On Corson's property (C) and Maharam type of measures

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Joint work with Grzegorz Plebanek

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For a Boolean algebra $\mathcal{A}$, $P(\mathcal{A})$ denotes the space of all probability finitely additive measures on $\mathcal{A}$ with the topology of pointwise convergence.

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Property (C) passes to closed subspaces, quotients and products.

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Tightness vs. convex tightness

Tightness of a topological space

Tightness of a topological space X, denoted by $\tau(X)$, is the least cardinal number such that for every $A \subseteq X$ and $x \in \overline{A}$ there is a set $A_0 \subseteq A$ with $|A_0| \leqslant \tau(X)$ and $x \in \overline{A_0}$.

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$P(K)$ has ***convex countable tightness*** if $P(K)$ fulfills condition (2) of Pol's theorem:

for every $\mathcal{M} \subseteq P(K)$ *and every* $\mu \in \overline{\mathcal{M}}$ *there exists countable* $\mathcal{N} \subseteq \mathcal{M}$ *such that* $\mu \in \overline{\mathrm{conv}}\,\mathcal{N}$.

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Assume $P(K)$ has convex countable tightness ($\equiv C(K)$ has (C)). Does this imply the countable tightness of $P(K)$?

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- YES under MA(ω_1).

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If $P(K \times K)$ has convex countable tightness, then for every $\mu \in P(K)$ the space $L_1(\mu)$ is separable.

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For every K, $C(K \times K)$ has property (C) if and only if $P(K \times K)$ has countable tightness.

Note that: $\tau(P(K \times K)) = \omega \Rightarrow \tau(P(K)) = \omega$.

$C(K) \in (C) \Rightarrow C(K \times K) \in (C)$?

Question

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Equivalently, μ has a countable type iff $L_1(\mu)$ is separable.

Motivational theorems

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If K carries a measure of uncountable type, then K can be continuously mapped onto $[0,1]^{\omega_1}$. Hence, $P(K)$ has uncountable tightness.

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Idea of the proof

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By the Maharam Theorem there is $\langle B_\xi \in Bor(K) : \xi < \omega_1 \rangle$ such that:

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For $\xi < \omega_1$ construct $\nu_\xi \in P(Bor(K) \otimes Bor(K))$ in such a way that:

- $\nu_\xi(B_\eta \times B_\eta) = 1/4$ for $\eta < \xi$,
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Topological dichotomy for $P(K \times K)$

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Either $P(K \times K)$ contains a $\mathbb{G}_\delta$ point or $P(K \times K)$ has uncountable tightness.

The end

Thank you for your attention.