A note on set-valued Henstock–McShane integral in Banach (lattice) space setting

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Let (T, d) be a compact metric Hausdorff topological space and Σ its Borel σ-algebra, let $\mu : \Sigma \to \mathbb{R}_0^+$ be a regular σ-additive (bounded) measure, so that (T, d, Σ, μ) is a Radon measure space.

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If $A_i \in cbf(X),\ i = 1, \ldots, n$, we denote by $\sum_{i=1}^n A_i$ the set

$$\sum_{i=1}^n A_i := \text{cl}(A_1 + \cdots + A_n). \tag{1}$$

where + is the Minkowski addition.

H-integrable functions

Definition 1

A function $f : T \to X$ is H-*integrable* if there exists $I \in X$ such that, for every $\varepsilon > 0$ there is a gauge $\gamma : T \to \mathbb{R}^+$ such that for every γ-fine partition of T, $\Pi = \{(E_i, t_i), i = 1, \ldots, q\}$, we have:

$$\|\sigma(f, \Pi) - I\| \leq \varepsilon.$$

We set $I = (\mathrm{H}) \int_T f d\mu$. Moreover the symbol $\sigma(f, \Pi) := \sum_{i=1}^{q} f(t_i)\mu(E_i)$.

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A *decomposition* Π of T is $\Pi = \{(E_i, t_i) : i = 1, \ldots, k\}$ of pairs such that $t_i \in E_i$, $E_i \in \Sigma$ (Perron) and $\mu(E_i \cap E_j) = 0$ for $i \neq j$. The points t_i, $i = 1, \ldots, k$, are called *tags*. If $\bigcup_{i=1}^{k} E_i = T$, Π is called a *partition*.

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Proposition 2

Let us assume, in the previous setting, that μ is nonatomic, i.e. $\mu(\{t\}) = 0$ for all $t \in T$. If $f : T \to X$ is H-integrable in T, then it is also Mc Shane-integrable.

H-integrable multifunctions

From now on we assume that μ is non atomic.

Definition 3

Given a multifunction $F : T \to cwk(X)$, F is H-*integrable* if there exists an element $J \in cwk(X)$: $\forall \varepsilon > 0$ there exists a gauge γ: for every γ-fine partition Π, the following holds: $d_H(\sum_\Pi F, J) \leq \varepsilon$, where d_H:= Hausdorff distance. Then

$$J := (\mathrm{H}) \int_T F d\mu.$$

Also in this case, existence of the integral in T implies existence in all measurable subsets E of T (which will be denoted by $J_E(F)$).

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Theorem 4

[a] *If $F : T \to cwk(X)$ is H-integrable, then $S^1_{F,H} \neq \emptyset$.*

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[a]B. Cascales, V. Kadets, J. Rodríguez, J. Funct. Anal. **256** (2009),3, 673–699. L. Di Piazza, K. Musiał, Monatsh. Math. **173** (2014), 459–470

Φ multi-valued integral

Definition 5

Let $F : T \to 2^X \setminus \emptyset$ be a multifunction, and $E \in \Sigma$. We call $(\|\cdot\|)$*-integral* of F on E the set

$$\Phi(F,E) \;=\; \{\, z \in X : \forall\, \varepsilon \in \mathbb{R}^+ \exists\, \gamma : T \to \mathbb{R}^+ : \inf_{c \in \sum_{\Pi_\gamma} F} \|z - c\| \le \varepsilon$$

$$\forall\ \gamma\text{-fine partition } \Pi_\gamma := \{(E_i, t_i) : i = 1, \ldots, k\} \text{ of } E.\}$$

[1] A. Boccuto, A. R. Sambucini, *A McShane Integral for Multifunctions*, J. Concr. Appl. Math. **2** (4) (2004), 307-325
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Alternatively, one can write[1]

$$\Phi(F,E) = \bigcap_n \bigcup_\gamma \bigcap_{P_{\gamma,E}} \left[\Sigma_\Pi F + \frac{B_X}{n} \right], \tag{2}$$

where $P_{\gamma,E}$ is the family of all Henstock γ-fine partitions of E.

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Some relations:

In case F is $cwk(X)$-valued and H-integrable on E then for every $E \in \Sigma$, the following inclusion holds:

1) $J_E(F) \subset \Phi(F, E)$;

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3) Moreover we know that if f is H-integrable for every $E \in \Sigma$, then f is McShane integrable and so Definition 5 is equivalent to the $(\star)$-integral given in (2).

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3) Moreover we know that if f is H-integrable for every $E \in \Sigma$, then f is McShane integrable and so Definition 5 is equivalent to the $(\star)$-integral given in (2).

4) If we suppose that X is a separable Banach space and that there exists a countable family $(x'_n)_n$ in X' which separates points of X then the following equalities follow, for any measurable and integrably bounded multifunction F with values in $cwk(X)$:

$$J_E(F) = (\mathrm{AH})\int_E F d\mu = \Phi(F, E).$$

[2]

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Definition 6

Let S be a nonempty set. The set S is said to be a *near vector space* provided that an *addition* $+ : S \times S \to S$ is defined, such that $(S, +)$ is a cancellative commutative semigroup, and endowed with a multiplication by non-negative scalars satisfying usual properties.

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Definition 7

If $(S, \leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars which verifies:

7.1) $x \vee y$ exists for all $x, y \in S$; (joint-semilattice)

7.2) $(x \vee y) + z = (x + z) \vee (y + z)$ for all $x, y, z \in S$

then S is called a near vector lattice.

- The space $(S, +, \cdot, \leq) = (cbf(X), \oplus, \cdot, \subseteq)$ is an Archimedean near vector lattice with the unit ball B_X as a vector unit and $\mathbf{0} = \{0\}$.

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- $cwk(X)$ is a sub-near vector lattice with respect to the operations and the order induced by $cbf(X)$.

Theorem 8

[a] *Let X be any Banach space. Then there exists a compact Stonian Hausdorff space Ω and a positively linear map $i : cwk(X) \to C(\Omega)$ such that*

1) $d_H(A, C) = \|i(A) - i(C)\|_\infty, \quad A, C \in cwk(X);$
2) $i(cwk(X)) = cl(i(cwk(X))) \quad \text{(norm closure)}.$
3) $i(\overline{co}(A \cup B)) = \max\{i(A), i(B)\}$ *for all A, C in $cwk(X)$.*

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[a] C. C. A. Labuschagne, A. L. Pinchuck, C. J. van Alten, *A vector lattice version of Rådstrőm's embedding theorem*, Quaest. Math. **30** (3) (2007), 285–308.

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3) *for every $E \in \Sigma$ and $\varepsilon > 0$ there exists a gauge γ such that $\|\sigma(i(F), \Pi_1) - \sigma(i(F), \Pi_2)\|_\infty \leq \varepsilon$ for every γ-fine partitions Π_1, Π_2 of E;*

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4) *F is the sum of a non-negative H-integrable multifunction G with values in $cwk(X)$ and an H-integrable single-valued function $f : T \to X$.*

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4) *F is the sum of a non-negative H-integrable multifunction G with values in $cwk(X)$ and an H-integrable single-valued function $f : T \to X$.*

Moreover, if X is reflexive then the previous statements are equivalent to:

5) *the family $W_F = \{s(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.*

- B. Cascales, V. Kadets, J. Rodríguez, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (2009), no. 3, 673 - 699.
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Proposition 10

Let F be H-integrable. Then, for every $E \in \Sigma$ we have $J_E(F) = \Phi(F, E)$.

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Let F be H-integrable. Then, for every $E \in \Sigma$ we have $J_E(F) = \Phi(F, E)$.

Corollary 11

Let $F : T \to cwk(X)$ be any H-integrable multifunction. Then, for every $E \in \Sigma$, $M(E) := \Phi(F, E)$ is a countably additive multimeasure. Moreover, in the topology of $C(\Omega)$, M is σ-additive and μ-absolutely continuous.

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Definition 12

[a] $f : T \to X$ is oH-integrable if and only if there exist an element $J \in X$, an (o)-sequence $(b_n)_n$ ($b_n \downarrow 0$) and a corresponding sequence $(\gamma_n)_n$ of gauges, such that

$$|\sigma(f, \Pi) - J| \le b_n$$

holds, for every γ_n-fine partition Π (existence of this integral implies also integrability of $f\chi_E$, for each measurable set E).

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D. Candeloro, A.R. Sambucini, *Order-type Henstock and Mc Shane integrals in Banach lattice setting*, submitted to SISY 2014, arXiv:1405.6502 [math.FA].

Proposition 13

Let $f :\to X$ *be any* oH*-integrable mapping. Then*

- f *is also* H*-integrable, and the two integrals agree;*
- $|f|$ *is*
- *if* X *is an L-space then* f *is Bochner integrable.*

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Example 14

The function $f : [0, 1] \to c_{00}$, defined by

$$f(x) = \begin{cases} u_n & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$$

Example 15 (Skvortsov, Solodov, Real Analysis Exchange, **24** 1998-99)

Define the function $f : [0,1] \longrightarrow X$ in the following way

$$f(t) = \begin{cases} 0, \text{ if } t \in C \text{ or } t = d_i^r, & r \geq 0, 1 \leq i \leq 2^r, \\ 2 \cdot 3^r x_i^r, \text{ if } t \in (a_i^r, d_i^r), & r \geq 0, 1 \leq i \leq 2^r, \\ -2 \cdot 3^r x_i^r, \text{ if } t \in (d_i^r, b_i^r), & r \geq 0, 1 \leq i \leq 2^r. \end{cases}$$

Definition 16

Let $F : T \to 2^X$ be a multifunction with non-empty values, and $E \in \Sigma$. We call *(o)-integral* of F on E the set

$$\begin{aligned}\Phi^o(F,E) \quad = \quad & \{\, z \in X : \exists \text{ an } (o)\text{-sequence } (b_n)_n : \forall n \in \mathbb{N}\ \exists \gamma : T \to \mathbb{R}^+ : \\ & \forall \gamma\text{-fine partition } P_\gamma := \{(E_i, t_i) : i = 1, \dots, k\} \\ & \text{of } E\ \exists\ c \in \sum_{i \le k} F(t_i)\, \mu(E_i) \text{ with } |z - c| \le b_n \,\}.\end{aligned}$$

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$$\Phi^o(F,E) := \bigcup_{(b_n)_n} \bigcap_{n} \bigcup_{\gamma_n} \bigcap_{P_{\gamma_n}} \mathcal{U}\left(\Sigma_\Pi(F), b_n\right),$$

where $(b_n)_n$ denotes any (o)-sequence, γ_n any *gauge*, P_{γ_n} the family of γ_n-fine partitions of E, and the symbol $\mathcal{U}(C,b)$ (for any set $C \in X$ and any positive element $b \in X$) denotes the set of all elements $z \in X$ such that $|z - y| \le b$ for some $y \in C$

Definition 17

Let $F : T \to cwk(X)$ be any multifunction. F is oH-*integrable* if, for every $E \in \Sigma$ there exist an element $J_E \in cwk(X)$, an (o)-sequence $(b_n)_n$ in X and a corresponding sequence $(\gamma_n)_n$ of gages in T, such that, for every n and every γ_n-fine partition Π of E, we have

$$\Sigma_\Pi(F) \subset \mathcal{U}(J_E, b_n), \quad \text{and} \quad J_E \subset \mathcal{U}(\Sigma_\Pi(F), b_n).$$

This in turn implies that $\Sigma_\Pi(F) \subset \mathcal{V}(J_E, b_n), \quad \text{and} \quad J_E \subset \mathcal{V}(\Sigma_\Pi(F), b_n)$, where $\mathcal{V}(A, b) = \{z \in X : \exists\, a_0 \in A \text{ with } z \le a_0 + b\}$, for every $(A, b) \in (cwk(X), X^{++})$.

Proposition 18

If $F : T \to cwk(X)$ is oH-integrable, then

- *its integral J_E is unique;*
- $\Phi^o(F, E) = J_E \in cwk(X)$.

From now on, we shall assume that $F(t)$ are order-bounded for every $t \in T$.

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Theorem 19

Let $F : T \to cwk(X)$ be oH-integrable, with integral J, and define

$$g(t) := \sup F(t), \quad S := \sup J.$$

Then, g is oH-integrable, and its integral is S.

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Theorem 20

Let $F : T \to cwk(X)$ be any oH-integrable function, such that $\sup F(t) \in F(t)$ for each $t \in T$. Then F is the sum of an oH-integrable single-valued $g : T \to X$ and an oH-integrable $G : T \to cwk(X)$: $s(x^*, G(t)) \geq 0$ for all $x^* \in X^*$ and $s(x^*, G(t)) = 0$ for all positive elements $x^* \in X^*$.

It is well-known that in M an order unit e exists, and an equivalent norm $\|\cdot\|_e$ can be introduced, as follows: $\|x\|_e := \inf\{\alpha > 0 : |x| \leq \alpha e\}$ for all $x \in M$. With this norm, can be applied, and this gives rise to the integral Φ_e.

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Proposition 21

Given $F : T \to cwk(M)$, for every $E \in \Sigma$ we have that

(1) $\Phi^o(F, E) = \Phi(F, E) = \Phi_e(F, E)$,

(2) $I_E = \Phi(F, E) = \Phi^o(F, E)$ *whenever F is H-integrable.*

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