Openness of measures and closedness of their range

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Outline

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Closedness of range of a measure

This well-known result was first establishd by Lyapunov and rediscovered by Halmos some years later.

Theorem (Lyapunov, 1940; Halmos, 1948)

The range of an $\mathbb{R}^N$-valued σ-additive measure defined on a σ-algebra is closed.

A. A. Lyapunov.
Sur les fonctions-vecteurs complètement additives (in Russian).
Izvestia Akad. Nauk SSSR Ser. Mat., 4 (1940), 465–478.

P. R. Halmos.
The range of a vector measure.
Bull. Amer. Math Soc., 54 (1948), 416–421.

Open Maps

Let X and Y be topological spaces, and let x be a point in X.

Definition

A map T of X onto Y is open at x if for each neighborhood U of x the image $T(U)$ is a neighborhood of $T(x)$.
We simply say that T is open if it is open at each point of X.
Equivalently: The image of every open set is open.

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Openness Lemma

If X, Y are 1st-countable (in particular pseudo-metrizable), a map T of X onto Y is open at $x \in X$ if and only if for every sequence $(y_n)_{n\in\mathbb{N}}$ in Y converging to $T(x)$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x and such that $T(x_n) = y_n$ for each $n \in \mathbb{N}$.

Openness of Measures

Let $\mathcal{A}$ be an algebra of sets, G a topological Abelian group, and $\mu\colon \mathcal{A} \to G$ a measure. It is well-known that there is the weakest FN-topology $\tau(\mu)$ which makes μ continuous, called μ-topology. Its neighborhoods at $\emptyset$ have the form

$$U_W = \{A \in \mathcal{A} \mid \forall B \subseteq A \quad \mu(B) \in W\},$$

where W is a neighborhood of 0 in G.

Theorem (A. Spakowski, 1988)

An $\mathbb{R}^N$-valued σ-additive nonatomic measure μ on a σ-algebra $\mathcal{A}$ is open when regarded as a map onto its image $\mu(\mathcal{A})$, where $\mathcal{A}$ is endowed with the μ-topology.

A. Spakowski.
Openness of vector measures and their integral maps.
J. Austral. Math. Soc. Ser. A, 45, no. 3 (1988), 351–359.

Outline

From Boolean Algebras to D-lattices

L: **Boolean algebra**
μ: measure
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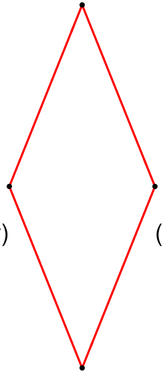

L: MV-algebra
μ: measure
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L: D-lattice
μ: modular measure

D-Posets and D-Lattices

Definition

A D-poset is a partially ordered set, having 0 and 1, endowed with a partially defined operation $\ominus$ such that:

1. $a \ominus b$ is defined if and only if $b \leq a$
2. If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$
3. If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$

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A D-lattice is a D-poset which is also a lattice.

- $a^{\perp} := 1 \ominus a$
- $a \perp b$ means $a \leq b^{\perp}$ (equivalently $b \leq a^{\perp}$)
- $a \oplus b := (a^{\perp} \ominus b)^{\perp}$ whenever $a \perp b$
  The sum $\oplus$ is commutative and associative.

From D-Lattices to Boolean Algebras

D-posets and D-lattices were introduced by Chovanec and Kôpka in 1994–1995.

In the same years, Bennett and Foulis defined independently the equivalent structure of effect algebra, for modelling unsharp measurement in a quantum mechanical system.

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- An orthomodular lattice is a D-lattice satisfying
  1. If $b \leq a^{\perp}$ then $a \wedge b = 0$.
- An MV-algebra is a D-lattice satisfying
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- A Boolean algebra is a D-lattice satisfying both 1 and 2.

Applications of Effect Algebras

These structures, and other generalizations of Boolean algebras, such as MV-algebras or orthomodular lattices, which are particular cases of effect algebras, have applications in

- Logic and Programming Languages:
  Effect algebras, MV-algebras
- Quantum Physics and Quantum Logic:
  Effect algebras, orthomdular lattices
- Mathematical Economics, Theory of Decision:
  Effect algebras, MV-algebras, orthomodular lattices
- Measure Theory:
  MV-algebras, orthomodular lattices, Boolean algebras

Orthogonal sequences

- The sum $\bigoplus_{k=1}^{n} a_k$ of a finite sequence is defined as: $\bigoplus_{k=1}^{1} a_k = a_1$, and $\bigoplus_{k=1}^{n+1} a_k = \left(\bigoplus_{k=1}^{n} a_k\right) \oplus a_{n+1}$ provided that the right-hand side exists.
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- If the sum $\bigoplus_{k=1}^{n} a_k$ exists, we say that the finite sequence $(a_1, \ldots, a_n)$ is orthogonal.

- We say that an infinite sequence $(a_n)_{n\in\mathbb{N}}$ is orthogonal if the sum $\bigoplus_{k=1}^{n} a_k$ exists for all $n \in \mathbb{N}$.
- The sum of an infinite orthogonal sequence $(a_n)_{n\in\mathbb{N}}$ is $\bigoplus_{k=1}^{+\infty} a_k := \sup_{n\in\mathbb{N}} \bigoplus_{k=1}^{n} a_k$ (if the supremum exists).

Modular Measures

L: D-lattice. G: Hausdorff topological Abelian group.

Definition

A map $\mu\colon L \to G$ is a modular measure if the following hold

1. $\forall a, b \in L$ with $a \perp b$ we have $\mu(a \oplus b) = \mu(a) + \mu(b)$
2. $\forall a, b \in L$ we have $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$

Let $\mu\colon L \to G$ be a modular measure.
We say that $a \in L$ is μ-negligible if $\mu(b) = 0$ whenever $b \leq a$.
The set of μ-negligible elements will be denoted by $I(\mu)$.

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Definition

A modular measure μ is σ-additive if, given any orthogonal sequence $(a_n)_{n\in\mathbb{N}}$ for which the sum exists, we have $\mu(\bigoplus_{k=1}^{+\infty} a_k) = \sum_{k=1}^{+\infty} \mu(a_k)$.

D-Uniformities

Definition

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Let G be a topological Abelian group.
Given a G-valued modular measure μ, there is always the weakest D-uniformity $\mathcal{U}(\mu)$ making μ uniformly continuous, called μ-uniformity.

The topology induced by $\mathcal{U}(\mu)$ is called the μ-topology.
In case L is a Boolean algebra, this definition of μ-topology agrees with the classical one.

Outline

Interior Points of the Range

L: D-lattice. $\mu\colon L \to \mathbb{R}^N$: modular measure.

Definition

We say that μ is nonatomic if given any $\varepsilon > 0$ there exists an orthogonal sequence $(a_1, \dots, a_m)$ with $\bigoplus_{k=1}^m a_k = 1$ such that for each $k \in \{1, \dots, m\}$ and every $b \leq a_k$ one has $\|\mu(b)\| \leq \varepsilon$.

Definition

L is σ-complete if every increasing sequence has a supremum.

Interior Points Theorem

Suppose L is σ-complete and μ is nonatomic. Let $a \in L$. If $\mu(a)$ is an interior point of the range $\mu(L)$, then μ is open at a.

The One-Dimensional Case

Let $\mu\colon L \to \mathbb{R}$ a modular measure, where L is σ-complete.

Closedness Theorem

If μ is σ-additive, the range $\mu(L)$ is a closed bounded interval.

Proof of the Closedness Theorem (sketch)

A generalization of Lyapunov theorem (G. Barbieri, 2004) ensures that $\mu(L)$ is convex. Moreover a σ-additive modular measure on a σ-complete D-lattice achieves its minimum and its maximum (H. Weber, 1996). □

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Openness Theorem

If μ is nonatomic, then it is an open map.

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Let L be a σ-complete D-lattice. A σ-additive modular measure $\mu\colon L \to \mathbb{R}^N$ has closed range. If μ is nonatomic then it is open.

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Example (a non-open σ-additive measure with convex range)

The σ-additive atomic measure μ, of $\mathcal{P}(\mathbb{N})$ onto $[0,1]$, defined by the formula $\mu(\{n\}) = 2^{-n}$ for $n \in \mathbb{N}$, is not open at $A = \{1\}$.

Proof of the Example.

If $A_n = \mathbb{N} \setminus \{1, n+1\}$ then $\lim_n \mu(A_n) = \lim_n \frac{1}{2} - \frac{1}{2^{n+1}} = \frac{1}{2} = \mu(A)$. Given $B_n \subseteq \mathbb{N}$ with $\mu(B_n) = \mu(A_n)$, we have $\mu(B_n) < \frac{1}{2}$, hence $1 \notin B_n$ and thus $\mu(B_n \triangle \{1\}) = \mu(B_n \cup \{1\}) \geq \frac{1}{2}$. Therefore B_n does not converge to $A = \{1\}$ with respect to the μ-topology. $\square$

Outline

Order Continuous Modular Measures

L: D-lattice. Y: Banach space.

If $(a_j)_{j\in J}$ is a decreasing net in L, and $a = \inf_{j\in J} a_j$, we write $a_j \searrow a$.

Definition

We say that a modular measure $\mu\colon L \to Y$ is

σ-order continuous (σ-o.c.) if $a_n \searrow 0$ implies $\lim_n \mu(a_n) = 0$, for every decreasing **sequence** $(a_n)_{n\in\mathbb{N}}$

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Theorem

A modular measure $\mu\colon L \to Y$ is σ-order continuous if and only if it is σ-additive.

Proof of the Interior Points Theorem

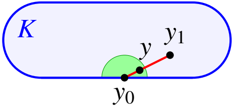
Interior Points Lemma

Let f be a map of a topological space X onto a subset K of a topological linear space Y. Fix $x_0 \in X$ and $y_0 := f(x_0)$. Suppose that for each $\varepsilon > 0$ there is a neighborhood V of 0 in Y such that for every $y \in (y_0 + V) \cap K$ we have $y_0 + \frac{1}{\varepsilon}(y - y_0) \in K$. Suppose also that there is $\gamma : X \times [0,1] \to X$ satisfying:

1. $\gamma(x, 0) = x_0$;
2. $\gamma(x, \cdot)$ *is equicontinuous at* 0 *(uniformly w.r.t.* $x \in X$*);*
3. $f(\gamma(x,t)) = (1-t)f(x_0) + tf(x)$ *for all* $x \in X$ *and* $t \in [0,1]$.

Then f is open at x_0.

The situation is illustrated in the figure at right where the set $(y_0 + V) \cap K$ is colored in green and the point $y_0 + \frac{1}{\varepsilon}(y - y_0)$ is denoted by y_1.



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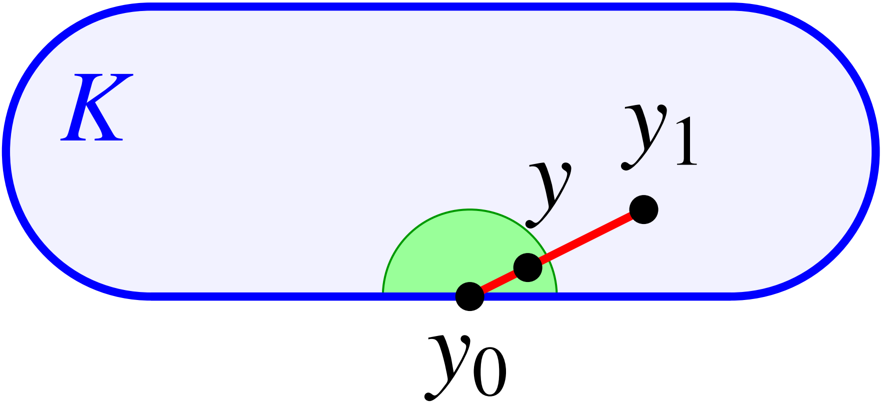

Uniformly Equicontinuous Paths

Path Theorem

Let $\mu\colon L \to \mathbb{R}^N$ be a modular measure, where L is σ-complete and μ is nonatomic. There exists $\gamma\colon L \times L \times [0,1] \to L$ such that, for all $a, b \in L$ and every $t \in [0,1]$:

1. $\gamma(a,b,0) = a, \quad \gamma(a,b,1) = b$;
2. $\mu\big(\gamma(a,b,t)\big) = (1-t)\mu(a) + t\mu(b)$;
3. $\gamma(a,b,\cdot)$ *is uniformly equicontinuous with respect to* $\mathcal{U}(\mu)$.

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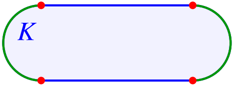
Applying the above theorem, and the Interior Points Lemma we obtain the Interior Points Theorem,
as well as the Openness Theorem in the one-dimensional case.

Exposed Points

Definition

Let K be a subset of a Banach space Y, and let $p \in K \subset Y$.
We say that p is an exposed point of K if there is $y^* \in Y^*$ such that $y^*(p) > y^*(y)$ for all $y \in K \setminus \{p\}$.
In this case we also say that K is exposed at p by y^*.

The exposed points are colored in green, while the four points marked in red are extreme points but not exposed points.



Exposed Points Lemma

Let L be a σ-complete D-lattice, $a \in L$, and Y a Banach space. If $\mu\colon L \to Y$ is a σ-additive modular measure with $\mu(L)$ exposed at $\mu(a)$ by a $y^* \in Y^*$, then μ is open at a. Also, given $(a_n)_{n\in\mathbb{N}}$ in L such that $\lim_n y^*(\mu(a_n)) = y^*(\mu(a))$, we have $\lim_n \mu(a_n) = \mu(a)$.

Decomposition by Central Elements

Definition

An element $c \in L$ is central if $\forall a \in L \quad (a \wedge c) \vee (a \wedge c^{\perp}) = a$.

The set $C(L)$ of central elements (center) is a Boolean algebra.

Decomposition Theorem

Let L be a complete D-lattice, Y a Banach space, and $\mu : L \to Y$ an o.c. modular measure with no nonzero μ-negligible elements. Given $y^* \in Y^* \setminus \{0\}$, there is a decomposition of μ into the sum of two modular measures h_1, h_2 such that:

1. $h_1 : a \mapsto \mu(a \wedge s)$ *and* $h_2 : a \mapsto \mu(a \wedge s^{\perp})$*, where* $s \in C(L)$*;*
2. $y^* \circ h_1 = 0$*;*
3. $h_2(L)$ *is exposed by* y^* *at* $h_2(p)$ *for some* $p \leq s^{\perp}$*.*

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   - Apply Decomposition Theorem to get $\mu = h_1 + h_2$, a central element s, and $p \leq s^\perp$. Let $(z_n)_{n \in \mathbb{N}}$ in Y with $\lim_n z_n = z$ and, for every n, let $a_n \in L$ such that $\mu(a_n) = z_n$. Since $y^*\big(h_2(p)\big) = y^*(z) = \lim_n y^*\big(\mu(a_n)\big) = \lim_n y^*\big(h_2(a_n)\big)$, by Exposed Points Lemma, $h_2(p) = \lim_n h_2(a_n)$, so $z - h_2(q) \in \overline{h_1(L)}$.

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   - Now let $a \in L$ such that $\mu(a) = z$. Then μ is open at a, because:

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   - Apply Decomposition Theorem to get $\mu = h_1 + h_2$, a central element s, and $p \leq s^\perp$. Let $(z_n)_{n\in\mathbb{N}}$ in Y with $\lim_n z_n = z$ and, for every n, let $a_n \in L$ such that $\mu(a_n) = z_n$. Since $y^*\big(h_2(p)\big) = y^*(z) = \lim_n y^*\big(\mu(a_n)\big) = \lim_n y^*\big(h_2(a_n)\big)$, by Exposed Points Lemma, $h_2(p) = \lim_n h_2(a_n)$, so $z - h_2(q) \in \overline{h_1(L)}$.
   - The range of h_1 has dimension $< N$, hence is closed. Thus we find $q \in L$ with $h_1(q) = z - h_2(p)$. Therefore $z = \mu\big((q \wedge s) \oplus p\big)$.
   - Now let $a \in L$ such that $\mu(a) = z$. Then μ is open at a, because:
     - $h_1(L)$ has dimension $< N$, hence h_1 is open at a;

Sketch of Proof of the Main Theorem

1. Up to a quotient, we assume L complete, μ o.c., and $I(\mu) = \{0\}$.
2. Consider the case when μ is nonatomic. Argue by induction: If $N = 1$ the result has been already established, so let $N > 1$.
   - In view of the Interior Points Theorem, it suffices to consider a boundary point z of $\mu(L)$: we will prove that $z \in \mu(L)$ and that μ is open at any $a \in L$ such that $\mu(a) = z$.
   - As $\mu(L)$ is convex, we find $y^* \in Y^* \setminus \{0\}$ with $y^*(z) = \sup_{x \in L} y^*\big(\mu(x)\big)$.
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   - The range of h_1 has dimension $< N$, hence is closed. Thus we find $q \in L$ with $h_1(q) = z - h_2(p)$. Therefore $z = \mu\big((q \wedge s) \oplus p\big)$.
   - Now let $a \in L$ such that $\mu(a) = z$. Then μ is open at a, because:
     - $h_1(L)$ has dimension $< N$, hence h_1 is open at a;
     - by Exposed Points Lemma, h_2 is open at a, too.

Sketch of Proof of the Main Theorem (continued)

3. In the general case (when μ is not necessarily nontomic) we have to prove that the range $\mu(L)$ is closed.

Sketch of Proof of the Main Theorem (continued)

3. In the general case (when μ is not necessarily nontomic) we have to prove that the range $\mu(L)$ is closed.
4. Use Hammer–Sobczyk Decomposition$^{(*)}$ to get $\mu = \lambda + \xi$, where λ, ξ are modular measures such that λ is nonatomic and ξ has compact range.

$(*)$

Sketch of Proof of the Main Theorem (continued)

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(*) A. Avallone, G. Barbieri, P. Vitolo and H. Weber.
Decomposition of effect algebras and the Hammer-Sobczyk theorem.
Algebra Universalis, 60, no. 1 (2009), 1–18.

Sketch of Proof of the Main Theorem (continued)

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5. Applying Step 2 to λ, we obtain that $\lambda(L)$ is closed.

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Sketch of Proof of the Main Theorem (continued)

3. In the general case (when μ is not necessarily nontomic) we have to prove that the range $\mu(L)$ is closed.
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5. Applying Step 2 to λ, we obtain that $\lambda(L)$ is closed.
6. Then $\mu(L)$ is closed because it coincides with $\lambda(L) + \xi(L)$.

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Sketch of Proof of the Main Theorem (continued)

3. In the general case (when μ is not necessarily nontomic) we have to prove that the range $\mu(L)$ is closed.
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Q.E.D.

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Thank you for your attention