

On Corson's property (C) and Maharam type of measures

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Joint work with Grzegorz Plebanek

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For a Boolean algebra \mathcal{A} , $P(\mathcal{A})$ denotes the space of all probability finitely additive measures on \mathcal{A} with the topology of pointwise convergence.

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Property (C) passes to closed subspaces, quotients and products.

Pol's characterization of (C)

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Tightness vs. convex tightness

Tightness of a topological space

Tightness of a topological space X , denoted by $\tau(X)$, is the least cardinal number such that for every $A \subseteq X$ and $x \in \overline{A}$ there is a set $A_0 \subseteq A$ with $|A_0| \leq \tau(X)$ and $x \in \overline{A_0}$.

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$P(K)$ has **convex countable tightness** if $P(K)$ fulfills condition (2) of Pol's theorem:

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Assume $P(K)$ has convex countable tightness ($\equiv C(K)$ has (C)). Does this imply the countable tightness of $P(K)$?

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If $P(K \times K)$ has convex countable tightness, then for every $\mu \in P(K)$ the space $L_1(\mu)$ is separable.

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For every K , $C(K \times K)$ has property (C) if and only if $P(K \times K)$ has countable tightness.

Note that: $\tau(P(K \times K)) = \omega \Rightarrow \tau(P(K)) = \omega$.

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Question

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Equivalently, the pseudo-metric space $(\text{Borel}(K), \rho_\mu)$ is separable, where $\rho_\mu(A, B) := \mu(A \triangle B)$ for every $A, B \in \text{Borel}(K)$.

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Equivalently, μ has a countable type iff $L_1(\mu)$ is separable.

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If K carries a measure of uncountable type, then K can be continuously mapped onto $[0, 1]^{\omega_1}$. Hence, $P(K)$ has uncountable tightness.

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By the Maharam Theorem there is $\langle B_\xi \in \text{Bor}(K) : \xi < \omega_1 \rangle$ such that:

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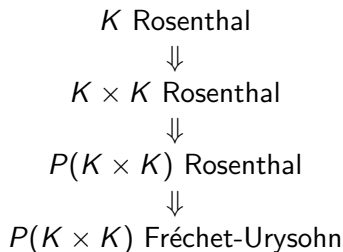
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$$\begin{array}{c} K \text{ Rosenthal} \\ \Downarrow \\ K \times K \text{ Rosenthal} \\ \Downarrow \\ P(K \times K) \text{ Rosenthal} \\ \Downarrow \\ P(K \times K) \text{ Fréchet-Urysohn} \\ \Downarrow \\ \tau(P(K \times K)) = \omega \end{array}$$

Topological dichotomy for $P(K \times K)$

Krupski, Plebanek '11

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The end

Thank you for your attention.