A LUSIN TYPE MEASURABILITY PROPERTY FOR VECTOR-VALUED FUNCTIONS

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Introduction

- In a Banach space, there are two basic notions of function measurability — the notions of *Bochner* (or *strong*) measurability and *scalar* (or *weak*) measurability.
- The Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued.
- Why do we need another notion of function measurability to deal with Riemann type integration theories, such as those of McShane and Henstock, in a Banach space?
- The above notions of function measurability diverge sharply for *non-separable* range spaces. Two classical examples illustrate some of the difficulties:

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- **Graves (1927)** Define $\varphi : [0,1] \to \ell^\infty[0,1]$ by $\varphi(t) = \chi_{[t,1]}$ for each t in $[0,1]$. Then φ is Riemann integrable but not Bochner measurable on $[0,1]$.
- **Phillips (1940)** Under the Continuum Hypothesis, there exists a bounded scalarly measurable function $\varphi : [0,1] \to \ell^\infty[0,1]$ such that Pettis' theory does not assign any integral to φ on $[0,1]$.
- It is well-known that the McShane and Henstock integrals can be defined without the use of Lebesgue measure as well as of any notion of function measurability.
- Which other integration theories are based on Riemann type sums?

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$$\left\| \sum_n f(t_n)\lambda(E_n) - w \right\| < \varepsilon.$$

- The Kolmogorov-Birkhoff theory of integration is not as simple and as useful as the Riemann type integration theories: the above definition uses Lebesgue measurable partitions as well as the notion of unconditional convergence of an infinite series of elements in a Banach space.

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- **Jeffery (1940)** *'measurable'* functions;
- **Kunisawa (1943)** $*$-*measurable* functions;
- **Snow (1958)** *almost-Riemann-integrable* functions;
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$$\left\| \sum_n \{f(t_n) - f(t'_n)\} \right\| \le R,$$
where t_n and t'_n are any two points in E_n.
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- **Jeffery (1940)** If $f : [a, b] \to X$ is both bounded and Jeffery-measurable $[a, b]$, then f is Birkhoff integrable on $[a, b]$.
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Notation

- $[a, b]$ will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval.
- A positive function defined on $[a, b]$ will be called a *gauge* on $[a, b]$.
- A *McShane partition* of $[a, b]$ is a finite collection $\mathscr{P} = \{(I_k, t_k)\}_{k=1}^{K}$ of interval-point pairs such that $\{I_k\}_{k=1}^{K}$ is a collection of pairwise non-overlapping intervals, $t_k \in [a, b]$ for each k, and $\{I_k\}_{k=1}^{K}$ covers $[a, b]$. $\mathscr{P}$ is *subordinate* to a gauge δ on $[a, b]$ if $I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each k.
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The McShane and Henstock integrals

Definition

A function $f : [a, b] \to X$ is *McShane integrable* (*Henstock integrable*) on $[a, b]$, with *McShane integral* (*Henstock integral*) $w \in X$, if for each $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{k=1}^{K} f(t_k)\lambda(I_k) - w \right\| < \varepsilon$$

whenever $\{(I_k, t_k)\}_{k=1}^{K}$ is a McShane partition (Henstock partition) of $[a, b]$ subordinate to δ.

The restricted versions

Definition

A function $f : [a, b] \to X$ is said to be *$\mathscr{M}$-integrable* (*$\mathscr{H}$-integrable*) on $[a, b]$ if it is McShane (Henstock) integrable on $[a, b]$ and for each $\varepsilon > 0$ there exists a *Lebesgue measurable* gauge δ on $[a, b]$ that corresponds to ε in the definition of the McShane (Henstock) integral of f on $[a, b]$.

- **Solodov (2005)** The $\mathscr{M}$-integral was first introduced for vector-valued functions.
- **Solodov (2005)** The $\mathscr{M}$-integral and the Kolmogorov-Birkhoff integral are equivalent.

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- Lusin measurability is equivalent to Bochner measurability.

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$$\left\| \sum_{k=1}^{K} \{f(t_k) - f(t_k')\} \cdot \lambda(I_k) \right\| < \varepsilon$$

whenever $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise non-overlapping intervals with $\max_k \lambda(I_k) < \delta$ and t_k, t_k' are any two points in $I_k \cap F$.

- Lusin measurability implies Riemann measurability.
- Riemann integrability implies Riemann measurability.

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The main results

Theorem

If $f : [a,b] \to X$ is $\mathscr{H}$-integrable on $[a,b]$, then f is Riemann measurable on $[a,b]$.

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If $f : [a,b] \to X$ is both bounded and Riemann measurable on $[a,b]$, then f is $\mathscr{M}$-integrable on $[a,b]$.

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If $f : [a,b] \to X$ is Riemann measurable on $[a,b]$, then f is scalarly measurable on $[a,b]$.

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If $f : [a,b] \to X$ is both McShane (Henstock) integrable and Riemann measurable on $[a,b]$, then f is $\mathscr{M}$-integrable ($\mathscr{H}$-integrable) on $[a,b]$.

Corollary

Let $f : [a,b] \to X$. Then f is Kolmogorov-Birkhoff integrable on $[a,b]$ if and only if f is both Pettis integrable and Riemann measurable on $[a,b]$.

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Let $f : [a,b] \to X$. Suppose that X is separable. If f is McShane (Henstock) integrable on $[a,b]$, then f is $\mathscr{M}$-integrable ($\mathscr{H}$-integrable) on $[a,b]$.

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Let $f : [a,b] \to X$. Suppose that X is separable. If f is McShane (Henstock) integrable on $[a,b]$, then f is $\mathscr{M}$-integrable ($\mathscr{H}$-integrable) on $[a,b]$.

Concluding remarks

- How wide the Riemann measurable function class for a non-separable range space may be?
- Let $f : [a, b] \to X$ be *bounded* on $[a, b]$. Then each of the following statements about f implies all the others.
  (i) f is $\mathscr{M}$-integrable integrable on $[a, b]$.
  (ii) f is Riemann measurable on $[a, b]$.
  (iii) f is Jeffery-measurable on $[a, b]$.
  (iv) f is $*$-measurable on $[a, b]$.
  (v) f is *almost-Riemann-integrable* on $[a, b]$.
  (vi) $Z_f = \{x^* f : x^* \in X^*, \|x^*\| \le 1\}$ has the Bourgain property.
- **Fremlin (2007)** There exists a bounded function $\varphi : [0, 1] \to \ell^\infty(\mathfrak{c})$ that is McShane integrable but not Birkhoff integrable on $[0, 1]$. As a result, φ is neither $\mathscr{M}$-integrable nor Riemann measurable on $[0, 1]$.

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Concluding remarks

- **Fremlin (2007)** Suppose that X is linearly isometric to a subspace of ℓ^∞. Then each McShane integrable X-valued function is Birkhoff integrable. Consequently, if $f : [a, b] \to X$ is McShane integrable on $[a, b]$, then f is Riemann measurable on $[a, b]$.
- **Fremlin and Mendoza (1994)** There exists a bounded function $\varphi : [0, 1] \to \ell^\infty$ that is Pettis integrable but not McShane integrable. In particular, φ is not Riemann measurable on $[0, 1]$.
- **Question:** Suppose that X is linearly isometric to a subspace of ℓ^∞. Is any Henstock integrable X-valued function necessarily Riemann measurable?

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