurable*, if $\{t \in [a,b] : \Gamma(t) \cap O \neq \emptyset\} \in \mathcal{L}$, for each open subset O of $\mathbb{R}^n$.
- Γ is said to be *scalarly measurable* if for every $x \in \mathbb{R}^n$, the map $s(x, \Gamma(\cdot))$ is measurable.
- A multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ is said to be *scalarly* (resp. *scalarly Henstock-Kurzweil*) *integrable on* $[a,b]$ if for each $x \in \mathbb{R}^n$ the real function $s(x, \Gamma(\cdot))$ is integrable (resp. Henstock-Kurzweil integrable) on $[a,b]$.

In case of $ck(\mathbb{R}^n)$-valued multifunctions the scalar measurability and the measurability are equivalent.

Definition

A function $f\colon [a,b] \to \mathbb{R}^n$ is called a *selection* of a multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ if, for every $t \in [a,b]$, one has $f(t) \in \Gamma(t)$. By $\boldsymbol{\mathcal{S}(\Gamma)}$ (resp. $\boldsymbol{\mathcal{S}_{\mathbf{H}}(\Gamma)}$) we denote the family of all measurable selections of Γ that are Bochner integrable (resp. Henstock integrable).

Definition

A measurable multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ is said to be *Aumann integrable on* $[a,b]$ if $\mathcal{S}(\Gamma) \neq \emptyset$. Then we define

$$\mathbf{(A)}\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\Gamma}(\mathbf{t})\,\mathbf{dt} := \left\{ \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}(\mathbf{t})\,\mathbf{dt} : \mathbf{f} \in \boldsymbol{\mathcal{S}(\Gamma)} \right\}.$$

Definition

A multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ is said to be *Pettis integrable* on $[a,b]$ if Γ is scalarly integrable on $[a,b]$ and for each $A \in \mathcal{L}$ there exists a set $W_A \in ck(\mathbb{R}^n)$ such that for each $x \in \mathbb{R}^n$, we have

$$\mathbf{s(x, W_A) = \int_A s(x, \Gamma(t))\, dt}.$$

Then we set $\mathbf{(P)} \int_{\mathbf{A}} \boldsymbol{\Gamma}\mathbf{(t)\, dt := W_A}$, for each $A \in \mathcal{L}$.

Given $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ and a partition $\mathcal{P} = \{(I_i, t_i) : \ i = 1, ..., p\}$ in $[a,b]$ we set

$$\boldsymbol{\sigma(\Gamma, \mathcal{P}) = \sum_{i=1}^{p} |I_i|\Gamma(t_i)}.$$

Definition

A multifunction $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ is said to be **Henstock** (resp. **McShane**) **integrable** on $[a,b]$ if there exists a set $W \in ck(\mathbb{R}^n)$ with the following property:
for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$, we have

$$d_H(W, \sigma(\Gamma, \mathcal{P})) < \varepsilon\,.$$

Pettis, McShane and Aumann integrals coincide for set-valued functions taking values in $ck(\mathbb{R}^n)$, with the same value of the integrals.

Theorem 1 (*L. Di Piazza and K. Musiał, Monatsh. Math. 148(2006), 119–126*)

Let $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ be a scalarly Henstock-Kurzweil integrable multifunction. Then the following conditions are equivalent:

(*i*) **Γ is Henstock integrable;**

(*ii*) **for every $f \in \mathcal{S}_H(\Gamma)$ the multifunction $G\colon [a,b] \to ck(\mathbb{R}^n)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable;**

(*iii*) **there exists $f \in \mathcal{S}_H(\Gamma)$ such that the multifunction $G\colon [a,b] \to ck(\mathbb{R}^n)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable;**

(*iv*) **every measurable selection of Γ is Henstock integrable.**

During this presentation I consider integrals, where functions are replaced by fuzzy-number valued functions.
Fuzzy Henstock integral has been introduced and studied by Wu and Gong in [17] (Fuzzy Sets and Systems 120 (2001), 523–532) and [18] (1994). It is an extension of the integrals introduced in [12] (M. Matloka, Proc. Polish Symp., Interval and Fuzzy Math. 1989, Poznan 163-170) and in [10] (O. Kaleva, Fuzzy sets and Systems, 24 (1987) 301-317).

Definition

The n-dimensional fuzzy number space $\mathbb{E}^n$ is defined as the set

$$\mathbb{E}^{\mathbf{n}} = \{\mathbf{u} \colon \mathbb{R}^{\mathbf{n}} \to [\mathbf{0}, \mathbf{1}] \colon \quad \mathbf{u} \text{ satisfies conditions } \mathbf{(1)}\text{–}\mathbf{(4)} \text{ below}\} :$$

(1) u is a normal fuzzy set, i.e. there exists $x_0 \in \mathbb{R}^n$, such that $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^n$, $t \in [0,1]$;

(3) u is upper semi-continuous (i.e. $\limsup_{x_k \to x} u(x_k) \leq u(x)$);

(4) supp $u = \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

For $r \in (0,1]$ and $u \in \mathbb{E}^n$ let

$$[\mathbf{u}]^{\mathbf{r}} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \ \mathbf{u}(\mathbf{x}) \geq \mathbf{r}\}$$

and

$$[\mathbf{u}]^{\mathbf{0}} = \overline{\bigcup_{\mathbf{s} \in (0,1]} [\mathbf{u}]^{\mathbf{s}}}.$$

In the sequel we will use the following representation theorem (see [1] and [19]).

Theorem 0 (FSS 29(1989),341-348)

If $u \in \mathbb{E}^n$, then

(*i*) $[u]^r \in ck(\mathbb{R}^n)$, for all $r \in [0,1]$;

(*ii*) $[u]^{r_2} \subset [u]^{r_1}$, for $0 \leq r_1 \leq r_2 \leq 1$;

(*iii*) if (r_k) is a nondecreasing sequence converging to $r > 0$, then

$$[u]^r = \bigcap_{k \geq 1} [u]^{r_k}.$$

Conversely,
if $\{\mathbf{A_r} : \mathbf{r} \in [0,1]\}$ is a family of subsets of $\mathbb{R}^\mathbf{n}$ satisfying (*i*)–(*iii*), then there exists a unique $\mathbf{u} \in \mathbb{E}^\mathbf{n}$ such that $[\mathbf{u}]^\mathbf{r} = \mathbf{A_r}$ for $\mathbf{r} \in (0,1]$ and $[\mathbf{u}]^0 = \overline{\cup_{0<\mathbf{r}\leq 1}[\mathbf{u}]^\mathbf{r}} \subset \mathbf{A_0}$.

For each $f:[a,b]\to\mathbb{R}^n$ define $\widetilde{f}:[a,b]\to\mathbb{E}^n$ by

$$[\widetilde{f}(t)](x) := \chi_{\{f(t)\}}(x) \quad \text{if } x\in\mathbb{R}^n,\ t\in[a,b].$$

We have

$$\forall\, 0<r\le 1\ [\widetilde{f}(t)]^r = \{x\in\mathbb{R}^n\colon [\widetilde{f}(t)](x)\ge r = \{f(t)\}$$

and

$$[\widetilde{f}(t)]^0 = \overline{\bigcup_{0<r\le 1} [\widetilde{f}(t)]^r} = \{f(t)\}.$$

Define $\mathbf{D}\colon \mathbb{E}^{\mathbf{n}} \times \mathbb{E}^{\mathbf{n}} \to \mathbf{R}^{+} \cup \{\mathbf{0}\}$ by the equation

$$\mathbf{D}(\mathbf{u},\mathbf{v}) = \sup_{\mathbf{r}\in[\mathbf{0},\mathbf{1}]} \mathbf{d_H}([\mathbf{u}]^{\mathbf{r}},[\mathbf{v}]^{\mathbf{r}}).$$

$(\mathbb{E}^{\mathbf{n}}, \mathbf{D})$ is a complete metric space (see [1] and [19]).
For $u,\ v \in \mathbb{E}^n$ and $k \in R$ the addition and the scalar multiplication are defined respectively by

$$[\mathbf{u}+\mathbf{v}]^{\mathbf{r}} := [\mathbf{u}]^{\mathbf{r}} + [\mathbf{v}]^{\mathbf{r}} \ \text{ and } \ [\mathbf{k}\mathbf{u}]^{\mathbf{r}} := \mathbf{k}[\mathbf{u}]^{\mathbf{r}}.$$

If $f, g : [a, b] \to \mathbb{R}^n$, then

$$\begin{aligned}\forall\, 0 < r \le 1\ & [\widetilde{f(t)}]^r + [\widetilde{g(t)}]^r \\ &= \{x \in \mathbb{R}^n : \chi_{\{f(t)\}}(x) \ge r\} + \{x \in \mathbb{R}^n : \chi_{\{g(t)\}}(x) \ge r\} \\ &= \{f(t) + g(t)\} = [\chi_{\{f(t)+g(t)\}}]^r = [\widetilde{f(t) + g(t)}]^r\end{aligned}$$

$$[\widetilde{f(t)}]^0 + [\widetilde{g(t)}]^0 = \{f(t) + g(t)\} = [\chi_{\{f(t)+g(t)\}}]^0 = [\widetilde{f(t) + g(t)}]^0.$$

Definition

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is said to be* **measurable** *if for every* $r \in [0,1]$ *the set valued function* $[\widetilde{\Gamma}]^r : [a,b] \to ck(\mathbb{R}^n)$ *is measurable. (Since the range space* $\mathbb{R}^n$ *is finite dimensional this is equivalent to the measurability of all support functions* $s(x, [\widetilde{\Gamma(\cdot)}]^r)\,,\ x \in S^{n-1}$*.)*
From now on we set

$$\widetilde{\boldsymbol{\Gamma}}_{\mathbf{r}}(\mathbf{t}) = [\widetilde{\boldsymbol{\Gamma}}(\mathbf{t})]^{\mathbf{r}}.$$

A fuzzy-number-valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is said to be* **scalarly** *(resp.* **scalarly Henstock-Kurzweil***)* **integrable** *on* $[a,b]$ *if for all* $r \in [0,1]$ *the multifunction* $\widetilde{\Gamma}_r \colon [a,b] \to ck(\mathbb{R}^n)$ *is scalarly (resp. scalarly Henstock-Kurzweil) integrable.*

Definition

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is said to be* **weakly fuzzy Henstock integrable on** $[a,b]$, *if for every* $r \in [0,1]$ *the multifunction* $\widetilde{\Gamma}_r$ *is Henstock integrable on* $[a,b]$ *and there exists a fuzzy number* $\widetilde{A} \in \mathbb{E}^n$ *such that for any* $r \in [0,1]$ *and for any* $x \in \mathbb{R}^n$ *we have*

$$\mathbf{s}(\mathbf{x}, [\widetilde{\mathbf{A}}]^{\mathbf{r}}) = (\mathbf{HK})\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{s}(\mathbf{x}, \widetilde{\Gamma}_{\mathbf{r}}(\mathbf{t}))\, \mathbf{dt}.$$

Definition

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is said to be* **weakly fuzzy Pettis** *or* **weakly fuzzy McShane integrable on** $[a,b]$, *if for every* $r \in [0,1]$ *the multifunction* $\widetilde{\Gamma}_r$ *is Pettis or McShane integrable on* $[a,b]$ *and there exists a fuzzy number* $\widetilde{A} \in \mathbb{E}^n$ *such that for any* $r \in [0,1]$ *and for any* $x \in \mathbb{R}^n$ *we have*

$$\mathbf{s}(\mathbf{x}, [\widetilde{\mathbf{A}}]^{\mathbf{r}}) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{s}(\mathbf{x}, \widetilde{\Gamma}_{\mathbf{r}}(\mathbf{t}))\, \mathbf{dt}.$$

Definition

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ is said to be **fuzzy Aumann integrable** *on $[a,b]$ if there exists a fuzzy number $\widetilde{A} \in \mathbb{E}^n$ such that for every $r \in [0,1]$ the multifunction $\widetilde{\Gamma}_r$ is Aumann integrable on $[a,b]$ and $[\widetilde{A}]^r = (A)\int_a^b \widetilde{\Gamma}_r(t)\,dt$. We write $(FA)\int_a^b \widetilde{\Gamma}(t)\,dt := \widetilde{A}$.*

Remark

Since Pettis, McShane and Aumann integrals coincide for set-valued functions taking values in a finite dimensional space, then also the fuzzy Aumann, the weakly fuzzy Pettis and the weakly fuzzy McShane integrals coincide.

Definition

(see [18]) A fuzzy-number-valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is said to be* **fuzzy Henstock** *(resp.* **fuzzy McShane***)* **integrable on $[a,b]$**, *if there exists a fuzzy number* $\tilde{A} \in \mathbb{E}^n$ *such that for every* $\varepsilon > 0$ *there is a gauge* δ *on* $[a,b]$ *such that for every* δ*-fine Perron partition (resp. partition)* $\mathcal{P}$ *of* $[a,b]$*, we have*

$$\mathbf{D}(\tilde{\mathbf{A}}, \sigma(\widetilde{\Gamma}, \mathcal{P})) = \mathbf{D}(\tilde{\mathbf{A}}, \sum_{\mathbf{i}=1}^{\mathbf{p}} |\mathbf{I_i}| \widetilde{\Gamma}(\mathbf{t_i})) < \varepsilon.$$

We write $\mathbf{(FH)}\int_{\mathbf{a}}^{\mathbf{b}} \widetilde{\Gamma}(\mathbf{t})\,\mathbf{dt} := \tilde{\mathbf{A}}$ *(resp.* $\mathbf{(FMc)}\int_{\mathbf{a}}^{\mathbf{b}} \widetilde{\Gamma}(\mathbf{t})\,\mathbf{dt} := \tilde{\mathbf{A}}$*).*

Using the notion of equi-integrability it is possible to characterize the fuzzy Henstock and the fuzzy McShane integrability.

Definition

A family $\{\mathbf{g}_\alpha\}$ *of real valued functions in* $\mathcal{HK}[a,b]$ *(resp.* $\mathcal{M}c[a,b]$*) is said to be* **Henstock-Kurzweil** *(resp.* **McShane***)* **equi-integrable on [a, b]** *whenever for every* $\varepsilon > 0$ *there is a gauge* δ *on* $[a,b]$ *such that*

$$\sup_{\alpha} \left| \sigma(\mathbf{g}_\alpha, \mathcal{P}) - (\mathbf{HK})\int_a^b \mathbf{g}_\alpha(t)\,dt \right| < \varepsilon\,.$$

$$\left(\text{resp. } \sup_{\alpha} \left| \sigma(\mathbf{g}_\alpha, \mathcal{P}) - \int_a^b \mathbf{g}_\alpha(t)\,dt \right| < \varepsilon\,.\right)$$

for each δ*-fine Perron partition (resp. partition)* $\mathcal{P}$ *of* $[a,b]$.

Proposition 0

Let $\Gamma\colon [a,b] \to ck(\mathbb{R}^n)$ be a Henstock-Kurzweil (McShane) scalarly (resp. scalarly) integrable multifunction. Then the following are equivalent:

(*j*) **Γ is Henstock (resp. McShane) integrable on $[a,b]$;**

(*jj*) **the collection $\left\{s(x,\Gamma(\cdot)) :\ x \in S^{n-1}\right\}$ is Henstock-Kurzweil (resp. McShane) equi-integrable.**

Proposition 1

Let $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ be a Henstock-Kurzweil (McShane) scalarly (resp. scalarly) integrable fuzzy-number-valued function. Then the following are equivalent:

- (*j*) **$\widetilde{\Gamma}$ is fuzzy Henstock (resp. McShane) integrable on $[a,b]$;**
- (*jj*) **the collection $\left\{s(x, \widetilde{\Gamma}_r(\cdot)) : \ x \in S^{n-1} \ \text{and} \ 0 \le r \le 1\right\}$ is Henstock-Kurzweil (resp. McShane) equi-integrable.**

Proof.

$(j) \Rightarrow (jj)$. According to Hörmander's equality and the definition of the metric D in $\mathbb{E}^n$ we have

$$
\begin{aligned}
D\left(\tilde{A}, \sum_{i=1}^{p} |I_i|\, \widetilde{\Gamma}(t_i)\right) &= \sup_{r\in[0,1]} d_H([\tilde{A}]^r, \sum_{i=1}^{p} |I_i|\, \widetilde{\Gamma}_r(t_i)) \\
&= \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} \left| s(x, [\tilde{A}]^r) - \sum_{i=1}^{p} s(x, \widetilde{\Gamma}_r(t_i))\, |I_i| \right| \\
&= \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} \left| \int_a^b s(x, \widetilde{\Gamma}_r(t))\, dt - \sum_{i=1}^{p} s(x, \widetilde{\Gamma}_r(t_i))\, |I_i| \right| .
\end{aligned}
$$

Thus, the implication holds true.

If $\{A_r : r \in [0,1]\}$ is a family of subsets of $\mathbb{R}^n$ satisfying the conditions

(i) $A_r \in ck(\mathbb{R}^n)$, for all $r \in [0,1]$;

(ii) $A_{r_2} \subset A_{r_1}$, for $0 \leq r_1 \leq r_2 \leq 1$;

(iii) if (r_k) is a nondecreasing sequence converging to $r > 0$, then

$$A_r = \bigcap_{k \geq 1} A_{r_k}.$$

Then there exists a unique $u \in \mathbb{E}^n$ such that

$$[u]^r = A_r \text{ for } r \in (0,1]$$

and

$$[u]^0 = \overline{\cup_{0<r\leq 1}[u]^r} \subset A_0.$$

$(jj) \Rightarrow (j)$. [HK-version] Let us fix $r \in [0,1]$. Since the collection $\left\{ s(x, \widetilde{\Gamma}_r(\cdot)) : \ x \in S^{n-1} \right\}$ is Henstock-Kurzweil equi-integrable, by Proposition 0 there exists $A_r \in ck(\mathbb{R}^n)$ such that for each $x \in S^{n-1}$

$$s(x, A_r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\, dt, \qquad (1)$$

Now we are going to prove that the family $\{A_r : \ r \in [0,1]\}$ satisfies properties (i)–(iii) of Theorem 0.

Since $A_r \in ck(\mathbb{R}^n)$ it remains to prove only (ii) and (iii).

Let $0 \leq r_1 \leq r_2 \leq 1$. By Theorem 0 we have $\widetilde{\Gamma}_{r_2}(t) \subset \widetilde{\Gamma}_{r_1}(t)$, for each $t \in [a, b]$. Therefore

$$\begin{aligned} s(x, A_{r_2}) &= (HK)\int_a^b s(x, \widetilde{\Gamma}_{r_2}(t))\, dt \\ &\leq (HK)\int_a^b s(x, \widetilde{\Gamma}_{r_1}(t))\, dt = s(x, A_{r_1}), \end{aligned}$$

for each $x \in \mathbb{R}^n$.
Then, as a consequence of the separation theorem for convex sets, we also infer the inclusion $A_{r_2} \subset A_{r_1}$ and property (ii) is satisfied.

If (r_k) is a nondecreasing sequence converging to $r > 0$, then for each $t \in [a, b]$ we have

$$\widetilde{\Gamma}_r(t) = \bigcap_{k \geq 1} \widetilde{\Gamma}_{r_k}(t) .$$

Consequently (see [16, Proposition 1])

$$s(x, \widetilde{\Gamma}_r(t)) = \lim_k s(x, \widetilde{\Gamma}_{r_k}(t)) ,$$

for each $t \in [a, b]$ and $x \in \mathbb{R}^n$.

By hypothesis, for each $x \in \mathbb{R}^n$, the sequence of real valued functions $\left(s(x, \widetilde{\Gamma}_{r_k}(\cdot))\right)$ is Henstock-Kurzweil equi-integrable. So we have (see [15])

$$s(x, A_r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt$$

$$= \lim_k (HK)\int_a^b s(x, \widetilde{\Gamma}_{r_k}(t))\,dt = \lim_k s(x, A_{r_k}) = s(x, \bigcap_{k\geq 1} A_{r_k}),$$

Since above equalities hold for each $x \in \mathbb{R}^n$, we obtain $A_r = \bigcap_{k\geq 1} A_{r_k}$ and property (iii) is satisfied.

Therefore according to Theorem 0 there exists a unique $\widetilde{A} \in \mathbb{E}^n$ such that

$$\mathbf{[\widetilde{A}]^r = A_r} \quad \text{for } \mathbf{r} \in \mathbf{(0,1]} \text{ and } \mathbf{[\widetilde{A}]^0} = \overline{\bigcup_{\mathbf{s}\in(\mathbf{0,1}]} \mathbf{[\widetilde{A}]^s}} \subset \mathbf{A_0}.$$

If $\varepsilon > 0$ is fixed and a gauge δ corresponds to the uniform equi-integrability, then taking into account $(1)[s(x, A_r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt]$ and the definition of the distance D we get

$$\begin{aligned} & D\left(\tilde{A}, \sum_{i=1}^{p} |I_i|\, \widetilde{\Gamma}(t_i)\right) \\ = & \sup_{r\in[0,1]} d_H([\tilde{A}]^r, \sum_{i=1}^{p} |I_i|\, \widetilde{\Gamma}_r(t_i)) \le \delta. \end{aligned}$$

and hence the fuzzy Henstock integrability of $\widetilde{\Gamma}$ on $[a, b]$ with the fuzzy Henstock integral equal to $\widetilde{A}$. □

As a direct consequence of Proposition 1 we have the following characterization of the fuzzy Henstock and fuzzy McShane integrability:

Corollary 1

A fuzzy-number-valued function $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ *is fuzzy Henstock (resp. fuzzy McShane) integrable on* $[a,b]$ *if and only if it is weakly fuzzy Henstock (resp. weakly fuzzy McShane) integrable on* $[a,b]$ *and the collection* $\left\{ s(x, \widetilde{\Gamma}_r(\cdot)) : \ x \in S^{n-1} \ \text{ and } \ 0 \leq \mathrm{r} \leq 1 \right\}$ *is Henstock-Kurzweil (resp. McShane) equi-integrable.*

Using the above Proposition one can show that the family of all weakly fuzzy Henstock (resp. McShane) integrable functions is wider than the family of all fuzzy Henstock (resp. McShane) integrable fuzzy-number-valued functions.

At first it may look strange since we are in $\mathbb{R}^n$ and the Henstock (resp. McShane) integral of $ck(\mathbb{R}^n)$-valued multifunctions defined with the help of support functions coincides with that defined with the help of the Hausdorff distance.
In particular, for each $0 \leq r \leq 1$ the family $\{s(x, \widetilde{\Gamma}_r(\cdot)) : \ x \in S^{n-1}\}$ is Henstock-Kurzweil (resp. McShane) equi-integrable.
But it is known that an infinite union of equi-integrable families may be not equi-integrable. Thus, the fuzzy approach may change the situation.
In fact, in the example below we show even more. We prove that there exists a weakly fuzzy McShane integrable fuzzy-number-valued function on $[0, 1]$ that is not fuzzy Henstock integrable (hence also not fuzzy McShane integrable).

Example

It is enough to show that such a function exists for $n = 1$.
Define $g_m = \chi_{[1-2^{-m},1]}$, $m = 1, 2, \ldots$ where χ_B denotes the characteristic function of the set B, and let $f_k = \sum_{m=1}^{k} g_m$, $k = 1, 2, \ldots$
Remark that $f_k(t) \le f_{k+1}(t)$, for $t \in [0,1]$, and set $\mathcal{O}_r(t) = [0, f_k(t)]$, $\mathcal{Q}_r = [0, 1 - 2^{-k}]$, for $(k+1)^{-1} < r \le k^{-1}$, $t \in [0,1]$ and $k \in N$, $\mathcal{O}_0(t) = \overline{\cup_{r\in(0,1]}\mathcal{O}_r(t)}$, $\mathcal{Q}_0 = [0,1]$.
It is easy to check that $\mathcal{O}_r(t)$ and $\mathcal{Q}_r$ satisfy conditions (i)–(iii) of Theorem 0, for any $t \in [0,1]$.
Then, from Theorem 0 it is possible to define a function $\widetilde{\Gamma}\colon [0,1] \to E^1$ and a fuzzy number $\widetilde{A}$ such that $\widetilde{\Gamma}_r(t) = \mathcal{O}_r(t)$ and $[\widetilde{A}]^r = \mathcal{Q}_r$ for all $0 < r \le 1$ and all $t \in [0,1]$.
The fuzzy-number-valued function $\widetilde{\Gamma}$ is weakly fuzzy McShane integrable but not fuzzy Henstock integrable. □

Decomposition Theorem

Let $\widetilde{\Gamma}\colon [a,b] \to \mathbb{E}^n$ be a fuzzy-number valued function on $[a,b]$. Then the following conditions are equivalent:

(A) $\widetilde{\Gamma}$ **is fuzzy Henstock integrable;**

(B) **For every Henstock integrable function $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$ the fuzzy-number valued function $\widetilde{G}\colon [a,b] \to \mathbb{E}^n$ defined by $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$ (where $\widetilde{f}(t) = \chi_{\{f(t)\}}$) is fuzzy McShane integrable on $[a,b]$ and**

$$\left[(FH)\int_a^b \widetilde{\Gamma}(t)\,dt\right]^r = \left[(FMc)\int_a^b \widetilde{G}(t)\,dt\right]^r + (H)\int_a^b f(t)\,dt, \tag{2}$$

for every $r \in [0,1]$;

Decomposition Theorem, cont.

(C) **There exists a Henstock integrable function $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$ such that the fuzzy-number valued function $\widetilde{G}\colon [a,b] \to \mathbb{E}^n$ defined by $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$ is fuzzy McShane integrable on $[a,b]$ and**

$$\left[(FH)\int_a^b \widetilde{\Gamma}(t)\,dt\right]^r = \left[(FMc)\int_a^b \widetilde{G}(t)\,dt\right]^r + (H)\int_a^b f(t)\,dt, \tag{3}$$

for every $r \in [0,1]$.

Equivalently,

$$(FH)\int_a^b \widetilde{\Gamma}(t)\,dt = (FMc)\int_a^b \widetilde{G}(t)\,dt + (H)\int_a^b f(t)\,dt.$$

$(C) \Rightarrow (A)$. Assume that $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G}$ is a fuzzy-number valued function fuzzy McShane integrable on $[a, b]$ and f is an Henstock integrable function $f \in \mathcal{S}_H([\widetilde{\Gamma}]^1)$. Then according to Proposition 1 we have that the collection

$$\mathbb{B} := \left\{ \mathsf{s}(\mathsf{x}, \widetilde{\mathsf{G}}_\mathsf{r}(\cdot)) : \ \mathsf{x} \in \mathsf{S}^{\mathsf{n}-1} \ \text{ and } \ 0 \le r \le 1 \right\}$$

is McShane equi-integrable. Therefore by the equality

$$s(x, \widetilde{\Gamma}_r(t)) = s(x, \widetilde{G}_r(t)) + \langle x, f(t) \rangle,$$

we infer that the collection

$$\left\{ \mathsf{s}(\mathsf{x}, \widetilde{\Gamma}_\mathsf{r}(\cdot)) : \ \mathsf{x} \in \mathsf{S}^{\mathsf{n}-1} \ \text{ and } \ 0 \le r \le 1 \right\}$$

is Henstock-Kurzweil equi-integrable. And applying once again Proposition 1 we obtain the fuzzy Henstock integrability of $\widetilde{\Gamma}$. $\square$

It is quite easy to define $\widetilde{G}$ and $\widetilde{f}$. It is however not so simple to show that $\widetilde{G}$ is fuzzy McShane integrable.

Before proving the Decomposition Theorem we need some preliminary results. It is well known that if $g\colon [a,b] \to R$ is a non negative Henstock-Kurzweil integrable function, then g is McShane integrable. So one could expect that if $\mathbb{A}$ is a family of non negative Henstock-Kurzweil equi-integrable functions, then $\mathbb{A}$ is also McShane equi-integrable. At the moment we don't know if this is true, however under additional suitable conditions next theorem gives the expected McShane equi-integrability. The idea of our proof is taken from Fremlin [6, Theorem 8].

Proposition 2

Let $S \neq \emptyset$ be an arbitrary set and let $\mathbb{A} = \{g_\alpha \colon [a, b] \to [0, \infty) \colon \alpha \in S\}$ be a family of functions satisfying the following conditions:

(a) $\mathbb{A}$ is Henstock-Kurzweil equi-integrable;

(b) $\mathbb{A}$ is totally bounded in the L^1 norm;

(c) $\mathbb{A}$ is pointwise bounded.

Then the family $\mathbb{A}$ is McShane equi-integrable.

We need yet the following fact that is a very special case of a general theorem proved in [13, Theorem 3.3].

Proposition 3

Let $G\colon [a,b] \to ck(\mathbb{R}^n)$ be a Pettis integrable multifunction whose support functions are non negative. Then the set

$$\mathbb{S} = \left\{ s(x, G(\cdot)) : \ x \in S^{n-1} \right\}$$

is compact in $L^1[a,b]$.

Proof. Let $M_G(E)$ be the Pettis integral of G on the set $E \in \mathcal{L}$. Moreover, let $\{x_n : n \in \mathbb{N}\} \subset S^{n-1}$ be an arbitrary sequence and let $\{x_{n_k}\}_k$ be a subsequence converging to x_0. We have then

$$\lim_k \int_E s(x_{n_k} - x_0, G(t))\, dt = \lim_k s(x_{n_k} - x_0, M_G(E)) = 0 \quad \text{for every } E \in \mathcal{L}$$

and the convergence of the sequence $\{s(x_{n_k} - x_0, M_G(E))\}_k$ is uniform on $\mathcal{L}$, because $M_G(E) \subseteq M_G(\Omega)$, for every $E \in \mathcal{L}$. Thus, the sequence $\{s(x_{n_k}, G)\}_k$ is convergent in $L_1(\mu)$ to $s(x_0, G)$ (cf. [14, Proposition II.5.3]). □

Proof of the Decomposition Theorem. $(A) \Rightarrow (B)$. Since $\widetilde{\Gamma}$ is fuzzy Henstock integrable, then for each $r \in [0,1]$ the set function $\widetilde{\Gamma}_r$ is Henstock integrable. So, according to Theorem 1, $\mathcal{S}_H(\widetilde{\Gamma}_1) \neq \emptyset$.

Let us fix $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$ and **define** a fuzzy-number valued function $\widetilde{f} \colon [a,b] \to \mathbb{E}^n$ as follows: $\widetilde{\mathbf{f}}(\mathbf{t}) = \chi_{\{\mathbf{f}(\mathbf{t})\}}$, for each $t \in [a,b]$.

Now define $\widetilde{G} \colon [a,b] \to \mathbb{E}^n$ setting $\widetilde{\mathbf{G}}(\mathbf{t}) := \widetilde{\boldsymbol{\Gamma}}(\mathbf{t}) - \widetilde{\mathbf{f}}(\mathbf{t})$. To prove that $\widetilde{G}(t)$ is fuzzy McShane integrable on $[a,b]$, by Proposition 1 it is enough to show that the collection

$$\mathbb{B} := \left\{ s(x, \widetilde{G}_r(\cdot)) : \ x \in S^{n-1} \ \ \text{and} \ \ 0 \leq \mathrm{r} \leq 1 \right\}$$

is McShane equi-integrable.

At the beginning we are going to prove that $\mathbb{B}$ fulfils the hypotheses of Proposition 2.

Since $\widetilde{\Gamma}$ is fuzzy Henstock integrable, it follows from Proposition 1 that the family of functions

$$\left\{ s(x, \widetilde{\Gamma}_r(\cdot)) : \; x \in S^{n-1} \; \text{ and } \; 0 \leq \mathrm{r} \leq 1 \right\}$$

is Henstock-Kurzweil equi-integrable.

Moreover, for each $r \in [0, 1]$ the set-function $\widetilde{\Gamma}_r$ is Henstock integrable and

$$\widetilde{\Gamma}_r(t) = \widetilde{G}_r(t) + f(t), \quad \text{for each} \quad t \in [a, b]. \tag{4}$$

Then, for $r \in [0, 1]$, $t \in [a, b]$ and $x \in \mathbb{R}^n$, we have

$$s(x, \widetilde{G}_r(t)) = s(x, \widetilde{\Gamma}_r(t)) - \langle x, f(t) \rangle.$$

Now applying Theorem 1 to each set-function $\widetilde{\Gamma}_r$, we obtain that, for every $r \in [0,1]$, the set function $\widetilde{G}_r$ is Pettis integrable.
Since the function f is Henstock integrable, we infer that the family $\mathbb{B}$ is Henstock-Kurzweil equi-integrable.
We observe that all support functions of $\widetilde{G}_r(t)$ are non negative.
Consequently, if $0 \le r_1 \le r_2 \le 1$, then $\widetilde{G}_{r_2}(t) \subset \widetilde{G}_{r_1}(t) \subset \widetilde{G}_0(t)$, and

$$0 \le s(x, \widetilde{G}_{r_2}(t)) \le s(x, \widetilde{G}_{r_1}(t)) \le s(x, \widetilde{G}_0(t)), \tag{5}$$

for every $x \in S^{n-1}$ and $t \in [a,b]$.
So the family $\mathbb{B}$ is pointwise bounded.

It remains to show that $\mathbb{B}$ is also totally bounded in $L^1[a, b]$.

Claim 1.

If $g_r(x) := \int_0^1 s(x, \widetilde{G}_r(t))\, dt$, for each $x \in S^{n-1}$ and $r \in [0, 1]$, then for each r the function g_r is continuous and the family $\{g_r \colon r \in [0, 1]\}$ is norm relatively compact in $C(S^{n-1})$, the space of real continuous functions on S^{n-1}.

Proof. Given $x, y \in S^{n-1}$ and $r \in [0,1]$, we have for $x \neq y$

$$\begin{aligned}
|g_r(x) &- g_r(y)| \leq \int_a^b \left| s(x, \widetilde{G}_r(t)) - s(y, \widetilde{G}_r(t)) \right| \, dt \\
&\leq \int_a^b \left[s\left(x - y, \widetilde{G}_r(t)\right) + s\left(y - x, \widetilde{G}_r(t)\right)\right] \, dt \\
&\leq \|x - y\| \int_a^b \left[s\left(\frac{x-y}{\|x-y\|}, \widetilde{G}_r(t)\right) + s\left(\frac{y-x}{\|x-y\|}, \widetilde{G}_r(t)\right)\right] \, dt \\
&\leq \|x - y\| \int_a^b \left[s\left(\frac{x-y}{\|x-y\|}, \widetilde{G}_0(t)\right) + s\left(\frac{y-x}{\|x-y\|}, \widetilde{G}_0(t)\right)\right] \, dt \\
&\leq 2\|x - y\| \sup_{\|z\| \leq 1} \int_a^b s(z, \widetilde{G}_0(t)) \, dt
\end{aligned}$$

But, since $\widetilde{G}_0$ is Pettis integrable, we have $\sup_{\|z\|\leq 1} \int_a^b s(z, \widetilde{G}_0(t))\,dt < \infty$ (cf. [5, Theorem 5.5]). It follows that g_r satisfies the Lipschitz condition.
Consequently the family $\{g_r \colon r \in [0,1]\,\}$ is equicontinuous.
Moreover, since $0 \leq g_r(x) \leq g_0(x)$ for each $r \in [0,1]$ and each $x \in [a,b]$, from Ascoli's theorem follows that the family $\{g_r : r \in [0,1]\}$ is norm relatively compact in $C(S^{n-1})$. □

Claim 2.

$\mathbb{B}$ *is totally bounded in* $L_1[a, b]$.

Proof. Let us fix $\varepsilon > 0$. It follows from Claim 1 that the family $\{g_r \colon r \in [0,1]\}$ is totally bounded in $C(S^{n-1})$. That is there exist reals $r_1, \ldots, r_m \in [0,1]$ such that

$$\forall\, r \in [0,1]\ \exists\, i \le m \quad : \quad \|g_r - g_{r_i}\|_{C(S^{n-1})} < \varepsilon/2.$$

$\left(g_r(x) := \int_0^1 s(x, \widetilde{G}_r(t))\, dt\right)$

But

$$\begin{aligned}\|g_r - g_{r_i}\|_{C(S^{n-1})} &= \sup_{x\in S^{n-1}} \left| \int_a^b s(x, \widetilde{G}_r(t))\, dt - \int_a^b s(x, \widetilde{G}_{r_i}(t))\, dt \right| \\ &= \sup_{x\in S^{n-1}} \left| \int_a^b \left[s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t)) \right] dt \right| \\ &= \sup_{x\in S^{n-1}} \int_a^b \left| s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t)) \right| dt,\end{aligned}$$

where the final equality follows from (5). Consequently, we have

$$\int_a^b \left| s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t)) \right| dt < \varepsilon/2, \quad \text{for every } x \in S^{n-1}.$$

But from Proposition 3 we know that for each $i \leq m$ the family $\left\{ s(x, \widetilde{G}_{r_i}) : \ x \in S^{n-1} \right\}$ is totally bounded in $L_1[a,b]$. Hence, there are points $\{x_{1i}, \ldots, x_{p_i}\} \subset S^{n-1}$ such that if $x \in S^{n-1}$ is arbitrary, then

$$\int_a^b \left| s(x, \widetilde{G}_{r_i}(t)) - s(x_{ji}, \widetilde{G}_{r_i}(t)) \right| dt < \varepsilon/2, \quad \text{for a certain } j \leq p_i.$$

It follows that the set $\left\{ \mathbf{s(x_{ji}, \widetilde{G}_{r_i}(\cdot)) : j \leq p_i, \ i \leq m} \right\}$ is an ε-mesh of $\mathbb{B}$ in the norm of $\mathbf{L_1[a,b]}$.

□

Then the collection $\mathbb{B}$ is McShane equi-integrable and, applying once again Proposition 1, we get that $\widetilde{G}$ is fuzzy McShane integrable on $[a,b]$.

The equality

$$\left[\mathbf{(FH)}\int_{\mathbf{a}}^{\mathbf{b}} \widetilde{\boldsymbol{\Gamma}}(\mathbf{t})\,\mathbf{dt}\right]^{\mathbf{r}} = \left[\mathbf{(FMc)}\int_{\mathbf{a}}^{\mathbf{b}} \widetilde{\mathbf{G}}(\mathbf{t})\,\mathbf{dt}\right]^{\mathbf{r}} + \mathbf{(H)}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}(\mathbf{t})\,\mathbf{dt}$$

follows at once from equality

$\widetilde{\Gamma}_r(t) = \widetilde{G}_r(t) + f(t), \quad$ for each $\; t \in [a, b]$.

The implication $(B) \Rightarrow (C)$ is obvious.

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