Descriptive Characterizations of Pettis and Bochner Integrals on *m*-Dimensional Compact Intervals

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Abstract

We give necessary and sufficient conditions for an additive interval function $F : \mathcal{I} \to X$ to be the primitive of a Pettis or Bochner integrable function $f : I_0 \to X$. We consider the additive interval functions defined on the family $\mathcal{I}$ of all non-degenerate closed subintervals of the unit interval $I_0 = [0, 1]^m$ in the Euclidean space $\mathbb{R}^m$ and taking values in a Banach space X.

Introduction I

There are well-known results about characterizations of the primitive $F : [0,1] \to X$ of a Pettis or Bochner integrable function $f : [0,1] \to X$ in terms of scalar derivative or differential of F.

Introduction II

- In the paper **"On Denjoy type extensions of the Pettis integral"**, K. M. Naralenkov has proved that F **is the primitive of a Pettis integrable function** f **if and only if** F **is absolutely continuous and** f **is a scalar derivative of** F.
- In the Monograph **"Topics in the Banach Space Integration"** of Š. Schwabik and Y. Guoju, there is a descriptive characterization of Bochner integral, (see Theorem 7.4.15). According to this result, F **is the primitive of a Bochner integrable function** f **if and only if** F **is strongly absolutely continuous and** $F'(t) = f(t)$ **at almost all** $t \in [0, 1]$.

Introduction III

Then a question arises naturally:

Are there similar results for the higher-dimensional case ?

- The main tool in the proof of the above characterizations relies on the **Vitali covering theorem** in the real line.
- In higher-dimensional Euclidean spaces, the Vitali covering theorem requires **regularity**, which is the source of difficulty.

We use the notions of the **cubic average range** and the **cubic derivative** of interval functions to overcome this difficulty. So, the descriptive characterizations of Pettis and Bochner integrals are given in terms of the cubic average range and the cubic derivative of their primitives.

Outline

Notation

The following basic notation will be used in this presentation

- X is a **real Banach space** with its norm $||.||$,
- X^* is the **topological dual** to X,
- $\mathbb{R}^m$ is the m**-dimensional Euclidean space** equipped with the maximum norm,
- Given an interval $I \in \mathcal{I}$, the ratio of its **shortest** side s_I to its **longest** side l_I, denoted by $\mathbf{reg}(I) = s_I / l_I$, is the **regularity** of I,
- λ^m is the **Lebesgue measure** on I_0; **the volume** of an interval $I \in \mathcal{I}$ is denoted by $|I|$,
- $\mathcal{M}$ is the **family of all Lebesgue measurable subset** of I_0,

The Cubic Average Range and the Cubic Derivative I

Assume that an interval function $F : \mathcal{I} \to X$, a point $t \in I_0$ and a real number $\alpha \in (0, 1]$ are given.

- $\mathcal{I}_{t,\alpha} = \{I \in \mathcal{I} : t \in I, \mathrm{reg}(I) \geq \alpha\}$
- $A_F(t, \delta, \alpha) = \{\frac{F(I)}{|I|} : I \in \mathcal{I}_{t,\alpha}, |I| < \delta\}$
- $A_F(t, \alpha) = \bigcap_{\delta>0} \overline{A_F}(t, \delta, \alpha)$, where $\overline{A_F}(t, \delta, \alpha)$ is the closure of $A_F(t, \delta, \alpha)$

The set $A_F(t, 1)$ is said to be the **cubic average range** of F at t.

The Cubic Average Range and the Cubic Derivative II

The interval function $F : \mathcal{I} \to X$ is said to be the **cubic derivable** at the point t if there is a vector $x \in X$ such that

$$\lim_{|C_t| \to 0} \frac{F(C_t)}{|C_t|} = x,$$

where $C_t \in \mathcal{I}$ is a cubic interval and t is a vertex of C_t. The vector x is said to be the **cubic derivative** of F at t.

A Scalar Derivative

A function $f : I_0 \to X$ is said to be a **scalar derivative** of the interval function $F : \mathcal{I} \to X$ if for each $x^* \in X^*$, we have

$$(x^* F)'(t) = (x^* f)(t) \quad \text{at almost all} \quad t \in I_0$$

(the exceptional set may vary with x^*).
For the notion of the derivative of a real-valued interval function, we refer to the Monograph **"Henstock-Kurzweil Integration on Euclidean Spaces"** of L. T. Yeong.

Additive Interval Function

- At first, two intervals $I, J \in \mathcal{I}$ are said to be **non-overlapping** if $I^o \cap J^o = \emptyset$, where I^o is the interior of I,
- A finite collection $\mathscr{D} = \{I_1, \ldots, I_p\}$ of pairwise non-overlapping intervals in $\mathcal{I}$ is said to be a **division** of the interval $I \in \mathcal{I}$ if $\bigcup_{j=1}^{p} I_j = I$,
- The interval function $F : \mathcal{I} \to X$ is said to be **additive** if for each interval $I \in \mathcal{I}$, we have

$$F(I) = \sum_{J \in \mathscr{D}} F(J),$$

whenever $\mathscr{D}$ is a division of the interval I.

Strong Absolute Continuity and Absolute Continuity

- The additive interval function $F : \mathcal{I} \to X$ is said to be **strongly absolutely continuous (sAC)** if for each $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\sum_{j=1}^{p} ||F(I_j)|| < \varepsilon$$

  whenever $\{I_1, \ldots, I_p\}$ is a finite collection of pairwise non-overlapping intervals in $\mathcal{I}$ with $\sum_{j=1}^{p} |I_j| < \eta$.
- Replacing the above inequality with $||\sum_{j=1}^{p} F(I_j)|| < \varepsilon$, we obtain the notion of the **absolute continuity (AC)**.

The main result here is the following theorem. Its proof is similar in spirit to CARATHEODORY-HAHN-KLUVANEK EXTENSION THEOREM, (see Theorem I.5.2 in the Monograph **"Vector Measures"** of J. Diestel and J. J. Uhl, Jr.)

Theorem (The First Extension Theorem)

If an additive interval function $F : \mathcal{I} \to X$ is absolutely continuous, then F has a unique extension to a countably additive λ^m-continuous vector measure $F_{\mathcal{M}} : \mathcal{M} \to X$.

In general, $F_{\mathcal{M}}$ is not of σ-finite variation, even if X has the weak Radon-Nikodym property.

- **There exists a countably additive λ-continuous vector measure $\nu : \mathcal{L} \to L^2$ such that there is no Pettis integrable function $f : [0,1] \to L^2$ satisfying**

$$\nu(E) = (P)\int_E f d\lambda \quad \text{for all} \quad E \in \mathcal{L},$$

- **and since L^2 has the weak Radon-Nikodym property, ν is not of σ-finite variation.**

(*see "On integration in vector spaces", B. J. Pettis (p.303) and "Integration of functions with values in a Banach space", G. Birkhoff (p.376)*)

- **Thus, the additive interval function F defined by equality**

$$F([a,b]) = \nu([a,b]) \quad \text{for all} \quad [a,b] \subset [0,1], (a < b),$$

is absolutely continuous and $F_{\mathcal{M}} = \nu$ is not of σ-finite variation.

(λ is *the Lebesgue measure on* $[0,1]$, $\mathcal{L}$ is *the family of all Lebesgue measurable subset of* $[0,1]$ and L^2 is *the space of Lebesgue square-integrable functions from* $[0,1]$ *to* $\mathbb{R}$).

Applying the First Extension Theorem to an additive interval function which is strongly absolutely continuous, we obtain the Second Extension Theorem.

Theorem (The Second Extension Theorem)

If an additive interval function $F : \mathcal{I} \to X$ is strongly absolutely continuous, then F has a unique extension to a countably additive λ^m-continuous vector measure $F_{\mathcal{M}} : \mathcal{M} \to X$. Moreover, the vector measure $F_{\mathcal{M}}$ is of bounded variation.

In other words, if the function F in The First Extension Theorem becomes strongly absolutely continuous, then its extension $F_{\mathcal{M}}$ becomes of bounded variation.

Let's now highlight the relationship between the cubic derivative and the cubic average range of an interval function $F : \mathcal{I} \to X$ at a point $t \in I_0$. First, we have

Lemma

If the interval function F is the cubic derivable at the point t, then

$$A_F(t, 1) = \{x_0\},$$

where x_0 is the cubic derivative of F at t.

It is a bit surprising that **the converse to this lemma is false.**

If we consider the family $\mathcal{I}_1$ of all closed non-degenerate subintervals of $I_1 = [-1, 1]^m$ and define the interval function

$$F : \mathcal{I}_1 \to l_p \quad (p > 1)$$

by equality

$$F(I) = \begin{cases} (0, \dots, 0, \dots) & \text{if} \quad |I| \neq \frac{1}{n} \\ (0, \dots, 0, \frac{1}{n}, 0, \dots) & \text{if} \quad |I| = \frac{1}{n} \end{cases} \qquad I \in \mathcal{I}_1,\ n \in \mathbb{N}$$

then

- F **is not cubic derivable at** $0 \in \mathbb{R}^m$,
- $A_F(0, 1) = \{(0)\}$.

Theorem (Descriptive Characterizations of Pettis Integral)

If $F : \mathcal{I} \to X$ is an additive interval function, then the following statements are equivalent.

(i) *F is the primitive of a Pettis integrable function $f : I_0 \to X$, i.e.,*

$$F(I) = (P)\int_I f d\lambda^m \quad \text{for all} \quad I \in \mathcal{I},$$

(ii) *F is AC and f is a scalar derivative of F,*

(iii) *F is AC and for each $x^* \in X^*$, we have*

$$(x^* f)(t) \in A_{x^* F}(t, 1) \quad \text{at almost all} \quad t \in I_0$$

(the exceptional set may vary with x^*).

The main tools in the proof of the first main theorem are

- The Descriptive Characterizations of Lebesgue Integral
- The First Extension Theorem
- The Relationship between the Cubic Average Range and the Cubic Derivative of a real-valued interval function.

By the same way as in the paper **"On Denjoy type extensions of the Pettis integral"** of K. M. Naralenkov, it can be proved that if

- X **has the weak Radon-Nikodym property**,
- $F : \mathcal{I} \to X$ is an additive interval function,

then, **in general, the following statements are not equivalent.**

(i) F is absolutely continuous,

(ii) F is the primitive of a Pettis integrable function.

Theorem (Descriptive Characterizations of Bochner Integral)

If $F : \mathcal{I} \to X$ is an additive interval function, then the following are equivalent.

(i) *F is the primitive of a Bochner integrable function $f : I_0 \to X$, i.e., $F(I) = (B)\int_I f d\lambda^m$ for all $I \in \mathcal{I}$,*

(ii) *F is sAC and*

$$\lim_{|C_t|\to 0} \frac{F(C_t)}{C_t} = f(t) \quad \text{at almost all} \quad t \in I_0,$$

where $C_t \in \mathcal{I}$ is a cubic interval and t is a vertex of C_t,

(iii) *F is sAC and $f(t) \in A_F(t,1)$ at almost all $t \in I_0$.*

The main tools in the proof of the second main theorem are

- The First Main Theorem,
- The Second Extension Theorem,
- The Relationship between the Cubic Average Range and the Cubic Derivative of an interval function $F : \mathcal{I} \to X$,
- A Generalization of Lebesgue Differentiation Theorem.

Finally, let's present two useful corollaries of the above theorem.

The First Corollary

It is well-known that **a function** $f : I_0 \to X$ **is Bochner integrable on** I_0 **if and only if** f **is strongly McShane integrable on** I_0**. Moreover**,

$$(B)\int_I f d\lambda^m = (M)\int_I f d\lambda^m \quad \text{for all} \quad I \in \mathcal{I}.$$

Thus, replacing in the Descriptive Characterization Theorem of Bochner Integral, the Bochner integral with the strong McShane integral, we obtain descriptive characterizations of the strong McShane integral.

The Second Corollary

If we have

- X has the Radon-Nikodym property,
- $F : \mathcal{I} \to X$ is an additive interval function,

then the following statements are equivalent

(i) F is strongly absolutely continuous,

(ii) F is the primitive of a Bochner integrable function,

(iii) F is the primitive of a strongly McShane integrable function.

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Thank You